Proof Theory of Classical Logic

Its Basics with an Emphasis on Quantitative Aspects

Short course at Notre Dame

Feb 15: A Much Shorter Proof Containing Cuts

Outline

Measuring proofs suggested by Solovay's method

Summary

Preliminaries

Theorem (about formulas I_n and J_n)

Let $J_n = \{\, x\,;\, \forall y{\in}I_n(y + x \in I_n) \,\}$, $I_{n+1} = J_n \cap \{\, x\,;\, \mathrm{E}(x) \in J_n \,\}$, and $I_0 = P$. Then the following 8 sentences are provable in PEX for each n.

(a) $J_n \subseteq I_n$, J_n contains 0 and is closed under S and $+$.
(b) $I_{n+1} \subseteq J_n$, I_{n+1} contains 0 and is closed under S.
(c) $\forall x(x \in I_{n+1} \rightarrow \mathrm{E}(x) \in J_n)$.

Lemma (identity theorem)

Let $\underline{x}$ and $\underline{y}$ denote $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, let $\varphi(\underline{x})$ be a formula whose all free variables are among $x_1, \ldots, x_k$. Then both sequents

$$\langle\, \mathrm{PEX} \Rightarrow \forall\underline{x}\forall\underline{y}(\underline{x} = \underline{y} \rightarrow (\varphi(\underline{x}) \rightarrow \varphi(\underline{x})))\,\rangle,$$
$$\langle\, \mathrm{PEX} \Rightarrow \forall\underline{x}\forall\underline{y}(\underline{x} = \underline{y} \rightarrow (\varphi(\underline{y}) \rightarrow \varphi(\underline{x})))\,\rangle,$$

where $\underline{x} = \underline{y}$ is $x_1 = y_1 \;\&\; \ldots \;\&\; x_k = y_k$, have a cut-free proof of depth $\mathcal{O}(\mathrm{d}(\varphi))$.

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Measuring proofs

Theorem

Each of the eight sentences in the theorem about I_n and J_n has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

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The sentence $\forall x(I_{n+1}(x) \rightarrow I_n(\mathrm{E}(x)))$ has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

Theorem (main theorem)

The sentence $\mathrm{P}(\mathrm{E}^{(n)}(0))$ has a proof with depth $\mathcal{O}(n)$ and rank $\mathcal{O}(n)$.

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Proof of main theorem

Proof

Let $\mathcal{S}_0$ be the sequent

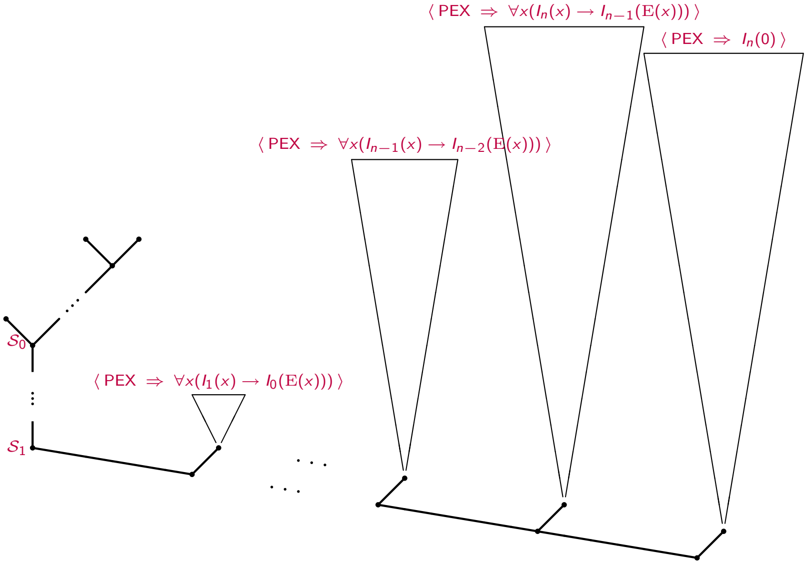
$$
\begin{aligned}
&\langle\, \mathsf{PEX}, I_n(0),\\
&\quad I_n(0) \to I_{n-1}(\mathrm{E}(0)),\\
&\qquad\qquad \vdots\\
&\quad I_1(\mathrm{E}^{(n-1)}(0)) \to I_0(\mathrm{E}^{(n)}(0)) \;\Rightarrow\; I_0(\mathrm{E}^{(n)}(0))\,\rangle.
\end{aligned}
$$

Then n $\forall$I inferences yield the following sequent $\mathcal{S}_1$:

$$
\begin{aligned}
&\langle\, \mathsf{PEX}, I_n(0),\\
&\quad \forall x(I_n(x) \to I_{n-1}(\mathrm{E}(x))),\\
&\qquad\qquad \vdots\\
&\quad \forall x(I_1(x) \to I_0(\mathrm{E}(x))) \;\Rightarrow\; I_0(\mathrm{E}^{(n)}(0))\,\rangle.
\end{aligned}
$$

Then $n+1$ cuts yield the desired proof of $\langle\, \mathsf{PEX} \;\Rightarrow\; \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$.
The whole proof looks as depicted on the following frame and has depth $\mathcal{O}(n)$. Since $\mathrm{d}(I_n) = 3n$ and $\mathrm{d}(J_n) = 3n + 2$, the proof has also rank $\mathcal{O}(n)$.

The proof constructed in the proof of main theorem



Summary

We have a benchmark sequent $\langle\, \mathsf{PEX} \Rightarrow \mathrm{P}(\mathrm{E}^{(n)}(0))\,\rangle$. The cut-eliminability theorem guarantees the existence of its midsequent proof having depth $2_{\mathcal{O}(n)}^{\mathcal{O}(n)}$. We know that all midsequent proof have depth 2_n^0. So some improvements might be possible, but the hyper-exponential growth in cut elimination theorem is necessary.