Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Feb 15: A Much Shorter Proof Containing Cuts

Outline

Preliminaries

Theorem (about formulas I_n and J_n)

Let $J_n = \{\, x\,;\ \forall y {\in} I_n (y + x \in I_n)\,\}$, $I_{n+1} = J_n \cap \{\, x\,;\ \mathrm{E}(x) \in J_n\,\}$, and $I_0 = P$. Then the following 8 sentences are provable in PEX for each n.

(a) $J_n \subseteq I_n$, J_n contains 0 and is closed under S and $+$.

(b) $I_{n+1} \subseteq J_n$, I_{n+1} contains 0 and is closed under S.

(c) $\forall x(x \in I_{n+1} \rightarrow \mathrm{E}(x) \in J_n)$.

Lemma (identity theorem)

Let $\underline{x}$ and $\underline{y}$ denote $x_1, .., x_k$ and $y_1, .., y_k$, let $\varphi(\underline{x})$ be a formula whose all free variables are among $x_1, .., x_k$. Then both sequents

$$\langle\, \mathsf{PEX} \;\Rightarrow\; \forall \underline{x} \forall \underline{y}(\underline{x} = \underline{y} \;\rightarrow\; (\varphi(\underline{x}) \;\rightarrow\; \varphi(\underline{x})))\,\rangle,$$
$$\langle\, \mathsf{PEX} \;\Rightarrow\; \forall \underline{x} \forall \underline{y}(\underline{x} = \underline{y} \;\rightarrow\; (\varphi(\underline{y}) \;\rightarrow\; \varphi(\underline{x})))\,\rangle,$$

where $\underline{x} = \underline{y}$ is $x_1 = y_1$ & .. & $x_k = y_k$, have a cut-free proof of depth $\mathcal{O}(\mathrm{d}(\varphi))$.

Measuring proofs

Theorem

Each of the eight sentences in the theorem about I_n and J_n has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

Theorem

The sentence $\forall x(I_{n+1}(x) \rightarrow I_n(\mathrm{E}(x)))$ has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

Theorem (main theorem)

The sentence $\mathrm{P}(\mathrm{E}^{(n)}(0))$ has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

Proof of main theorem

Proof

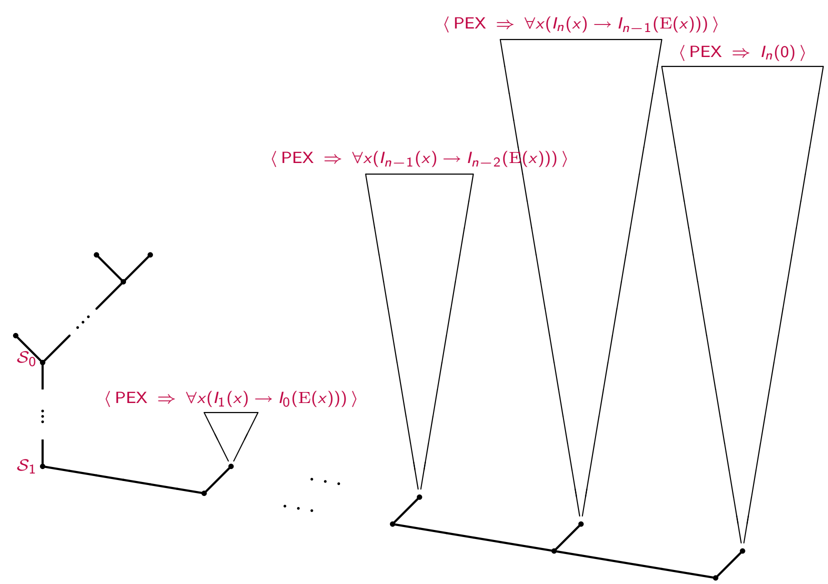
Let $\mathcal{S}_0$ be the sequent

$\langle\, \mathsf{PEX}, I_n(0),$
$I_n(0) \rightarrow I_{n-1}(\mathrm{E}(0)),$
$\vdots$
$I_1(\mathrm{E}^{(n-1)}(0)) \rightarrow I_0(\mathrm{E}^{(n)}(0)) \;\Rightarrow\; I_0(\mathrm{E}^{(n)}(0))\,\rangle$. Then n $\forall$I inferences yield the following sequent $\mathcal{S}_1$:

$\langle\, \mathsf{PEX}, I_n(0),$
$\forall x(I_n(x) \rightarrow I_{n-1}(\mathrm{E}(x))),$
$\vdots$
$\forall x(I_1(x) \rightarrow I_0(\mathrm{E}(x))) \;\Rightarrow\; I_0(\mathrm{E}^{(n)}(0))\,\rangle$.

Then $n+1$ cuts yield the desired proof of $\langle\, \mathsf{PEX} \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$. The whole proof looks as depicted on the following frame and has depth $\mathcal{O}(n)$. Since $\mathrm{d}(I_n) = 3n$ and $\mathrm{d}(J_n) = 3n+2$, the proof has also rank $\mathcal{O}(n)$.

The proof constructed in the proof of main theorem



Summary

We have a benchmark sequent $\langle \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0)) \rangle$. The cut-eliminability theorem guarantees the existence of its midsequent proof having depth $2^{\mathcal{O}(n)}_{\mathcal{O}(n)}$. We know that all midsequent proof have depth 2^{0}_{n}. So some improvements might be possible, but the hyper-exponential growth in cut elimination theorem is necessary.