

Do We Need Recursion?

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Outline

Recursion in various situations. Is its use necessary?

The expressive power of bounded conditions and formulas

Arithmetization of syntactic notions without recursion

Primitive recursion, course-of-values recursion

The equations $z^0 = 1$ and $z^{x+1} = z^x \cdot z$ derive the exponential function $[x, z] \mapsto z^x$ by **primitive recursion** from g and h where $g(z) = 1$ and $h(v, x, z) = v \cdot z$.

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The equation $f(x) = g(\mu v (g(v) \notin \{f(0), \dots, f(x-1)\}))$ derives f from g by **course-of-values recursion** (and minimization). If $\text{Rng}(g)$ is infinite, then f is one-to-one and $\text{Rng}(f) = \text{Rng}(g)$.

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Consider the definition: *t is a term in the arithmetic language if t is the constant 0, or t is a variable, or t has one of the forms $s(t_1)$, $+(t_1, t_2)$ or $\cdot(t_1, t_2)$ where t_1 and t_2 are terms.*

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Once *variables* are defined (say, as strings like v_{1011}), a programmer can write a procedure that decides what is and what is not a term by making calls to itself.

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Coding of syntactic objects: the term $+(v1, 0)$ is the number $43 \cdot 128^6 + 40 \cdot 128^5 + 118 \cdot 128^4 + 49 \cdot 128^3 + 44 \cdot 128^2 + 48 \cdot 128 + 41$. The codes 43, 40, 118, ... are taken from modified ascii table.

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If terms are (identified with) natural numbers, then the above definition is an application of course-of-values recursion.

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3. In the definitions of syntactic notions in logic:

terms, formulas, free and bound occurrences of variables, substitutability of terms, the substitution operation itself.

How to get rid of recursion in definitions? Why?

If f is derived from g by minimization, $f(x) = \mu v(g(x, v) = 0)$,
then $y = f(x) \Leftrightarrow g(x, y) = 0 \ \& \ \forall v < y (g(x, v) \neq 0)$.

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Another option

Using Δ_0 conditions.

Δ_0 conditions, Δ_0 -formulas

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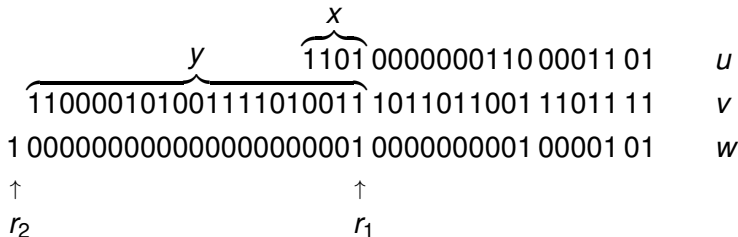
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Is the set $\{ y ; \exists x (y = 2^x) \}$ bounded? Answer: $\exists x (y = 2^x)$ is equivalent to $\forall v \leq y (v \mid y \rightarrow (v = 1 \vee 2 \mid v))$.

Exponentiation, i.e. the condition $y = z^x$

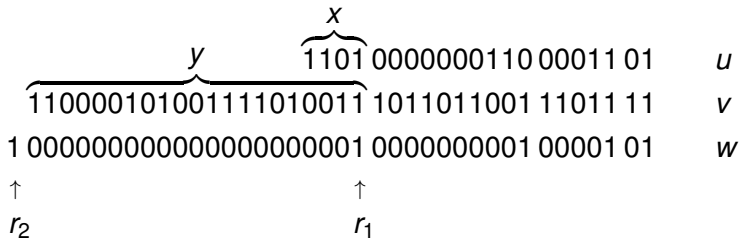
	$\overbrace{\hspace{10em}}^y \overbrace{\hspace{3em}}^x$	
	110000101001111010011	u
	10110110011101111	v
	$100000000000000000000100000000010000101$	w
\uparrow		\uparrow
r_2		r_1

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Let $\text{ExpW}(y, x, z, u, v, w)$ be a formula (which obviously is Δ_0) that describes this data structure. It says that if an item in u is t and the corresponding item in v is s , then either the next items are $2t$ and s^2 , or they are $2t + 1$ and $s^2 \cdot z$, etc.

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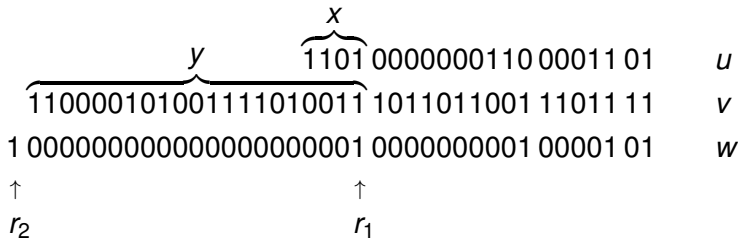


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Number of occurrences of a character: if w is the number $83 \cdot 128^6 + 40 \cdot 128^5 + 83 \cdot 128^4 + 40 \cdot 128^3 + 48 \cdot 128^2 + 41 \cdot 128 + 41$, then $\text{NOcc}(48, w) = 1$ and $\text{NOcc}(41, w) = 2$.

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Terms (in the arithmetic language)

A string w is **balanced** if $\text{Lh}(w) \geq 2$, $\text{NOcc}((), w) = \text{NOcc}(), w$, and $\text{NOcc}((), u) > \text{NOcc}(), u$ for any proper initial segment u of w . Example: $(()())$. Non-examples: $\forall 1011$ and $()()$.

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A quasiterm t is a **term** (abbreviated $Term(t)$) if every balanced substring (w) of t is either immediately preceded by the letter s and w is a quasiterm, or it is immediately preceded by $+$ or \cdot and w has the form u, v where u and v are quasiterms.

Terms (in the arithmetic language)

A string w is **balanced** if $Lh(w) \geq 2$, $\text{NOcc}((, w) = \text{NOcc}() , w)$, and $\text{NOcc}((, u) > \text{NOcc}() , u)$ for any proper initial segment u of w . Example: $(() ())$. Non-examples: $\forall 1011$ and $() ()$.

Quasiterm is any variable, the single-letter string 0 , or any string of the form $s(w)$, $+(w)$ or $\cdot(w)$ where (w) is a balanced string. Examples: $+((0))$ and $s(() () ())$.

A quasiterm t is a **term** (abbreviated $\text{Term}(t)$) if every balanced substring (w) of t is either immediately preceded by the letter s and w is a quasiterm, or it is immediately preceded by $+$ or \cdot and w has the form u, v where u and v are quasiterms.

Properties of terms provable in PA: Any variable and the string 0 are terms. If t_1 and t_2 are terms, then $s(t_1)$, $+(t_1, t_2)$ and $\cdot(t_1, t_2)$ are terms. Any term has one the forms $s(t_1)$, $+(t_1, t_2)$ or $\cdot(t_1, t_2)$ unless it is a variable or the string 0 .

Appendix: the number of positive bits






Work with a *summation tree* w for a number x :

0000000	$\overbrace{1011100101110101111001111}^x$	0
0000000101	1000011001011010001010	1
000001011	001011011010100	2
0001	010001100110	3
00101	01100	4
10001		5

where the bits (of the single number w) are split to several lines for better readability. It can be checked that $y = \text{NPB}(x)$ is a Δ_0 -formula.

In the above example, the summation tree witnesses the fact that the number of positive bits in the number 24 308 687 is 17.

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