Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Feb 1: The Cut-Eliminability Theorem

Outline

Reduction and cut lowering lemmas

Cut eliminability, consequences

Essential steps in cut elimination

Lemma 5 (reduction)

Consider a regular proof $\mathcal{P}_0$:

$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \langle\, \Gamma \Rightarrow \Delta, \theta \,\rangle\end{array} \qquad \begin{array}{c}\mathcal{P}_2\\ \langle\, \Pi, \theta \Rightarrow \Lambda \,\rangle\end{array}}{\langle\, \Gamma, \Pi \Rightarrow \Delta, \Lambda \,\rangle}$$

such that $\mathrm{r}(\mathcal{P}_1) \leq \mathrm{d}(\theta)$ and $\mathrm{r}(\mathcal{P}_2) \leq \mathrm{d}(\theta)$. Then $\langle\, \Gamma, \Pi \Rightarrow \Delta, \Lambda \,\rangle$ has a proof of rank at most $\mathrm{d}(\theta)$ and depth at most $\mathrm{d}(\mathcal{P}_1) + \mathrm{d}(\mathcal{P}_2)$.

Lemma 6 (cut lowering)

Let $\mathcal{P}$ be a regular proof with $\mathrm{r}(\mathcal{P}) > 0$.
Then there exists a proof $\mathcal{P}'$ of the same sequent satisfying $\mathrm{r}(\mathcal{P}') < \mathrm{r}(\mathcal{P})$ and $\mathrm{d}(\mathcal{P}') \leq 2^{\mathrm{d}(\mathcal{P})}$.

Essential steps in cut elimination

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Cut-eliminability, upper bounds

Definition (hyper-exponential function)

$2^y_0 = y, \qquad 2^y_{x+1} = 2^{2^y_x}.$

Theorem (cut eliminability)

For every proof $\mathcal{P}$ of a regular sequent there exists a cut-free proof $\mathcal{P}'$ of the same sequent satisfying $\mathrm{d}(\mathcal{P}') \leq 2^{\mathrm{d}(\mathcal{P})}_{\mathrm{r}(\mathcal{P})}$.

Theorem (Gentzen's midsequent theorem)

Every provable regular sequent containing only prenex formulas has a cut-free proof containing a sequent $\mathcal{S}_0$ such that

- no quantifier inferences are above $\mathcal{S}_0$,
- no propositional inferences are below $\mathcal{S}_0$.

Hypothesis (or, unfinished calculation)

A regular cut-free proof of depth n can be converted to a "midsequent proof" of depth $(n-1) + 2^{n-1}$. So of depth 2^n.

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Hilbert-Ackermann theorem

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Let φ be an open (i.e., quantifier-free) formula such that $\exists x \varphi$ is logically valid. Then there exists a number n and terms $t_1, \ldots, t_n$ such that $\varphi_x(t_1) \vee \ldots \vee \varphi_x(t_n)$ is a tautology.

Homework

Show that the theorem is not true for arbitrary formulas (possibly containing quantifiers).
Show that one cannot insist on $n = 1$.

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