On Interplay of Quantifiers in Gödel-Dummett Fuzzy Logics

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Abstract

Axiomatization of Gödel-Dummett predicate logics S2G, S3G, and PG, where PG is the weakest logic in which all prenex operations are sound, and the relationships of these logics to logics known from the literature are discussed. Examples of non-prenexable formulas are given for those logics where some prenex operation is not available. Inter-expressibility of quantifiers is explored for each of the considered logics.

1 Introduction

Mathematical *fuzzy logics* are particular many-valued logics, usually having the real interval $[0,1]$ as its (standard) set of truth values (truth value set). So does *Gödel* (*Gödel-Dummett*, in some sources) *predicate fuzzy logic* BG (also denoted G$\forall$, $G_{\mathbf{R}}$, or $G_{[0,1]}$). Truth function of implication $\rightarrow$ of the logic BG is the function $\Rightarrow$ where $a \Rightarrow b = 1$ if $a \leq b$ and $a \Rightarrow b = b$ otherwise. Truth functions of symbols &, $\vee$, $\forall$, and $\exists$ are in BG defined naturally as min, max, inf, and sup respectively. Further important Gödel-Dummett logics are obtained by retaining the truth functions but restricting the truth value set. For example the logic G_m for natural $m \geq 2$ is based on finite truth value set containing, besides the extremal values 0 and 1, only $m-2$ intermediate values, thus G_2 being the classical two-valued logic.

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It is known that BG is axiomatizable, see e.g. [10]. Its Hilbert-style calculus can be obtained by adding two axiomatic schemas, the propositional *prelinearity schema* $(\varphi \to \psi) \vee (\psi \to \varphi)$ and one quantifier schema

$$\mathrm{S}_1: \qquad \forall x(\psi \vee \varphi(x)) \to \psi \vee \forall x\varphi(x),$$

where x is not free in ψ, to the Hilbert calculus for intuitionistic predicate logic, as defined e.g. in [5]. Each of the logics G_m is axiomatizable as well; its axiomatization can be obtained by adding the schema

$$(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee .. \vee (\varphi_m \to \varphi_{m+1})$$

to the calculus for the logic BG (see [7]).

Baaz, Preining, and Zach in the paper [1], which is an essential source for the present paper, define two more Gödel-Dummett logics $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$ based on the countably infinite sets $\{0\} \cup \{\, \frac{1}{k}\,;\, k \geq 1 \,\}$ and $\{1\} \cup \{\, 1 - \frac{1}{k}\,;\, k \geq 1 \,\}$ respectively, and prove that $\mathrm{BG} \subseteq \mathrm{G}_\downarrow \subseteq \mathrm{G}_\uparrow = \bigcap_{m\geq 2} \mathrm{G}_m$. They also consider two more schemas

$$\mathrm{S}_2: \qquad (\psi \to \exists x\varphi(x)) \to \exists x(\psi \to \varphi(x)),$$

$$\mathrm{S}_3: \qquad (\forall x\varphi(x) \to \psi) \to \exists x(\varphi(x) \to \psi),$$

where again x is not free in ψ, and show that S_2 is sound w.r.t. $\mathrm{G}_\downarrow$ while both schemas S_2 and S_3 are sound w.r.t. $\mathrm{G}_\uparrow$. It follows from results in [1] about axiomatizability (of prenex fragments) of logics based on a truth value set $V \subseteq [0, 1]$ that $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$ neither are axiomatizable nor have an axiomatizable prenex fragment. These results were sharpened by P. Hájek in [4], who proved that $\mathrm{G}_\downarrow$ is non-arithmetical while $\mathrm{G}_\uparrow$ is Π_2-complete.

Recall that in classical logic *prenex operations* are expressed in terms of eight equivalences, i.e. sixteen implications. Out of these implications, thirteen are sound in intuitionistic logic, and the schemas S_1–S_3 are the only three of them that are not intuitionistically sound. This paper is motivated by study of axiomatically given logics based on the schemas S_1–S_3, i.e. of logics suggested by axioms of prenexability. We introduce logics S2G, S3G, and PG defined by adding the schema S_2, or the schema S_3, or both of them, to the axioms of the basic logic BG. Since we find natural to stipulate that the class of Gödel-Dummett logics be closed under schematic extensions, we relax the definition of Gödel-Dummett logic so that it includes the logics S2G, S3G, and PG (and DNS, also defined below). So we understand the expression "Gödel-Dummett logics" of [1] as "Gödel-Dummett logics based on a truth value set".

We explore the relationships between our axiomatically given logics and the logics known from the literature, justifying the diagram in Fig. 1 below. To do so we use semantical methods regardless of the fact that we have no completeness theorems for most of the logics we consider.

Concerning prenex operations, the paper [1] says that "$G_\uparrow$ is the only Gödel logic in which every formula is equivalent to a prenex formula". Evidently, this statement is to be understood as saying that among the *infinite*-valued logics G_V considered in [1], i.e. based on an infinite truth value set V, the logic $G_\uparrow$ is the only one in which all *prenex operations are sound.* In our setting it is the logic PG which is the weakest logic in which all prenex operations are sound. However, we still find prenexability an interesting topic, because (i) non-soundness of some prenex operation does not automatically entail an existence of a formula which is not prenexable, and (ii) the existence of a formula which is not prenexable in the logic $G_\downarrow$ does not seem to be directly deducible from [1]. We give examples of formulas that are not prenexable in the logics $G_\downarrow$ and S3G. The question whether there exists a logic G which allows prenexability while some of the classical prenex operations is not sound in G is one of the problems we leave open.

Finally we consider *inter-expressibility of quantifiers* in the logics in question. We show that the quantifier $\forall$ is not expressible in terms of $\exists$ and logical connectives even in the three-valued logic G_3. On the other hand, and this is perhaps a little bit more surprising, we show that in a quite wide class of logics the quantifier $\exists$ *is expressible* in terms of $\forall$ and logical connectives, while in some other logics including the basic logic BG it is not expressible.

2 Preliminaries, semantics, examples

We deal with *predicate formulas* built up from atomic formulas using propositional symbols and quantifiers $\forall$ and $\exists$. Atomic formulas are as in classical first-order predicate logic, built from function and predicate symbols belonging to certain first-order language L. Propositional symbols are $\bot$, $\rightarrow$, $\&$, and $\vee$. We also use symbols for negation and equivalence: $\neg\varphi$ and $\varphi \equiv \psi$ stand for $\varphi \rightarrow \bot$ and $(\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$ respectively. For omitting parenthesis, we assign the implication $\rightarrow$ lower priority than conjunction $\&$ and disjunction $\vee$, but higher than equivalence $\equiv$. So, for example, $\varphi \equiv \psi_1 \,\&\, \psi_2 \rightarrow \chi$ is a shorthand for $\varphi \equiv ((\psi_1 \,\&\, \psi_2) \rightarrow \chi)$.

A *truth value set* is defined as a closed subset V of the real interval $[0, 1]$ such that $\{0, 1\} \subseteq V$. A *many-valued realization* of (or a *fuzzy structure* for) a language L *based on* the truth value *set* V is a pair $\mathcal{J} = (D, \mathbf{s})$ where D is a non-empty *domain* and $\mathbf{s}$ a *valuation function* which maps n-ary relation symbols to functions from D^n to V and n-ary function symbols to functions from D^n to D. Let $\mathcal{J}(\varphi)[e]$ denote the truth value of a formula φ under an evaluation e of (free) variables. This function is defined naturally on atomic formulas, and extends to all formulas via the *truth functions* of logical symbols. In Gödel-Dummett logics, the truth function of implication $\rightarrow$ is the function $\Rightarrow$ mentioned above: $a \Rightarrow b = 1$ if $a \leq b$ and $a \Rightarrow b = b$ otherwise. Truth functions of

&, $\vee$, $\forall$, and $\exists$ are min, max, inf, and sup respectively, and truth function of the formula $\bot$ is 0. Note that closedness of the truth value set ensures the existence of infima and suprema, and see [1] for more details. If $\varphi(x_1,..,x_n)$ is a formula whose free variables are among $x_1,..,x_n$ and e an evaluation of variables sending $x_1,..,x_n$ to $a_1,..,a_n$ respectively then we write $\mathcal{J}(\varphi(x_1,..,x_n))[a_1,..,a_n]$ or simply $\mathcal{J}(\varphi(a_1,..,a_n))$ instead of $\mathcal{J}(\varphi)[e]$. Thus letters from the beginning of Latin alphabet denote elements of a domain of a fuzzy structure, while letters from the end denote variables. We also write e.g. $\underline{a}$ instead of $a_1,..,a_n$. So expressions like $\varphi(\underline{x})$ or $\mathcal{J}(\varphi(\underline{a}))$ are also possible.

A formula φ is *valid* in a fuzzy structure $\mathcal{J}$ if $\mathcal{J}(\varphi)[e] = 1$ for each evaluation e of variables. A *counter-example* for a formula φ *based on a truth value set* V is a fuzzy structure $\mathcal{J}$ based on V and an evaluation e of variables such that $\mathcal{J}(\varphi)[e] < 1$. A *logic* is any deductively closed set, i.e. any set of formulas closed under the modus ponens and generalization rules. It is easy to verify that the set of all formulas valid in a fuzzy structure $\mathcal{J}$ (or valid in a Kripke structure, as defined below in Section 4) is a logic in this sense. Also the set of all formulas having no counter-example based on a set V is a logic; we denote this logic G_V and call a *(Gödel-Dummett) logic based on a truth value set* V. The set G_V is the set of all formulas that are *logically true* w.r.t. the set V, i.e. that are *logical truths* of the logic G_V. If a formula or all instances of a schema are logically true w.r.t. V then we say that the formula or the schema is *sound in* (or *sound w.r.t.*) the logic G_V. Evidently, if $V_2 \subseteq V_1$ then $G_{V_1} \subseteq G_{V_2}$. As already noted, the logics BG, $\mathrm{G}_\downarrow$, $\mathrm{G}_\uparrow$, and G_m are the logics based on the sets

$$V_{\mathbf{R}} = [0,1], \qquad V_\downarrow = \{0\} \cup \{\, \tfrac{1}{k}\,;\, k \geq 1\,\},$$
$$V_\uparrow = \{1\} \cup \{\, 1 - \tfrac{1}{k}\,;\, k \geq 1\,\}, \qquad V_m = \{0, \tfrac{1}{2}, \tfrac{2}{3}, .., \tfrac{m-2}{m-1}, 1\},$$

respectively. We define a *Gödel-Dummett logic* as any logic containing the logic BG.

It is straightforward to verify that the usual axioms and rules of intuitionistic Hilbert-style calculus, as well as the schema S_1 and the prelinearity schema, are logically true w.r.t. the set $V_{\mathbf{R}}$, i.e. the corresponding calculus is sound w.r.t. the logic BG. Moreover this calculus is known to be complete w.r.t. the logic BG, so BG is an axiomatizable logic. One can also verify that the schema S_3 is logically true w.r.t. every set V such that any infimum in V is simultaneously a minimum; thus S_3 is sound in $\mathrm{G}_\uparrow$. Similarly, the schema S_2 is logically true w.r.t. every V such that any supremum in V, except possibly 1, is simultaneously a maximum; thus S_2 is sound in both $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$.

Example 1 Let $\{P\}$ be a language with a single unary predicate. Take an infinite domain D, take $V = \{0, \frac{1}{2}\} \cup \{\, \frac{1}{2} + \frac{1}{k}\,;\, k \geq 2\,\}$ and let a valuation function be defined so that $\mathcal{J}(P(a))$ assumes all possible values from V except 0 and $\frac{1}{2}$. Then $\mathcal{J}(\forall x P(x)) = \frac{1}{2}$, and $\mathcal{J}(P(a) \to \forall y P(y)) = \frac{1}{2}$ for each $a \in D$. So $\mathcal{J}$ is a fuzzy structure which is a counter-example for the sentence $\exists x(P(x) \to \forall y P(y)) \vee \neg\forall x P(x)$. On the other hand, since any supremum in V is simultaneously a maximum, every instance of S_2 is valid in $\mathcal{J}$. One can further check that $\exists x(P(x) \to \forall y P(y)) \vee \neg\forall x P(x)$ is a logical truth of $G_\downarrow$.

Example 2 While $A \vee \neg A$, i.e. the Principle of Excluded Middle, is generally not a logical truth of the logic BG, the principle $\neg\neg A \vee \neg A$ can be proved using the prelinearity schema. So if $\varphi(x)$ is any formula then the formulas $\forall x(\neg\neg\varphi(x) \vee \neg\varphi(x))$ and $\forall x(\neg\neg\varphi(x) \vee \exists y\neg\varphi(y))$ are provable in BG. Using the schema S_1 we obtain $\forall x\neg\neg\varphi(x) \vee \exists x\neg\varphi(x)$. From this and the fact that $\forall x\neg$ is intuitionistically equivalent to $\neg\exists x$ we easily get $\neg\neg\exists x\neg\varphi(x) \to \exists x\neg\varphi(x)$, which is thus an interesting example of a logically true schema of the logic BG.

3 Some more schemas and logics

Besides the schemas S_1–S_3 defined in the introduction we also consider the following:

$C_\downarrow$: $\quad \exists x(\exists y\varphi(y) \to \varphi(x))$,

DNS: $\quad \forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$,

E: $\quad \forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x)) \to \exists x\varphi(x)$,

$C_\uparrow$: $\quad \exists x(\varphi(x) \to \forall y\varphi(y))$.

The schemas $C_\downarrow$ and $C_\uparrow$ are taken from [1], while DNS is a known principle, Double Negation Shift. The only really new schema is the schema E. We show that over the logic BG all the schemas we have mentioned so far fall into only three non-equivalent groups. Some of the implications are known or folklore.

Theorem 1 *(a) The schemas* S_2, $C_\downarrow$, *and* E *are equivalent over the intuitionistic predicate logic.*
(b) Also the schemas S_3 *and* $C_\uparrow$ *are equivalent over the intuitionistic logic.*
(c) The schemas DNS *and* $\neg\forall x\varphi(x) \to \exists x\neg\varphi(x)$ *are equivalent over the logic* BG *and follow from the schema* S_3.

Proof The proofs of $S_2 \Rightarrow C_\downarrow$ and $S_3 \Rightarrow C_\uparrow$ are straightforward (and mentioned in [1]). We prove $C_\downarrow \Rightarrow$ E and E $\Rightarrow S_2$, leaving $C_\uparrow \Rightarrow S_3$ as an exercise. We only give informal proofs; the reader should have no difficulties with formalizing them in the Hilbert calculus.

$C_\downarrow \Rightarrow$ E. Assume that $\forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x))$. Apply this assumption to a z satisfying $\exists y\varphi(y) \to \varphi(z)$; such a z exists by $C_\downarrow$. So $\forall y(\varphi(y) \to \varphi(z)) \to \varphi(z)$. An intuitionistically sound prenex operation yields $(\exists y\varphi(y) \to \varphi(z)) \to \varphi(z)$. This and the assumption $\exists y\varphi(y) \to \varphi(z)$ yields $\varphi(z)$. So indeed, $\exists x\varphi(x)$.

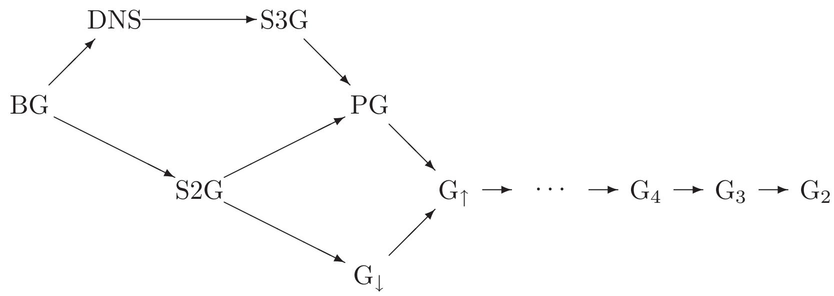

Figure 1: Relationships between Gödel-Dummett logics

$E \Rightarrow S_2$. Assume that $\psi \to \exists x\varphi(x)$, we want to verify that $\exists x(\psi \to \varphi(x))$. By E, it is sufficient to verify that

$$\forall x(\forall y((\psi \to \varphi(y)) \to (\psi \to \varphi(x))) \to (\psi \to \varphi(x))),$$

which is the same as

$$\forall x(\forall y(\varphi(y) \,\&\, \psi \to \varphi(x)) \,\&\, \psi \to \varphi(x)).$$

So let x satisfying $\forall y((\varphi(y) \,\&\, \psi) \to \varphi(x)) \,\&\, \psi$ be given. We want to verify that $\varphi(x)$. From ψ and the assumption $\psi \to \exists x\varphi(x)$ we have a z such that $\varphi(z)$, and we can apply $\forall y(..)$ to this z. So $\varphi(z) \,\&\, \psi \to \varphi(x)$. Since both $\varphi(z)$ and ψ, we indeed have $\varphi(x)$.

Finally the implication $\Leftarrow$ in (c) is easy and the implication $\Rightarrow$ follows from the principle $\neg\neg\exists x\neg(..) \to \exists x\neg(..)$ mentioned in Example 2 and from the fact that $\forall x\neg(..)$ is intuitionistically equivalent to $\neg\exists x(..)$. Also $\neg\forall x\varphi(x) \to \exists x\neg\varphi(x)$ follows easily from S_3 by substituting $\bot$ for ψ. ■

Having Theorem 1, we define some more Gödel-Dummett predicate logics. The logics S2G and S3G are the extensions of BG by the schemas S_2 and S_3 respectively. The logic PG (where P stands for "prenex") is the extension of BG by both S_2 and S_3. The logic DNS is the extension of BG by the schema DNS.

Theorem 2 *The relationships between the logics* BG, S2G, DNS, S3G, PG, $G_\downarrow$, $G_\uparrow$, *and* G_m *where* $m \geq 2$ *are as shown in Fig. 1, where arrows indicate strict inclusion.*

Proof The fact that DNS $\subseteq$ S3G was stated in Theorem 1(c). The remaining inclusions are immediate. As to non-inclusions, DNS $\not\subseteq G_\downarrow$, S3G $\not\subseteq$ DNS, and S2G $\not\subseteq$ S3G are evident or known from [1]. Also the formula from Example 1 is a formula in S3G − DNS. It remains to prove that $G_\downarrow \not\subseteq$ PG. We already know that if $\mathcal{J}$ is based on a truth value set V in which all infima are minima and all suprema except possibly 1 are maxima then both S_2 and S_3 are valid in $\mathcal{J}$. The problem is that if V satisfies the above condition on infima and suprema then V is finite or isomorphic to $V_\uparrow$, and then in turn $\mathcal{J}$ is a model of $G_\uparrow$, which is not what we need. So we have to use a more subtle method than (thinking about characteristic classes of logics) mentioned above or in [1]. We construct a fuzzy structure $\mathcal{J}_0$ based on a set V such that S_2 is valid in $\mathcal{J}_0$ despite the fact that V has a subset whose supremum is not a maximum. Then we exhibit a sentence λ such that $\mathcal{J}_0$ is a counter-example for λ, but λ is a logical truth of the logic $G_\downarrow$.

As in our other examples, $\mathcal{J}_0$ will be a structure for the language $\{P\}$ with a single unary predicate symbol. Let the domain D of $\mathcal{J}_0$ be the set $\{a_i; i \in \mathrm{N}\} \cup \{a_\omega\}$, where $a_\omega = \frac{1}{2}$ and $a_i = \frac{1}{2} - \frac{1}{i+3}$. Put $\mathcal{J}_0(P(a_\alpha)) = a_\alpha$ for $\alpha \leq \omega$. So each $\mathcal{J}_0(P(b))$ equals neither 0 nor 1 and a_ω is the only limit point of the range of the function $b \mapsto \mathcal{J}_0(P(b))$. The set D, and also the set $V = D \cup \{0, 1\}$ of all truth values, if equipped with the natural topology, are compact spaces. The set D in fact plays two roles: it is the domain of the structure $\mathcal{J}_0$, and it simultaneously is the set of all intermediate truth values. The key property of the structure $\mathcal{J}_0$ is the following: if the truth values $\frac{1}{2}$ and 1 are identified then the truth functions of all logical symbols, and in turn the truth functions defined by all formulas, are continuous.

More precisely, let Q be the function from V to $D \cup \{0\}$ defined by $Q(1) = \frac{1}{2}$, $Q(b) = b$ for $b \leq \frac{1}{2}$. We claim that if $\varphi(x_1, .., x_n)$ is any formula then the function

$$[b_1, .., b_n] \mapsto Q(\mathcal{J}_0(\varphi(\underline{b})))$$

is continuous as a function from D^n to $D \cup \{0\}$. This fact is evidently true if $\varphi(x_1, .., x_n)$ is one of the formulas $\bot$ and $P(x_j)$ where $1 \leq j \leq n$. The rest is an induction on the number of symbols in φ. Assume first that $\varphi(\underline{x})$ is $\varphi_1(\underline{x}) \vee \varphi_2(\underline{x})$. Then

$$\begin{aligned} Q(\mathcal{J}_0(\varphi(\underline{b}))) &= Q(\max\{\mathcal{J}_0(\varphi_1(\underline{b})), \mathcal{J}_0(\varphi_2(\underline{b}))\}) \\ &= \max\{Q(\mathcal{J}_0(\varphi_1(\underline{b}))), Q(\mathcal{J}_0(\varphi_2(\underline{b})))\}. \end{aligned}$$

Since the function $[d_1, d_2] \mapsto \max\{d_1, d_2\}$ as a function from $(D \cup \{0\})^2$ to $D \cup \{0\}$ is continuous, and continuous functions are closed on pairing and substitution, the function $[b_1, .., b_n] \mapsto Q(\mathcal{J}_0(\varphi(\underline{b})))$ is continuous. The same argument applies if the outermost symbol of φ is &. If φ is $\varphi_1 \to \varphi_2$ then

$$\begin{aligned} Q(\mathcal{J}_0(\varphi(\underline{b}))) &= Q(\mathcal{J}_0(\varphi_1(\underline{b})) \Rightarrow \mathcal{J}_0(\varphi_2(\underline{b}))) \\ &= Q(Q(\mathcal{J}_0(\varphi_1(\underline{b}))) \Rightarrow Q(\mathcal{J}_0(\varphi_2(\underline{b})))), \end{aligned}$$

for one can easily check that $Q(d_1 \Rightarrow d_2) = Q(Q(d_1) \Rightarrow Q(d_2))$. Since the function $[d_1, d_2] \mapsto Q(d_1 \Rightarrow d_2)$ is continuous (indeed, this is the crucial point in the whole proof), the reasoning is again similar. If φ is $\exists v\psi(\underline{x}, v)$ then we have

$$Q(\mathcal{J}_0(\varphi(\underline{b}))) = Q(\sup_d \mathcal{J}_0(\psi(\underline{b}, d))) = \sup_d Q(\mathcal{J}_0(\psi(\underline{b}, d))).$$

By the induction hypothesis the function $[\underline{b}, d] \mapsto Q(\mathcal{J}_0(\psi(\underline{b}, d)))$ is continuous. So $Q(\mathcal{J}_0(\varphi(\underline{b}))) = \max_d Q(\mathcal{J}_0(\psi(\underline{b}, d)))$. The rest follows from the fact that if K and L are compact spaces and $f : K \times L \to [0, 1]$ is continuous, then also the function g defined by $g(x) = \max_v f(x, v)$, from K to $[0, 1]$, is continuous. In our case $K = D^n$ and $L = D$. Reasoning in the case where φ is $\forall v\psi$ is similar.

Since the function $d \mapsto Q(\mathcal{J}_0(\varphi(\underline{b}, d)))$ where φ is any formula and $b_1, \ldots, b_n$ any parameters is continuous as a function from D to $D \cup \{0\}$, its range is a closed set. The function $d \mapsto \mathcal{J}_0(\varphi(\underline{b}, d))$ may be non-continuous and its range may be not closed. However, its range must have a maximum. From this fact validity of the schema S_2 in our structure $\mathcal{J}_0$ follows. Validity of S_3 is immediate since each infimum in the truth value set is a minimum.

Now as in [1], let $x \prec y$ be the formula $(P(y) \to P(x)) \to P(y)$. Then $\mathcal{J}(a \prec b)$, i.e. the truth value of the formula $(P(y) \to P(x)) \to P(y)$ under an evaluation mapping x and y to a and b respectively, equals 1 if $\mathcal{J}(P(a)) < \mathcal{J}(P(b))$ and equals $\mathcal{J}(P(b))$ otherwise, and this is true in all fuzzy structures $\mathcal{J}$. Consider the formula $\exists z(a \prec z \ \& \ z \prec b)$. Its truth value is 1 if there exists a d satisfying $\mathcal{J}(P(a)) < \mathcal{J}(P(d)) < \mathcal{J}(P(b))$, and its value is $\mathcal{J}(P(b))$ otherwise. More specifically, "otherwise" includes two cases: $\mathcal{J}(P(a)) \geq \mathcal{J}(P(b))$, and $\mathcal{J}(P(a)) < \mathcal{J}(P(b))$ with no d such that $\mathcal{J}(P(d))$ is between $\mathcal{J}(P(a))$ and $\mathcal{J}(P(b))$. Let $\mathrm{LeftLim}(y)$ be the formula $\forall x(x \prec y \to \exists z(x \prec z \ \& \ z \prec y))$. The truth value of the formula $\mathrm{LeftLim}(b)$ is 1 if $\mathcal{J}(P(b))$ is minimal among all $\mathcal{J}(P(d))$, or if $\mathcal{J}(P(b))$ is a limit of values lower than $\mathcal{J}(P(b))$. In the remaining cases the truth value of $\mathrm{LeftLim}(b)$ is $\mathcal{J}(P(b))$. Now let λ be the sentence

$$\forall y(\mathrm{LeftLim}(y) \to P(y) \vee (P(y) \to \forall v P(v))).$$

This sentence is valid in any structure based on the set $V_{\downarrow}$, but our structure $\mathcal{J}_0$ is a counter-example for it. ∎

4 Prenexability, inter-expressibility

In this section we use, in some cases, Kripke structures as an alternative semantics for fuzzy logics. We do not need to investigate the relationship between fuzzy structures and Kripke semantics. The Kripke semantics defined below is in fact Kripke semantics for intuitionistic predicate logic simplified for the purpose of Gödel-Dummett fuzzy logics.

A *Kripke frame* is a pair $\langle W, R\rangle$ such that W is a non-empty set (of *nodes*, or *worlds*) and R a relation on W which is reflexive, transitive, and quasi-linear.

If $\langle W, R\rangle$ is a Kripke frame and $\alpha R\beta$ then we say that β *is accessible from* α or that α *sees* β; the relation R is the *accessibility relation* of a frame $\langle W, R\rangle$ or of a Kripke structure $\langle W, R, \mathbf{s}\rangle$. *Quasi-linearity* means that no node α sees two incomparable nodes β_1 and β_2. A *Kripke structure* for a language L is a triple $\langle W, R, \mathbf{s}\rangle$ such that $\langle W, R\rangle$ is a Kripke frame and $\mathbf{s}$ a valuation function for the frame $\langle W, R\rangle$ and the language L. A *valuation function* $\mathbf{s}$ for a frame $\langle W, R\rangle$ and a language L is a function defined on W such that each $\mathbf{s}(\alpha)$ for $\alpha \in W$ is a structure (in the classical sense) for the language L and moreover the following two conditions are satisfied. First, all structures $\mathbf{s}(\alpha)$, for $\alpha \in W$, have the same domain and the same realizations $F^{\mathbf{s}(\alpha)}$ of all function symbols $F \in L$. Second, whenever $R \in L$ is a predicate symbol and $\alpha, \beta \in W$ nodes such that $\alpha R\beta$ then the realizations $R^{\mathbf{s}(\alpha)}$ and $R^{\mathbf{s}(\beta)}$ of the symbol R in the structures $\mathbf{s}(\alpha)$ and $\mathbf{s}(\beta)$ satisfy $R^{\mathbf{s}(\alpha)} \subseteq R^{\mathbf{s}(\beta)}$.

An example of a Kripke structure for a language $\{P, Q\}$ with two unary predicates P and Q is in Fig. 2. It consists of nodes α_i and β_i for $i \in \omega$, and α_ω and β_ω. The accessibility relation is indicated by arrows, where the automatic arrows implied by reflexivity and transitivity are not shown. So e.g. α_ω sees itself and all α_i, and nothing else. Ovals indicate the realizations of the symbol P, while dotted arcs indicate the realizations of the symbol Q.

A *forcing relation* of a Kripke structure is a relation $\Vdash$ between nodes, formulas and evaluations of variables. We read $\alpha \Vdash \varphi[e]$ as "φ is *satisfied* (*forced*) by e in α". Forcing relation is defined by $\alpha \Vdash \varphi[e] \Leftrightarrow \alpha \models \varphi[e]$, where $\models$ has the classical meaning, for atomic formulas φ. It extends to formulas whose outermost symbol is a conjunction, disjunction or *any* quantifier by conditions like

$$\alpha \Vdash (\varphi \,\&\, \psi)[e] \Leftrightarrow \alpha \Vdash \varphi[e] \text{ and } \alpha \Vdash \psi[e],$$

(preservability conditions), and it extends to formulas whose outermost symbol is an implication by a little bit more complicated condition

$$\alpha \Vdash (\varphi \to \psi)[e] \Leftrightarrow \forall\beta(\alpha R\beta \;\&\; \beta \Vdash \varphi[e] \Rightarrow \beta \Vdash \psi[e]).$$

For example, in the structure from Fig. 2 we have $\gamma \Vdash \exists x Q(x)$ for each γ except for $\gamma = \alpha_\omega$; we also have $\beta_\omega \not\Vdash \exists x P(x)$ and so $\beta_\omega \not\Vdash \exists x Q(x) \to \exists x P(x)$. On the other hand $\alpha_\omega \Vdash \exists x Q(x) \to \exists x P(x)$ because α_ω does not see β_ω.

A basic fact about Kripke structures is the *persistency condition*: if $\alpha, \beta \in W$ are nodes such that $\alpha R\beta$, and $\alpha \Vdash \varphi[e]$, then $\beta \Vdash \varphi[e]$.

A formula φ is *valid* in a Kripke structure $\langle W, R, \mathbf{s}\rangle$ if $\alpha \Vdash \varphi[e]$ for each node $\alpha \in W$ and each evaluation of variables e. If φ is not valid in $\langle W, R, \mathbf{s}\rangle$ then $\langle W, R, \mathbf{s}\rangle$ is a *counter-example* for φ. If all logical truths of a logic G are valid in a structure $\langle W, R, \mathbf{s}\rangle$ then we say that $\langle W, R, \mathbf{s}\rangle$ is a *model* of G. One can check that any Kripke structure is a model of the logic BG; in case of the schema S_1 the argument refers to the fact that all values of the valuation function $\mathbf{s}$ have the same domain (we deal with Kripke structures with

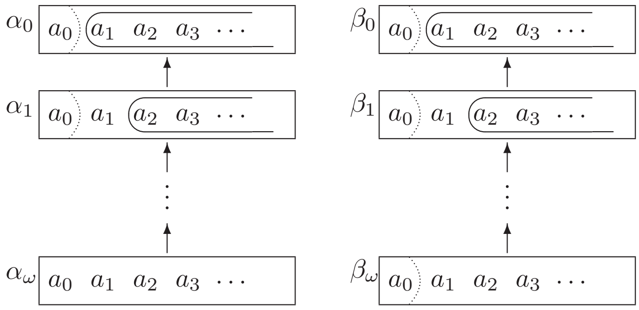

Figure 2: Structure $\mathbf{K_1}$ for the logic S3G

constant domains), while in case of the prelinearity schema the argument refers to quasi-linearity of the accessibility relation. Note that for the sole purpose of constructing counter-examples quasi-linearity could be replaced by mere linearity of Kripke frames. However, for our purposes the frames with incomparable nodes will be quite useful.

Theorem 3 *There are formulas that are not prenexable in the logic* S3G*. In particular,* $\exists x Q(x) \to \exists x P(x)$ *is such a formula.*

Proof If $\exists x Q(x) \to \exists x P(x)$ is equivalent to any prenex φ then it is also equivalent to a prenex φ in the language $\{P, Q\}$. If so then for any node γ of any Kripke structure $\langle W, R, \mathbf{s}\rangle$ which is a model of the logic S3G we have $\gamma \Vdash (\exists x Q(x) \to \exists x P(x))[e]$ iff $\gamma \Vdash \varphi[e]$. We will show that it is not the case: the structure $\mathbf{K_1}$ in Fig. 2 is a model of the logic S3G, the formula $\exists x Q(x) \to \exists x P(x)$ is forced in α_ω (hence, by persistency, in all α_i) and not forced in β_ω, while no prenex formula has this property.

First we verify that all instances of the schema S_3 are valid in $\mathbf{K_1}$. Let formulas $\varphi(x)$ and ψ and an evaluation e of variables (which we do not indicate if convenient) be given. We verify that if γ is any node of $\mathbf{K_1}$ such that $\gamma \Vdash \forall x \varphi \to \psi$ then $\gamma \Vdash \exists x(\varphi \to \psi)$. If $\gamma \Vdash \psi$ then $\gamma \Vdash \varphi(a) \to \psi$ for any a, and so $\gamma \Vdash \exists x(\varphi(x) \to \psi)$. So assume $\gamma \not\Vdash \psi$. Let γ_0 be the maximal node accessible from γ such that $\gamma_0 \not\Vdash \psi$. From $\gamma \Vdash \forall x \varphi \to \psi$ and $\gamma_0 \not\Vdash \psi$ we have $\gamma_0 \not\Vdash \forall x \varphi$. So there exists an a such that $\gamma_0 \not\Vdash \varphi(a)$. Now we can verify that $\gamma \Vdash \varphi(a) \to \psi$: if δ is such that $\gamma R \delta$ and $\delta R \gamma_0$ then $\delta \not\Vdash \varphi(a)$ by persistency, and if $\gamma_0 R \delta$ and $\delta \neq \gamma_0$ then $\delta \Vdash \psi$ because γ_0 was maximal such that $\gamma_0 \not\Vdash \psi$. So indeed $\gamma \Vdash \varphi(a) \to \psi$, and thus $\gamma \Vdash \exists x(\varphi \to \psi)$.

Now it is evident that *if* $i \in \omega$*,* e *an evaluation of variables and* φ *an open formula then* $\alpha_i \Vdash \varphi[e]$ *iff* $\beta_i \Vdash \varphi[e]$. Let us save this fact as Sublemma 1 and verify the following Sublemma 2: *if* φ *is open and* e *an evaluation of variables then* $\forall j {\in} \omega(\alpha_j \Vdash \varphi[e])$ *iff* $\beta_\omega \Vdash \varphi[e]$. Indeed, $\alpha_j \Vdash P(a)$ is true for at most finitely many j and simultaneously $\beta_\omega \not\Vdash P(a)$, also $\alpha_j \Vdash Q(a_0)$ and $\beta_\omega \Vdash Q(a_0)$, and finally for $b \neq a_0$ we have $\alpha_j \not\Vdash Q(b)$ and $\beta_\omega \not\Vdash Q(b)$. So our Sublemma 2 holds for atomic φ. We continue by an induction on complexity of φ. If $\beta_\omega \Vdash \varphi$ then $\forall j(\beta_j \Vdash \varphi)$ by persistency, and $\forall j(\alpha_j \Vdash \varphi)$ by Sublemma 1; so the implication $\Leftarrow$ does not need the induction hypothesis. Assume that φ is $\psi \to \chi$ and $\beta_\omega \not\Vdash \varphi$. Then there is a γ accessible from β_ω such that $\gamma \Vdash \psi$ and $\gamma \not\Vdash \chi$. If $\gamma = \beta_i$ then $\alpha_i \not\Vdash \psi \to \chi$ by Sublemma 1. If $\gamma = \beta_\omega$ then, by the induction hypothesis, ψ is forced in all α_i while χ is not; so, using Sublemma 1 again, it is not the case that $\forall j(\alpha_j \Vdash \psi \to \chi)$. The reasoning in cases where φ is a conjunction or a disjunction is straightforward. So Sublemma 2 is proved.

The final step is to verify Sublemma 3: *if φ is prenex and e an evaluation of variables and $\alpha_\omega \Vdash \varphi[e]$ then $\beta_\omega \Vdash \varphi[e]$.* This is proved by induction on the number of quantifiers in the quantifier prefix of φ. If this number is zero then the statement follows from Sublemma 2. The rest is left to the reader. ∎

Theorem 4 *There are formulas that are not prenexable in the logic $\mathrm{G}_\downarrow$. In particular, $\neg\forall x P(x)$ is such a formula.*

Proof The reasoning is similar as in the proof of Theorem 3; now we use the structure $\mathbf{K_2}$ in Fig. 3, with nodes α_i where $i \in \omega$ and a separate node β. Realizations of the symbol P is indicated by arcs and an oval. We first (superfluously) verify that all instances of the schema S_2 are valid in $\mathbf{K_2}$. So let formulas $\varphi(x)$ and ψ and a node γ such that $\gamma \Vdash \psi \to \exists x\varphi$ be given. If ψ is forced in no node accessible from γ then $\gamma \Vdash \psi \to \varphi(a)$ for arbitrary a and so $\gamma \Vdash \exists x(\psi \to \varphi(x))$. Otherwise let γ_0 be minimal node accessible from γ such that $\gamma_0 \Vdash \psi$. From $\gamma \Vdash \psi \to \exists x\varphi$ and $\gamma_0 \Vdash \psi$ we have an a such that $\gamma_0 \Vdash \varphi(a)$. It is easy to verify that $\gamma \Vdash \psi \to \varphi(a)$. So indeed $\gamma \Vdash \exists x(\psi \to \varphi(x))$.

It is evident that, in the structure $\mathbf{K_2}$, the formula $\neg\forall x P(x)$ is forced in α_0 (and thus in all α_i) and not forced in β. We show that no prenex formula in the language $\{P\}$ has this property. This follows from the following two sublemmas. Sublemma 1 says that *if φ is open and e an evaluation of variables then $\exists j(\alpha_j \Vdash \varphi[e])$ iff $\beta \Vdash \varphi[e]$.* Then Sublemma 2 for prenex formulas says that *if φ is prenex and e an evaluation of variables such that $\exists j(\alpha_j \Vdash \varphi[e])$ then $\beta \Vdash \varphi[e]$.* Verification of these sublemmas is left to the reader.

Consider a fuzzy structure with domain $\{ a_i ; i \in \omega \}$ and a valuation function defined by $\mathcal{J}(P(a_i)) = \frac{1}{i+1}$ where $i \geq 0$. One can verify that *if φ is any formula, e an evaluation of variables and i least index such that, in $\mathbf{K_2}$, $\alpha_i \Vdash \varphi[e]$ then $\mathcal{J}(\varphi[e]) = \frac{1}{i+1}$. If φ is nowhere forced then $\mathcal{J}(\varphi[e]) = 0$.* It follows that any formula which is not forced in some α_i has a fuzzy counter-example based on the truth value set $V_\downarrow$. For similar reasons, any formula not satisfied in β has a (different) fuzzy counter-example based on the set $\{0, 1\} \subseteq V_\downarrow$. Put together

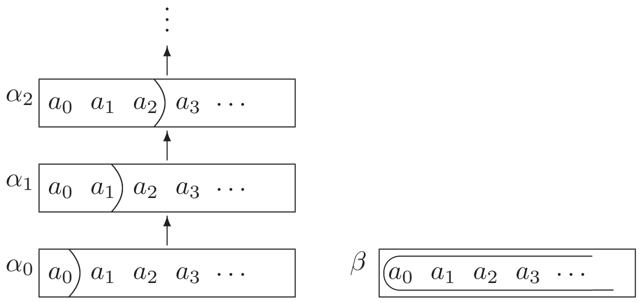

Figure 3: A structure $\mathbf{K_2}$ for the logic S2G

and contraposed, any formula having no fuzzy counter-example based on the set $V_\downarrow$ is valid in $\mathbf{K_2}$. So $\mathbf{K_2}$ is a model of the logic $G_\downarrow$. ■

Theorem 5 *(a) In the logic* S3G *the quantifier* $\exists$ *in not expressible in terms of the remaining logical symbols.*
(b) In the logic S2G*, however, the quantifier* $\exists$ *is expressible in terms of the remaining symbols.*
(c) The quantifier $\forall$ *is not expressible in terms of the remaining logical symbols in the logic* G_3*.*

Proof (b) In S2G the formulas $\exists x\varphi(x)$ and $\forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x))$ are equivalent; the implication $\to$ is easy, the implication $\leftarrow$ follows from Theorem 1.

(a) Turn back to the structure $\mathbf{K_1}$ from Fig. 2 and look at the left part with bottom α_ω. The formula $\exists x P(x)$ is forced in all α_i for $i \in \omega$, not forced in α_ω. One can verify by induction on complexity of φ that *if* φ *is a formula in the language* $\{P\}$ *and without an occurrence of* $\exists$ *then* $\{\ \alpha_\iota\ ;\ \alpha_\iota \Vdash \varphi\ \}$ *either is finite or contains all nodes from the left part of* $\mathbf{K_1}$ *including* α_ω*.* Since the structure $\mathbf{K_1}$ is a model of S3G, the formula $\exists x P(x)$ is not S3G-equivalent to any formula not containing the existential quantifier.

(c) Consider a domain $D = \{a, b\}$, a language with a single unary predicate P and a valuation function defined by $\mathcal{J}(P(a)) = 1$, $\mathcal{J}(P(b)) = \frac{1}{2}$. *Let* φ *be a formula in the language* $\{P\}$ *not containing* $\forall$*, let* e_0 *be an evaluation of variables such that* $\mathcal{J}(\varphi[e_0]) \neq 0$*. Then* $\mathcal{J}(\varphi[e]) \neq 0$ *for all* e*; moreover* $\mathcal{J}(\varphi[e]) = 1$ *whenever all values of* e *are* a*.* This fact can be proved by induction on complexity of φ. Since the formula $\forall x P(x)$ violates the condition in the claim it is not equivalent to any formula not containing $\forall$. ■

Remark 1 In connection with the fact that in G_3 the quantifier $\forall$ is not expressible in terms of the remaining logical symbols it can be of some interest that also none of the symbols $\to$ and & is in (propositional) G_3 expressible in terms of the remaining symbols. The proof can be obtained by analyzing the proof given for the logic BG in [9], for an alternative proof see [8]. M. Dummett discovered that disjunction is expressible in terms of $\rightarrow$ and &: the formula $A \vee B$ is BG-equivalent to $((A \rightarrow B) \rightarrow B)\ \&\ ((B \rightarrow A) \rightarrow A)$. The formula we used in Theorem 5 to express the quantifier $\exists$ in terms of $\forall$ can be viewed as reproducing the Dummett trick in predicate logic.

Remark 2 It seems an interesting problem to develop some satisfactory semantics for the logics S2G, S3G, and PG, with respect to which the logics would be complete. Perhaps the construction in Theorem 2 could be viewed as a hint for the definition of semantics for the logic PG.

Remark 3 We do not know whether there exists a Gödel-Dummett logic (in our sense or in the more restrictive sense of the Vienna school) in which each formula is equivalent to a prenex one while some of the classical prenex operations is not valid. So we do not know what is the weakest Gödel-Dummett logic which allows prenexability.

References

[1] M. Baaz, N. Preining, and R. Zach. Characterization of the axiomatizable prenex fragments of first-order Gödel logics. In *33rd International Symposium on Multiple-valued Logic, May 16–19, 2003*, pages 175–180, Tokyo, 2003. IEEE Computer Society Press.

[2] M. Dummett. A propositional calculus with denumerable matrix. *J. Symb. Logic*, 25:97–106, 1959.

[3] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer, 1998.

[4] P. Hájek. A non-arithmetical Gödel logic. *Logic J. of the IGPL*, 13(4):435–441, 2005. A special issue devoted to selected papers presented at the Challenge of Semantics workshop, Vienna, July 2004.

[5] S. C. Kleene. *Introduction to Metamathematics.* D. van Nostrand, 1952.

[6] B. Kozlíková. Sémantické metody v intuicionistické predikátové logice (Semantical Methods in Intuitionistic Predicate Logic). Master's thesis, Philosophical Faculty of Charles University, Department of Logic, 2004.

[7] N. Preining. *Complete Recursive Axiomatizability of Gödel Logics.* PhD thesis, Vienna University of Technology, Austria, 2003.

[8] V. Švejdar. Note on inter-expressibility of logical connectives in finitely-valued Gödel-Dummett logics. *Soft Computing*, 10(7):629–630, 2006.

[9] V. Švejdar and K. Bendová. On inter-expressibility of logical connectives in Gödel fuzzy logics. *Soft Computing*, 4(2):103–105, 2000.

[10] M. Takano. Another proof of strong completeness of the intuitionistic fuzzy logic. *Tsukuba J. of Math.*, 11:101–105, 1987.