Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Jan 29: Steps in Cut Elimination

Outline

Depth of formulas and proofs, cut rank

Some technical lemmas

The essential lemmas: inversion and reduction

Depth and cut rank, regularization

Addenda

From now on, we assume the following modification of the calculus GK: principal formulas of *initial sequent* must be *atomic*.

Definition

A proof is regular of no variable is simultaneously free and bound in it, and if moreover, an eigenvariable of a generalization inference never occurs outside the subproof of $\mathcal{P}$ generated by that inference.

Definition

Depth of a proof $\mathcal{P}$ is denoted $\mathrm{d}(\mathcal{P})$. Depth $\mathrm{d}(\varphi)$ of a formula φ is depth of φ written as a tree. (Cut) rank $\mathrm{r}(\mathcal{P})$ of a proof $\mathcal{P}$ is $\sup\{\, 1+\mathrm{d}(\varphi)\,;\, \varphi$ a cut formula in $\mathcal{P}\,\}$.

Lemma 1 (regularization)

For every proof of a regular sequent there exists a regular proof of the same sequent having the same depth and rank.

Inversion

Lemma 4 (inversion)

(a) In each line of the following table, if the left sequent has a regular proof, then the right sequent (both right sequents) has a proof of same or lower cut rank and depth:

$\langle\,\Gamma \Rightarrow \Delta, \varphi \to \psi\,\rangle$	$\langle\,\Gamma, \varphi \Rightarrow \Delta, \psi\,\rangle$
$\langle\,\Gamma \Rightarrow \Delta, \varphi \,\&\, \psi\,\rangle$	$\langle\,\Gamma \Rightarrow \Delta, \varphi\,\rangle, \langle\,\Gamma \Rightarrow \Delta, \psi\,\rangle$
$\langle\,\Gamma \Rightarrow \Delta, \varphi \vee \psi\,\rangle$	$\langle\,\Gamma \Rightarrow \Delta, \varphi, \psi\,\rangle$
$\langle\,\Gamma \Rightarrow \Delta, \neg\varphi\,\rangle$	$\langle\,\Gamma, \varphi \Rightarrow \Delta\,\rangle$
$\langle\,\Gamma, \varphi \to \psi \Rightarrow \Delta\,\rangle$	$\langle\,\Gamma \Rightarrow \Delta, \varphi\,\rangle, \langle\,\Gamma, \psi \Rightarrow \Delta\,\rangle$
$\langle\,\Gamma, \varphi \,\&\, \psi \Rightarrow \Delta\,\rangle$	$\langle\,\Gamma, \varphi, \psi \Rightarrow \Delta\,\rangle$
$\langle\,\Gamma, \varphi \vee \psi \Rightarrow \Delta\,\rangle$	$\langle\,\Gamma, \varphi \Rightarrow \Delta\,\rangle, \langle\,\Gamma, \psi \Rightarrow \Delta\,\rangle$
$\langle\,\Gamma, \neg\varphi \Rightarrow \Delta\,\rangle$	$\langle\,\Gamma \Rightarrow \Delta, \varphi\,\rangle$.

(b) If $\mathcal{P}$ is a regular proof of $\langle\,\Gamma \Rightarrow \Delta, \forall x\varphi\,\rangle$ (or $\langle\,\Gamma, \exists x\varphi \Rightarrow \Delta\,\rangle$) and if no variable of term t is generalized or quantified in $\mathcal{P}$, then $\langle\,\Gamma \Rightarrow \Delta, \varphi_x(t)\,\rangle$ (or $\langle\,\Gamma, \varphi_x(t) \Rightarrow \Delta\,\rangle$, respectively) has a proof of same or lower cut rank and depth.

Substitution and weakening

Lemma 2 (substitution)

Assume that z is a variable that is not generalized in a proof $\mathcal{P}$, and no variable of a term t is generalized or quantified in $\mathcal{P}$. Then $\mathcal{P}_x(t)$, the result of substitution of t for all occurences of z in $\mathcal{P}$, is a proof.

Lemma 3 (weakening)

Let $\mathcal{P}$ be a proof of a sequent $\langle\,\Gamma \Rightarrow \Delta\,\rangle$, let no variable free in $\Gamma \cup \Delta$ be generalized in $\mathcal{P}$. Then adding Π to all antecedents, and adding Λ to all succedents, yields a proof.

Reduction

Lemma 5 (reduction)

Consider a regular proof $\mathcal{P}_0$:

$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \langle\,\Gamma \Rightarrow \Delta, \theta\,\rangle\end{array} \qquad \begin{array}{c}\mathcal{P}_2\\ \langle\,\Pi, \theta \Rightarrow \Lambda\,\rangle\end{array}}{\langle\,\Gamma, \Pi \Rightarrow \Delta, \Lambda\,\rangle}$$

such that $\mathrm{r}(\mathcal{P}_1) \leq \mathrm{d}(\theta)$ and $\mathrm{r}(\mathcal{P}_2) \leq \mathrm{d}(\theta)$. Then $\langle\,\Gamma, \Pi \Rightarrow \Delta, \Lambda\,\rangle$ has a proof of rank at most $\mathrm{d}(\theta)$ and depth at most $\mathrm{d}(\mathcal{P}_1) + \mathrm{d}(\mathcal{P}_2)$.