Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Feb 5: Pudlak's Exponential Arithmetic

Outline

Pudlak's exponential arithmetic PEX

Analysing unprovability and provability in PEX

Addenda

Upper bound for midsequent theorem

A proof of depth n has at most 2^{n-1} quantifier inferences (we temporarily define $2^{-1} = 0$).
So, a regular cut-free proof of depth n, whose final sequent consists of prenex formulas only, can be converted to a midsequent proof of depth $n + 2^{n-1}$. So of depth 2^n.
So, a proof $\mathcal{P}$ of of regular prenex sequent can be converted to a midsequent proof having depth $\leq 2^{\mathrm{d}(\mathcal{P})}_{\mathrm{r}(\mathcal{P})+1}$.

Tautologies in predicate logic

A predicate formula is a tautology if it can be obtained from a propositional tautology by substituting predicate formulas for its atoms (same formulas for same atoms).

Numerals

Having a language containing 0 and S, the term $\mathrm{S}(\mathrm{S}(\ldots\mathrm{S}(0)\ldots))$ with n occurences of S is denoted $\overline{n}$ and called (n-th) numeral.
Thus 0 and $\overline{0}$ are the same terms.

Motivating questions, models

Questions

Can $\mathsf{PEX} \vdash \forall x \mathrm{P}(x)$? Can $\mathsf{PEX} \vdash \mathrm{P}(\mathrm{E}^{(60)}(0))$?
In our setting, these are questions about provability of sequents $\langle\, \mathsf{PEX} \Rightarrow \forall x\mathrm{P}(x)\,\rangle$ and $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(60)}(0))\,\rangle$,
where PEX is a set containing the 10 axioms of the theory PEX plus 7 identity axioms (listed on next frame).

Some models of PEX

1. $\mathbf{M_0} = \langle \mathrm{N}, 0^{\mathbf{N}}, (a \mapsto a+1)^{\mathbf{N}}, +^{\mathbf{N}}, (a \mapsto 2^a)^{\mathbf{N}}, \mathrm{N}\rangle$.
2. Let $\mathbf{M} = \langle M, 0^{\mathbf{M}}, \mathrm{S}^{\mathbf{M}}, +^{\mathbf{M}}, \cdot^{\mathbf{M}}\rangle$ be a non-standard model of PA. Take $\langle M, 0^{\mathbf{M}}, \mathrm{S}^{\mathbf{M}}, +^{\mathbf{M}}, (a \mapsto 2^a)^{\mathbf{M}}, \mathrm{N}\rangle$.
3. In the same model, change the last component as follows. Fix $a_0 \in M$, and send P to the set $a_0 + \mathrm{N} = \{\, a\,;\, \exists n{\in}\mathrm{N}(a \leq^{\mathbf{M}} a_0 +^{\mathbf{M}} n)\,\}$.
4. $\langle \mathrm{R} \cap [0,\infty), 0^{\mathbf{R}}, (a \mapsto a+1)^{\mathbf{R}}, +^{\mathbf{R}}, (a \mapsto 2^a)^{\mathbf{R}}, \mathrm{N}\rangle$.

Pudlak's exponential arithmetic PEX

Axioms about 0, S, and $+$

Q1: $\forall x \forall y(\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y)$,
Q2: $\forall x(\mathrm{S}(x) \neq 0)$,
Q4: $\forall x(x + 0 = x)$,
Q5: $\forall x \forall y(x + \mathrm{S}(y) = \mathrm{S}(x+y))$,
$\forall x \forall y(x + y = y + x)$,
$\forall x \forall y \forall z(x + (y+z) = (x+y)+z)$,

Axioms about E

$\mathrm{E}(0) = \mathrm{S}(0)$,
$\forall x(\mathrm{E}(\mathrm{S}(x)) = \mathrm{E}(x) + \mathrm{E}(x))$,

Axioms about P

$\mathrm{P}(0)$,
$\forall x(\mathrm{P}(x) \rightarrow \mathrm{P}(\mathrm{S}(x)))$.

Identity axioms

Seven identity axioms in the set PEX

$\forall x(x = x)$,
$\forall x \forall y(x = y \rightarrow y = x)$,
$\forall x \forall y \forall z(x = y \rightarrow (y = z \rightarrow x = z))$,
$\forall x \forall y(x = y \rightarrow (\mathrm{S}(x) = \mathrm{S}(y)))$,
$\forall x_1 \forall x_2 \forall y_1 \forall y_2(x_1 = y_1 \;\&\; x_2 = y_2 \rightarrow (x_1 + x_2 = y_1 + y_2))$,
$\forall x \forall y(x = y \rightarrow (\mathrm{E}(x) = \mathrm{E}(y)))$,
$\forall x \forall y(x = y \rightarrow (\mathrm{P}(x) \rightarrow \mathrm{P}(y)))$.

Example proof

Constructing a proof of $\langle\, \text{PEX} \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(60)}(0))\,\rangle$

1. Find number m such that $\overline{m} = \mathrm{E}^{(60)}(0)$.
2. Prove the sequent $\langle\, \text{PEX} \;\Rightarrow\; \overline{m} = \mathrm{E}^{(60)}(0)\,\rangle$.
3. Prove the sequent
   $\langle\, \text{PEX}, \mathrm{P}(0), \mathrm{P}(\bar{0}) \rightarrow \mathrm{P}(\bar{1}), \ldots, \mathrm{P}(\overline{m-1}) \rightarrow \mathrm{P}(\overline{m}) \;\Rightarrow\; \mathrm{P}(\overline{m})\,\rangle$.
4. Un-substitute $0, \ldots, \overline{m-1}$, and obtain $\langle\, \text{PEX} \;\Rightarrow\; \mathrm{P}(\overline{m})\,\rangle$.
5. Prove the sequent
   $\langle\, \text{PEX}, \overline{m} = \mathrm{E}^{(60)}(0) \;\rightarrow\; (\mathrm{P}(\overline{m}) \;\rightarrow\; \mathrm{P}(\mathrm{E}^{(60)}(0))),$
   $\mathrm{P}(\overline{m}), \overline{m} = \mathrm{E}^{(60)}(0) \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(60)}(0))\,\rangle$.
6. Un-substitute, i.e. use $\forall$I twice, to obtain
   $\langle\, \text{PEX}, \mathrm{P}(\overline{m}), \overline{m} = \mathrm{E}^{(60)}(0) \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(60)}(0))\,\rangle$.
7. Cuts on sequents from 2 and 4 yield $\langle\, \text{PEX} \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(60)}(0))\,\rangle$.