Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Jan 29: Steps in Cut Elimination

Outline

Depth and cut rank, regularization

Addenda

From now on, we assume the following modification of the calculus GK: principal formulas of *initial sequent* must be *atomic.*

Definition

A proof is regular of no variable is simultaneously free and bound in it, and if moreover, an eigenvariable of a generalization inference never occurs outside the subproof of $\mathcal{P}$ generated by that inference.

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Depth of a proof $\mathcal{P}$ is denoted $\mathrm{d}(\mathcal{P})$. Depth $\mathrm{d}(\varphi)$ of a formula φ is depth of φ written as a tree. (Cut) rank $\mathrm{r}(\mathcal{P})$ of a proof $\mathcal{P}$ is $\sup\{\, 1 + \mathrm{d}(\varphi)\,;\, \varphi \text{ a cut formula in } \mathcal{P}\,\}$.

Lemma 1 (regularization)

For every proof of a regular sequent there exists a regular proof of the same sequent having the same depth and rank.

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For every proof of a regular sequent there exists a regular proof of the same sequent having the same depth and rank.

Substitution and weakening

Lemma 2 (substitution)

Assume that z is a variable that is not generalized in a proof $\mathcal{P}$, and no variable of a term t is generalized or quantified in $\mathcal{P}$. Then $\mathcal{P}_x(t)$, the result of substitution of t for all occurences of z in $\mathcal{P}$, is a proof.

Lemma 3 (weakening)

Let $\mathcal{P}$ be a proof of a sequent $\langle \Gamma \Rightarrow \Delta \rangle$, let no variable free in $\Gamma \cup \Delta$ be generalized in $\mathcal{P}$. Then adding Π to all antecedents, and adding Λ to all succedents, yields a proof.

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Inversion

Lemma 4 (inversion)

(a) In each line of the following table, if the left sequent has a regular proof, then the right sequent (both right sequents) has a proof of same or lower cut rank and depth:

$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \rightarrow \psi \,\rangle$	$\langle\, \Gamma, \varphi \;\Rightarrow\; \Delta, \psi \,\rangle$
$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\&\, \psi \,\rangle$	$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle, \langle\, \Gamma \;\Rightarrow\; \Delta, \psi \,\rangle$
$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle$	$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi, \psi \,\rangle$
$\langle\, \Gamma \;\Rightarrow\; \Delta, \neg\varphi \,\rangle$	$\langle\, \Gamma, \varphi \;\Rightarrow\; \Delta \,\rangle$
$\langle\, \Gamma, \varphi \rightarrow \psi \;\Rightarrow\; \Delta \,\rangle$	$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle, \langle\, \Gamma, \psi \;\Rightarrow\; \Delta \,\rangle$
$\langle\, \Gamma, \varphi \,\&\, \psi \;\Rightarrow\; \Delta \,\rangle$	$\langle\, \Gamma, \varphi, \psi \;\Rightarrow\; \Delta \,\rangle$
$\langle\, \Gamma, \varphi \vee \psi \;\Rightarrow\; \Delta \,\rangle$	$\langle\, \Gamma, \varphi \;\Rightarrow\; \Delta \,\rangle, \langle\, \Gamma, \psi \;\Rightarrow\; \Delta \,\rangle$
$\langle\, \Gamma, \neg\varphi \;\Rightarrow\; \Delta \,\rangle$	$\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle$.

(b) If $\mathcal{P}$ is a regular proof of $\langle\, \Gamma \;\Rightarrow\; \Delta, \forall x\varphi \,\rangle$ (or $\langle\, \Gamma, \exists x\varphi \;\Rightarrow\; \Delta \,\rangle$) and if no variable of term t is generalized or quantified in $\mathcal{P}$, then $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_x(t) \,\rangle$ (or $\langle\, \Gamma, \varphi_x(t) \;\Rightarrow\; \Delta \,\rangle$, respectively) has a proof of same or lower cut rank and depth.

Reduction

Lemma 5 (reduction)

Consider a regular proof $\mathcal{P}_0$:

$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \langle\, \Gamma \Rightarrow \Delta, \theta \,\rangle\end{array} \qquad \begin{array}{c}\mathcal{P}_2\\ \langle\, \Pi, \theta \Rightarrow \Lambda \,\rangle\end{array}}{\langle\, \Gamma, \Pi \Rightarrow \Delta, \Lambda \,\rangle}$$

such that $\mathrm{r}(\mathcal{P}_1) \leq \mathrm{d}(\theta)$ and $\mathrm{r}(\mathcal{P}_2) \leq \mathrm{d}(\theta)$. Then $\langle\, \Gamma, \Pi \Rightarrow \Delta, \Lambda \,\rangle$ has a proof of rank at most $\mathrm{d}(\theta)$ and depth at most $\mathrm{d}(\mathcal{P}_1) + \mathrm{d}(\mathcal{P}_2)$.