Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Feb 8: Minimal Depth of a Cut-Free Proof

Outline

Analysis of cut-free proofs of $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$

Inductive formulas and Solovay's shortening method

Some summary

(a) A proof $\mathcal{P}$ of a regular prenex sequent (one containing prenex formulas only) can be converted to a midsequent proof
(a cut-free proof in which all propositional steps precede all quantifier steps) of depth $2^{\mathrm{d}(\mathcal{P})}_{\mathrm{r}(\mathcal{P})+1}$.
(b) The set PEX contains 10 "mathematical" axioms and 7 identity axioms, prenex formulas only. $\langle\, \mathsf{PEX} \Rightarrow \forall x \mathrm{P}(x) \,\rangle$ and $\langle\, \mathsf{PEX} \Rightarrow \forall x \forall y (\mathrm{P}(x) \,\&\, \mathrm{P}(y) \rightarrow \mathrm{P}(x+y)) \,\rangle$ are examples of unprovable sequents. The sequent $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0)) \,\rangle$ is provable, but we need more information about its proof(s).

Minimal depth of a proof of $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$

Theorem

Any midsequent proof of $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$ has depth at least 2^0_n.

Proof

Let $\mathcal{P}$ be a midsequent proof of $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$.
Let $\mathcal{S}$ be its midsequent, and denote $m = 2^0_n$.
Succedent of $\mathcal{S}$ must be $\{\mathrm{P}(\mathrm{E}^{(n)}(0))\}$, and all inferences below $\mathcal{S}$, i.e. all quantifier inferences in entire $\mathcal{P}$, must be $\forall$l. We can assume that $\mathcal{P}$ contains no free variables. So antecedent of $\mathcal{S}$ contains open sentences of 17 kinds, substitutional instances of 17 axioms of PEX. Each atomic subformula of a formula in $\mathcal{S}$ has the form $t_1 = t_2$ or $\mathrm{P}(t)$, where t_1, t_2, and t are closed terms. Let $|t|$ denote the "true" value of a term t, i.e. the number m such that $\mathbf{M_0} \models \overline{m} = t$ where $\mathbf{M_0}$ is the standard (or any) model of PEX.

Minimal depth of a proof of $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$

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Minimal depth ...

Proof (continued)

Put

$$X = \{ \, |t| \, ; \, \mathrm{P}(t) \rightarrow \mathrm{P}(\mathrm{S}(t)) \text{ occurs in } \mathcal{S} \text{ and } |t| < m \, \}.$$

If $X \neq \{0, .., m-1\}$, fix $j_0 < m$, $j_0 \notin X$, and define a truth evaluation v as follows:

$$v(t_1 = t_2) = 1 \Leftrightarrow |t_1| = |t_2|,$$
$$v(\mathrm{P}(t)) = 1 \Leftrightarrow |t| \leq j_0.$$

Then $v(\mathrm{P}(\mathrm{E}^{(n)}(0))) = 0$, but one can verify that all formulas φ in the antecedent of $\mathcal{S}$ have $v(\varphi) = 1$. This is a contradiction, $\mathcal{S}$ is a tautological sequent.
So $X = \{0, m-1\}$, there are at least m different sentences of the form $\mathrm{P}(t) \rightarrow \mathrm{P}(\mathrm{S}(t))$ in $\mathcal{S}$, and the path from $\mathcal{S}$ down to $\langle \, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0)) \, \rangle$ has depth at least m, i.e. 2^0_n.

Working with inductive formulas

Definition

Let T have a language containing 0 and S. A formula $\varphi(x)$ is inductive in T if $T \vdash \varphi(0) \mathbin{\&} \forall x(\varphi(x) \to \varphi(\mathrm{S}(x)))$.

Definition

Formulas $I_0, I_1, I_2, \ldots$ and $J_0, J_1, J_2, \ldots$ are defined as follows:

$$\begin{aligned} I_0(x) &\equiv \mathrm{P}(x), \\ J_n(x) &\equiv \forall y(I_n(y) \to I_n(y+x)), \\ I_{n+1}(x) &\equiv J_n(x) \mathbin{\&} J_n(\mathrm{E}(x)). \end{aligned}$$

Theorem (Solovay shortening)

For each n, the following 8 sentences are provable in PEX.

(a) $J_n \subseteq I_n$, J_n contains 0 and is closed under S and $+$.

(b) $I_{n+1} \subseteq J_n$, I_{n+1} contains 0 and is closed under S.

(c) $\forall x(x \in I_{n+1} \to \mathrm{E}(x) \in J_n)$.

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