Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Jan 22: Propositional Gentzen Calculus

Outline

What is Gentzen calculus?

What are its main properties?

Propositional Gentzen (sequent) calculus

Definition

A sequent is a pair of sets of formulas. A sequent consisting of sets Γ and Δ is written as $\langle \Gamma \Rightarrow \Delta \rangle$. The sets Γ and Δ are called antecedent and succedent of the sequent $\langle \Gamma \Rightarrow \Delta \rangle$.

Rules, part 1

A: $/ \ \langle \Gamma, A \Rightarrow \Delta, A \rangle$,

W: $\langle \Gamma \Rightarrow \Delta \rangle \ / \ \langle \Gamma, \Pi \Rightarrow \Delta, \Lambda \rangle$,

&r: $\langle \Gamma \Rightarrow \Delta, A \rangle, \langle \Gamma \Rightarrow \Delta, B \rangle \ / \ \langle \Gamma \Rightarrow \Delta, A \,\&\, B \rangle$,

$\vee$r: $\langle \Gamma \Rightarrow \Delta, A \rangle \ / \ \langle \Gamma \Rightarrow \Delta, A \vee B \rangle$,

$\langle \Gamma \Rightarrow \Delta, B \rangle \ / \ \langle \Gamma \Rightarrow \Delta, A \vee B \rangle$,

$\neg$r: $\langle \Gamma, A \Rightarrow \Delta \rangle \ / \ \langle \Gamma \Rightarrow \Delta, \neg A \rangle$,

$\neg$l: $\langle \Gamma \Rightarrow \Delta, A \rangle \ / \ \langle \Gamma, \neg A \Rightarrow \Delta \rangle$,

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Gentzen calculus, continuation

Rules, part 2

$$
\begin{aligned}
\&\mathrm{l}\colon\ & \langle \Gamma, A \Rightarrow \Delta \rangle \ / \ \langle \Gamma, A \,\&\, B \Rightarrow \Delta \rangle, \\
& \langle \Gamma, B \Rightarrow \Delta \rangle \ / \ \langle \Gamma, A \,\&\, B \Rightarrow \Delta \rangle, \\
\vee\mathrm{l}\colon\ & \langle \Gamma, A \Rightarrow \Delta \rangle, \langle \Gamma, B \Rightarrow \Delta \rangle \ / \ \langle \Gamma, A \vee B \Rightarrow \Delta \rangle, \\
\to\mathrm{r}\colon\ & \langle \Gamma, A \Rightarrow \Delta, B \rangle \ / \ \langle \Gamma \Rightarrow \Delta, A \to B \rangle, \\
\to\mathrm{l}\colon\ & \langle \Gamma \Rightarrow \Delta, A \rangle, \langle \Pi, B \Rightarrow \Lambda \rangle \ / \ \langle \Gamma, \Pi, A \to B \Rightarrow \Delta, \Lambda \rangle, \\
\mathrm{Cut}\colon\ & \langle \Gamma \Rightarrow \Delta, A \rangle, \langle \Pi, A \Rightarrow \Lambda \rangle \ / \ \langle \Gamma, \Pi \Rightarrow \Delta, \Lambda \rangle.
\end{aligned}
$$

Example proof

$$
\dfrac{\dfrac{\dfrac{\langle A \Rightarrow A, B \rangle}{\langle A, \neg A \Rightarrow B \rangle}}{\langle \neg A \Rightarrow A \to B \rangle} \qquad \dfrac{\langle A, B \Rightarrow B \rangle}{\langle B \Rightarrow A \to B \rangle}}{\langle \neg A \vee B \Rightarrow A \to B \rangle}
$$

Gentzen calculus, continuation

Rules, part 2

&l: $\langle \Gamma, A \Rightarrow \Delta \rangle \;/\; \langle \Gamma, A \,\&\, B \Rightarrow \Delta \rangle$,

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$\vee$l: $\langle \Gamma, A \Rightarrow \Delta \rangle, \langle \Gamma, B \Rightarrow \Delta \rangle \;/\; \langle \Gamma, A \vee B \Rightarrow \Delta \rangle$,

$\rightarrow$r: $\langle \Gamma, A \Rightarrow \Delta, B \rangle \;/\; \langle \Gamma \Rightarrow \Delta, A \rightarrow B \rangle$,

$\rightarrow$l: $\langle \Gamma \Rightarrow \Delta, A \rangle, \langle \Pi, B \Rightarrow \Lambda \rangle \;/\; \langle \Gamma, \Pi, A \rightarrow B \Rightarrow \Delta, \Lambda \rangle$,

Cut: $\langle \Gamma \Rightarrow \Delta, A \rangle, \langle \Pi, A \Rightarrow \Lambda \rangle \;/\; \langle \Gamma, \Pi \Rightarrow \Delta, \Lambda \rangle$.

Example proof

$$\dfrac{\dfrac{\dfrac{\langle A \Rightarrow A, B \rangle}{\langle A, \neg A \Rightarrow B \rangle}}{\langle \neg A \Rightarrow A \rightarrow B \rangle} \qquad \dfrac{\langle A, B \Rightarrow B \rangle}{\langle B \Rightarrow A \rightarrow B \rangle}}{\langle \neg A \vee B \Rightarrow A \rightarrow B \rangle}$$

Some terminology

Terminology about sequents

antecedent, succedent, cedent.

About proofs

initial sequent (leaf), endsequent (final sequent).

About rules, or steps in proofs

initial sequent, structural rules, weak structural rules, strong rules (i.e. propositional rules plus the cut rule), upper and lower sequent, principal formula, side formula, auxiliary formula.

Variants of Gentzen calculi

Sets or sequences, context sensitive or context insensitive rules.

Definition

A proof of a formula A is a proof of the sequent $\langle \Rightarrow A \rangle$.

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A proof of a formula A is a proof of the sequent $\langle \ \Rightarrow \ A \rangle$.

Are there unprovable sequents?

Definition

A cut-free proof is a proof containing no cuts.

Theorem (subformula property)

Any formula in a cut-free proof $\mathcal{P}$ is a subformula of some formula in the final sequent of $\mathcal{P}$.

Definition

A counter-example to a sequent $\langle \Gamma \Rightarrow \Delta \rangle$ is a truth evaluation v such that $v(A) = 1$ for all $A \in \Gamma$ and $v(A) = 0$ for all $A \in \Delta$.
A sequent $\langle \Gamma \Rightarrow \Delta \rangle$ is tautological if it has no counter-example, i.e. if for each truth evaluation that evaluates all formulas in Γ by 1 there exists a formula $B \in \Delta$ such that $v(A) = 1$.

Theorem (soundness)

Every provable sequent is tautological.

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Every provable sequent is tautological.

Main results and conclusions

Theorem (completeness)

A sequent is tautological if and only if it is provable.

Theorem (consequence of proof)

Every tautological sequent containing n occurences of logical connectives has a cut-free proof of depth at most $2n$, which contains at most $2^{n+1} - 1$ sequents.

Theorem (cut eliminability)

Every provable sequent also has a cut-free proof.

Homework

Assume that also equivalence $\equiv$ is accepted as a basic connective, and design rules for it.
Soundness, completeness, and subformula property should still be valid.

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