Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Jan 25: Predicate Gentzen Calculus

Outline

Quantifier rules of the calculus GK

Addendum to last talk

A proof of a formula φ is defined as a proof of the sequent $\langle \Rightarrow \varphi \rangle$. A proof of φ from a set Σ of assumptions is a proof of a sequent $\langle \Gamma \Rightarrow \varphi \rangle$ where $\Gamma \subseteq \Sigma$ is finite.

Two specification rules

$\exists$r: $\dfrac{\langle \Gamma \Rightarrow \Delta, \varphi_x(t) \rangle}{\langle \Gamma \Rightarrow \Delta, \exists x \varphi \rangle}$ $\forall$l: $\dfrac{\langle \Gamma, \varphi_x(t) \Rightarrow \Delta \rangle}{\langle \Gamma, \forall x \varphi \Rightarrow \Delta \rangle}$

where t is a term substitutable for x in φ.

Two generalization rules

$\exists$l: $\dfrac{\langle \Gamma, \varphi_x(y) \Rightarrow \Delta \rangle}{\langle \Gamma, \exists x \varphi \Rightarrow \Delta \rangle}$ $\exists$l: $\dfrac{\langle \Gamma \Rightarrow \Delta, \varphi_x(y) \rangle}{\langle \Gamma \Rightarrow \Delta, \forall x \varphi \rangle}$

where the variable y is substitutable for x in φ, has no free occurences in the principal formula, and has no free occurences in $\Gamma \cup \Delta$.

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Example, soundness

Example proof

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Homework

Consider a language $\{P, F\}$ with a unary predicate and a unary function. Find a proof of the sentence $\exists x (P(F(x)) \vee \neg P(x))$.

Definition

A counter-example to a sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ is a first-order structure $\mathbf{D}$ and an evaluation of variables e such that $\mathbf{D} \models \varphi[e]$ for each $\varphi \in \Gamma$, and $\mathbf{D} \not\models \varphi[e]$ for each $\varphi \in \Delta$.
A sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ is logically valid if it has no counter-example.

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The subformula property

Definition

Subformulas are defined as one would expect, but $\varphi_x(t)$, where t is a term substitutable for x in φ, is considered a subformula of both $\forall x \varphi$ and $\exists x \varphi$.

Theorem (subformula property)

Any formula in a cut-free proof $\mathcal{P}$ is a subformula of some formula in the final sequent of $\mathcal{P}$. Moreover, if rules for $\to$ and $\neg$ are never used in $\mathcal{P}$, then any formula in an antecedent (succedent) of $\mathcal{P}$ is a subformula of some formula in antecedent (or succedent, respectively) of the final sequent of $\mathcal{P}$.

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Regular stuff, cut rank

Definition

A sequent is regular of no variable is simultaneously free and bound in it.

Definition

A proof is regular of no variable is simultaneously free and bound in it, and if moreover, an eigenvariable of a generalization inference never occurs outside the subtree of $\mathcal{P}$ generated by that inference.

Definition

Depth of a proof $\mathcal{P}$ is denoted $\mathrm{d}(\mathcal{P})$. Depth $\mathrm{d}(\varphi)$ of a formula φ is depth of φ written as a tree. (Cut) rank $\mathrm{r}(\mathcal{P})$ of a proof $\mathcal{P}$ is $\sup\{\, 1 + \mathrm{d}(\varphi)\,;\, \varphi$ a cut formula in $\mathcal{P}\,\}$.

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First steps

Lemma 1 (regularization)

For every proof of a regular sequent there exists a regular proof of the same sequent having the same depth and rank.

Lemma 2 (substitution)

Assume that z is a variable that is not generalized in a proof $\mathcal{P}$, and no variable of a term t is generalized or quantified in $\mathcal{P}$. Then $\mathcal{P}_x(t)$, the result of substitution of t for all occurences of z in $\mathcal{P}$, is a proof.

Lemma 3 (weakening)

Let $\mathcal{P}$ be a proof of a sequent $\langle \Gamma \Rightarrow \Delta \rangle$, let no variable free in $\Gamma \cup \Delta$ be generalized in $\mathcal{P}$. Then adding Π to all antecedents, and adding Λ to all succedents, yields a proof.

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