Gödel-Dummett Predicate Logics and Axioms of Prenexability

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Outline

Introduction: What is Gödel-Dummett logic?

Logics and axioms of prenexability

Properties of Gödel-Dummett logics, problems

Gödel-Dummett propositional logic BG

Semantical definition

Truth values are numbers from the real interval $[0, 1]$; truth function of implication $\rightarrow$ is the function $\Rightarrow$ where $a \Rightarrow b = 1$ if $a \leq b$, and $a \Rightarrow b = b$ otherwise; truth functions of & and $\vee$ are min and max; tautologies are formulas with value 1 under any truth evaluation.

Properties

- Axiomatized by intuitionistic Hilbert-style calculus enhanced by the prelinearity schema: $(A \rightarrow B) \vee (B \rightarrow A)$.
- FMP,
- G_m, for $m \geq 2$, is the extension of BG where only $m - 2$ intermediate truth values are possible: $BG \subseteq \ldots \subseteq G_4 \subseteq G_3 \subseteq G_2$,
- $A \vee B$ is equivalent to $((A \rightarrow B) \rightarrow B) \& ((B \rightarrow A) \rightarrow A)$.

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Semantical definition

A formula φ in a multi-valued structure $\mathcal{J}$ under an evaluation of variables e has a truth value $\mathcal{J}(\varphi)[e] \in [0,1]$; quantifiers $\forall$ and $\exists$ get evaluated using inf and sup; φ is a logical truth if $\mathcal{J}(\varphi)[e] = 1$ for each $\mathcal{J}$ and e.

Properties

- Axiomatized by the propositional calculus for BG plus S_1: $\forall x(\psi \vee \varphi(x)) \to \psi \vee \forall x \varphi(x)$, x not free in ψ.
- FMP is not true. Consider, e.g., $\exists x(\exists y P(y) \to P(x))$.
- An infinite truth value set V may determine a logic different from BG and and also from all G_m. Thus, it makes sense to define multi-valued structure based on a set V and the notion of logical truth of a set V.

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Prenex operations, a reminder

Prenex operations are the following equivalences, x is not free in ψ:

$(\psi \mathbin{\&} \forall x\varphi(x)) \equiv \forall x(\psi \mathbin{\&} \varphi(x))$,

$(\psi \mathbin{\&} \exists x\varphi(x)) \equiv \exists x(\psi \mathbin{\&} \varphi(x))$,

$(\psi \vee \forall x\varphi(x)) \equiv \forall x(\psi \vee \varphi(x))$, note that $\leftarrow$ is the schema S_1,

$(\psi \vee \exists x\varphi(x)) \equiv \exists x(\psi \vee \varphi(x))$,

$(\psi \rightarrow \forall x\varphi(x)) \equiv \forall x(\psi \rightarrow \varphi(x))$,

$(\psi \rightarrow \exists x\varphi(x)) \equiv \exists x(\psi \rightarrow \varphi(x))$, let $\rightarrow$ be called S_2,

$(\forall x\varphi(x) \rightarrow \psi) \equiv \exists x(\varphi(x) \rightarrow \psi)$, let $\rightarrow$ be called S_3,

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The logics we consider

Definition

(a) Let S2G be the logic BG plus the schema S_2:

S_2: $(\psi \to \exists x \varphi(x)) \to \exists x(\psi \to \varphi(x))$,

let S3G be the logic BG plus the schema S_3:

S_3: $(\forall x \varphi(x) \to \psi) \to \exists x(\varphi(x) \to \psi)$,

let PG be the logic BG plus both S_2 and S_3.

(b) Let $\mathrm{G}_\uparrow$ and $\mathrm{G}_\downarrow$ be the logics of the truth value sets $\mathrm{V}_\uparrow = \{1\} \cup \{\, 1 - \frac{1}{k} \,;\, k \geq 1 \,\}$ and $\mathrm{V}_\downarrow = \{0\} \cup \{\, \frac{1}{k} \,;\, k \geq 1 \,\}$ respectively.

Questions

- What are the properties of these logics?
- What are their relationships?

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Properties of logics, their relationships

Theorem (Basic properties of S2G)

Over BG, *the logic* S2G *is equivalently axiomatized by* $\exists x(\exists y \varphi(y) \rightarrow \varphi(x))$ *or by* $\forall x(\forall y(\varphi(y) \rightarrow \varphi(x)) \rightarrow \varphi(x)) \rightarrow \exists x \varphi(x)$*.*
Its characteristic class is the class of all truth value sets where no value except possibly 1 *is a limit of lower values.*

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The relationships between the logics we consider are as shown in the following figure:

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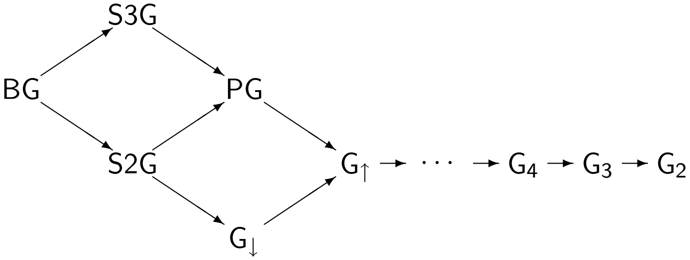
Relationships between Gödel-Dummett logics



Proof

S2G, S3G $\subseteq$ PG is evident.
S2G $\subseteq$ $G_\downarrow$ follows from $V_\downarrow \in \mathrm{Char}(\mathrm{S2G})$. Similarly,
PG $\subseteq$ $G_\uparrow$ follows from $V_\uparrow \in \mathrm{Char}(\mathrm{PG})$.
$G_\downarrow \subseteq G_\uparrow$ follows from $G_\uparrow = \bigcap_{m\geq 2} G_m$, a result by [BPZ03].
S3G $\not\subseteq$ $G_\downarrow$ follows from $V_\downarrow \notin \mathrm{Char}(\mathrm{S3G})$.
S2G $\not\subseteq$ S3G follows from $\mathrm{Char}(\mathrm{S3G}) \not\subseteq \mathrm{Char}(\mathrm{S2G})$. However,
$G_\downarrow \not\subseteq$ PG is difficult.

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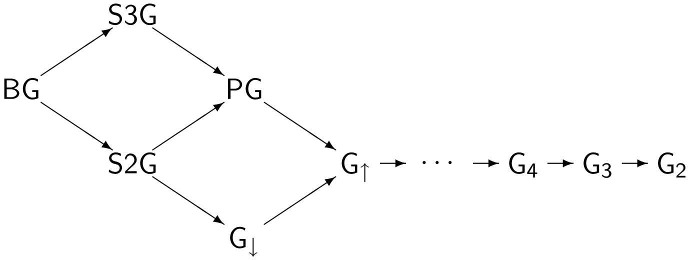

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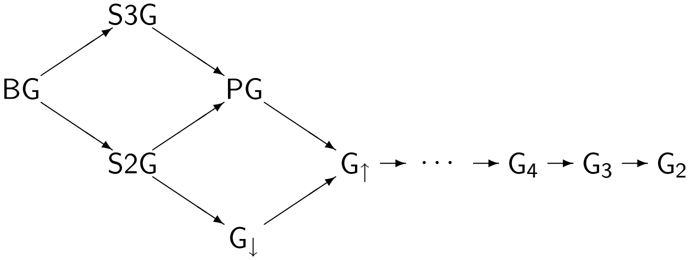

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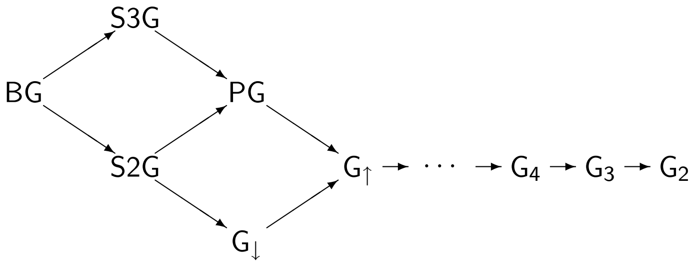
S2G, S3G ⊆ PG is evident.

S2G ⊆ $G_\downarrow$ follows from $\mathrm{V}_\downarrow \in \mathrm{Char}$(S2G). Similarly,
PG ⊆ $G_\uparrow$ follows from $\mathrm{V}_\uparrow \in \mathrm{Char}$(PG).
$G_\downarrow \subseteq G_\uparrow$ follows from $G_\uparrow = \bigcap_{m\geq 2} G_m$, a result by [BPZ03].
S3G ⊈ $G_\downarrow$ follows from $\mathrm{V}_\downarrow \notin \mathrm{Char}$(S3G).
S2G ⊈ S3G follows from Char(S3G) ⊈ Char(S2G). However,
$G_\downarrow$ ⊈ PG is difficult.

Relationships between Gödel-Dummett logics



Proof

S2G, S3G $\subseteq$ PG is evident.
S2G $\subseteq G_\downarrow$ follows from $\mathrm{V}_\downarrow \in \mathrm{Char}(\text{S2G})$. Similarly,
PG $\subseteq G_\uparrow$ follows from $\mathrm{V}_\uparrow \in \mathrm{Char}(\text{PG})$.
$G_\downarrow \subseteq G_\uparrow$ follows from $G_\uparrow = \bigcap_{m\geq 2} G_m$, a result by [BPZ03].
S3G $\not\subseteq G_\downarrow$ follows from $\mathrm{V}_\downarrow \notin \mathrm{Char}(\text{S3G})$.
S2G $\not\subseteq$ S3G follows from $\mathrm{Char}(\text{S3G}) \not\subseteq \mathrm{Char}(\text{S2G})$. However,
$G_\downarrow \not\subseteq$ PG is difficult.

Properties of logics (continued), problems

Theorem (Inter-expressibility of quantifiers)

The quantifier $\forall$ is not expressible in terms of the remaining logical symbols in the logic G_3*.*
In the logic S3G*, the quantifier $\exists$ is not expressible in terms of the remaining logical symbols. In the logic* S2G*, however, it* ***is*** *expressible.*

Problems

- Is the logic S2G (or S3G, or PG) complete with respect to some reasonable semantics?
- What is the weakest Gödel-Dummett logic in which each formula is equivalent to a prenex formula?

Properties of logics (continued), problems

Theorem (Inter-expressibility of quantifiers)

The quantifier ∀ is not expressible in terms of the remaining logical symbols in the logic G_3.
In the logic S3G, *the quantifier ∃ is not expressible in terms of the remaining logical symbols. In the logic* S2G, *however, it* *is* *expressible.*

Problems

- Is the logic S2G (or S3G, or PG) complete with respect to some reasonable semantics?
- What is the weakest Gödel-Dummett logic in which each formula is equivalent to a prenex formula?

Properties of logics (continued), problems

Theorem (Inter-expressibility of quantifiers)

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Properties of logics (continued), problems

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Problems

- Is the logic S2G (or S3G, or PG) complete with respect to some reasonable semantics?
- What is the weakest Gödel-Dummett logic in which each formula is equivalent to a prenex formula?

References

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Appendix 1: A multi-valued structure

Example

Truth value set: $V = \{0, \frac{1}{2}, 1\} \cup \{ \frac{1}{2} - \frac{1}{k} \,;\, k \geq 3 \}$;

language: $L = \{P\}$, with a single unary predicate symbol P;

domain of $\mathcal{J}$: $D = \{d_3, d_4, d_5, \dots\}$;

realization of the symbol P: $\mathcal{J}(P(x))[d_k] = \frac{1}{2} - \frac{1}{k}$.

Then we have $\mathcal{J}(\exists y P(y)) = \frac{1}{2}$, $\mathcal{J}(\exists y P(y) \to P(x))[a_k] = \frac{1}{2} - \frac{1}{k}$, $\mathcal{J}(\exists x(\exists y P(y) \to P(x))) = \frac{1}{2}$. So the sentence $\exists x(\exists y P(y) \to P(x))$ is not a logical truth of this particular set V.

Fact

If the truth value set V contains a value $a < 1$ which is a limit of lower values then the structure $\mathcal{J}$ can be chosen so that $\mathcal{J}(\exists x(\exists y P(y) \to P(x))) < 1$.
If not then the schema $\exists x(\exists y \varphi(y) \to \varphi(x))$ is a logical truth of V.

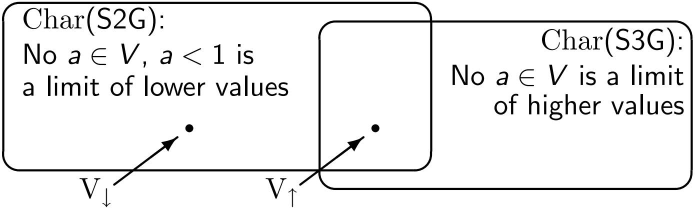
Appendix 2: Characteristic classes

Definition

A characteristic class of a logic S is the class of all truth value sets V such that S is valid in all structures based on V.

Fact *If* $S_1 \subseteq S_2$ *then* $\mathrm{Char}(S_2) \subseteq \mathrm{Char}(S_1)$.

Characteristic classes of S2G, S3G, and PG



Fact

All sets in $\mathrm{Char}(\mathrm{PG}) = \mathrm{Char}(\mathrm{S2G}) \cap \mathrm{Char}(\mathrm{S3G})$ *are finite or isomorphic to* $\mathrm{V}_\uparrow$.