Note on Inter-Expressibility of Logical Connectives in Finitely-Valued Gödel-Dummett Logics

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Abstract

Let G_m be the m-valued Gödel-Dummett fuzzy logic. If $m \geq 3$ then neither conjunction nor implication is in G_m expressible in terms of the remaining connectives. This fact remains true even if the propositional language is enriched by propositional constants for all truth values.

Gödel-Dummett fuzzy *propositional logic* can be defined as an extension of the intuitionistic propositional logic by the *prelinearity schema* $(A \to B) \vee (B \to A)$. This logic is known to be complete w.r.t. Kripke semantics with linearly ordered frames. Alternatively, it is (even better) known to be complete w.r.t. fuzzy semantics where the truth values can be numbers from the real interval $[0, 1]$, truth functions of conjunction and disjunction are the functions min and max, and the truth function of implication is the function $\Rightarrow$ defined by $a \Rightarrow b = 1$ if $a \leq b$ and $a \Rightarrow b = b$ otherwise. Negation $\neg$ can be considered a basic symbol in Gödel-Dummett logic, or the formula $\neg A$ can be understood as a shorthand for $A \to \bot$, where $\bot$ is the symbol for falsity. In any case the truth function of negation is the function $a \mapsto a \Rightarrow 0$; its value is 1 for $a = 0$ and its value is 0 for any other a.

The *m-valued Gödel-Dummett logic* G_m for $m \geq 2$ is defined as Gödel-Dummett logic with an additional restriction that, besides the extremal truth values 0 and 1, only $m - 2$ *intermediate* truth values are possible. By convention, the set $\{\, 1 - \frac{1}{k}\, ; 2 \leq k < m \,\}$ is usually taken for the set of intermediate truth values. If it is the case and if $m > 2$ then $\frac{1}{2}$ is the least intermediate value. Fig. 1 shows the truth functions of &, $\vee$, $\to$, $\neg$, in respective order, in the logic G_3 having

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only one intermediate truth value. The logic G_2, with no intermediate truth values, is the classical two-valued logic.

It is known that none of the binary logical connectives &, $\vee$, $\rightarrow$ is in the intuitionistic logic expressible in terms of the remaining propositional symbols, see e.g. [3] (the thesis [2] contains a careful elaboration in Czech). As to Gödel-Dummett logic, M. Dummett discovered that disjunction $\vee$ *is expressible* in this logic in terms of & and $\rightarrow$. On the other hand, it is shown in [7] that the results about intuitionistic non-expressibility of & and $\rightarrow$ can be adjusted for the Gödel-Dummett logic (see also [1]). In connection with the fact that in predicate logic the quantifier $\forall$ is not expressible in terms of $\exists$ already in the three-valued logic G_3, see [6], it is of some interest that & and $\rightarrow$ are not expressible in terms of the remaining connectives also already in the (propositional) logic G_3. Proofs can be obtained by careful analysis of [7].

In this paper I offer a simple direct proof of the fact that neither conjunction nor implication is expressible in terms of the remaining connectives in the three-valued logic G_3. I sharpen this fact in two directions. First, I give a proof which works for any number m of truth values such that $m \geq 3$, finite or infinite. Second and more important, I show that the result remains true if the propositional language is enriched by *constants for all truth values.* So to be specific, we consider a language with binary connectives &, $\vee$, and $\rightarrow$, a unary connective $\neg$, and symbols (constants) for all truth values. As to the constants, we do not actually need to introduce a notation for them, except that $\perp$ is a constant for the value 0. As to the negation symbol $\neg$, its presence is important because we claim that $\rightarrow$ is not expressible in terms of the remaining symbols *including negation.*

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Theorem 1 *(a) In Gödel-Dummett propositional logic* G_m*, where* $m \geq 3$*, conjunction* & *is not expressible in terms of* $\rightarrow$*,* $\vee$*,* $\neg$ *and symbols for all truth values.*
(b) In the same logics, implication $\rightarrow$ *is not expressible in terms of* &*,* $\vee$*,* $\neg$ *and symbols for all truth values.*

Proof Let a truth value set V containing, besides 0 and 1, at least one other truth value be given. We show that the formula $p \,\&\, q$ is not equivalent, in the logic having V as the truth value set, to any formula built up using only $\rightarrow$, $\vee$, $\neg$ and constants. Then we show that the formula $p \rightarrow q$ is not equivalent, in the same logic having V as the truth value set, to any formula built up using only &, $\vee$, $\neg$ and constants. It is evident that it suffices to think about formulas $A(p, q)$, without atoms different from p and q, as possible candidates for an equivalent formula. Each such formula $A(p, q)$ represents, in an obvious sense, a truth

	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

0	1
$\frac{1}{2}$	0
1	0

Figure 1: Truth functions

function from V^2 to V. We can think of f as a table like those in Fig. 1, where rows correspond to p while columns correspond to q. However f may be infinite. Let *border values* of a truth function f be the values $f(0,b)$ and $f(a,0)$ for arbitrary a and b. Let *interior points* be pairs $[a,b]$ of truth values such that $a \neq 0$ and $b \neq 0$, and let *interior values* of f be the values in interior points.

We split the proof into three claims. Claim 1 says that *if $A(p,q)$ is any formula and if f is the truth function represented by A then if some interior value of f is* 0 *then all interior values of f are* 0. This claim is easily proved by induction on the number of logical symbols (atoms, constants, and connectives) in A. Assume that A is $B \vee C$ and that B and C represent functions f_1 and f_2 respectively. Let $f(a_0, b_0) = 0$ where both a_0 and b_0 are nonzero and $f = \max\{f_1, f_2\}$. Then $f_1(a_0, b_0) = f_2(a_0, b_0) = 0$. By the induction hypothesis, $f_i(a, b) = 0$ for all pairs $[a, b]$ such that $a \neq 0$ and $b \neq 0$ and for both i. So $f(a, b) = \max\{f_1(a, b), f_2(a, b)\} = 0$ for all interior $[a, b]$. Reasoning in the case where A has the form $B \,\&\, C$ is similar. Assume that A is $B \to C$ and that A, B, and C represent functions f, f_1, and f_2 respectively, where $f = f_1 \Rightarrow f_2$. If $f(a_0, b_0) = 0$ for an interior point $[a_0, b_0]$ then $f_1(a_0, b_0) \neq 0$ and $f_2(a_0, b_0) = 0$. By the induction hypothesis, $f_1(a, b) \neq 0$ and $f_2(a, b) = 0$ for all interior points $[a, b]$. So $f(a, b) = 0$ for all interior points $[a, b]$. The case where A is $\neg B$ is similar, and actually can be omitted since negation is expressible in terms of the remaining symbols.

Now Claim 2 says that *if $A(p,q)$ contains no occurrences of the symbol & and if f is the truth function represented by A, then if f has an intermediate value (i.e. a value different from* 0 *and* 1*) in some interior point then some border value of f is nonzero.* This claim is evidently true for atoms and constants. Assume that A is $B \vee C$, the formulas A, B, and C represent functions f, f_1, and f_2 respectively, and $0 < f(a_0, b_0) < 1$ for an interior point $[a_0, b_0]$. Then the induction hypothesis is applicable to that of the two functions f_i, for which $f_i(a_0, b_0) = f(a_0, b_0)$. So f_i and in turn also $f = \max\{f_1, f_2\}$ has a nonzero border value. Reasoning in the case where A is $B \to C$ is similar: if the function $f_1 \Rightarrow f_2$ has an intermediate value in an interior point then the induction hypothesis is applicable to f_2. The case where A is $\neg B$ is evident for more than one reasons, one of which being that the function represented by A has no intermediate values.

Since the formula $p \,\&\, q$ represents a truth function f with 0 as the only border value and with intermediate truth value in some (in fact, in at least three) interior points, it is not equivalent to a formula $A(p,q)$ not containing &.

In (b) we use the following Claim 3: *if $A(p,q)$ contains no occurrences of the symbol $\rightarrow$ then the restriction of f to the set of all interior points is non-decreasing in any of its two arguments.* Again this claim is proved by induction on the number of symbols in A. The only a little interesting case occurs when A is $\neg B$. Assume that e.g. $f(a_1,b) > f(a_2,b)$ where a_1, a_2, b are nonzero. Since $a \Rightarrow 0$ can only be 0 or 1, we have $f(a_1,b) = 1$ and $f(a_2,b) = 0$. Let g be the truth function represented by B. We have $g(a_1,b) = 0$ and $g(a_2,b) \neq 0$, a contradiction with Claim 1.

Let a be a fixed intermediate truth value. Since for the truth function f represented by the formula $p \rightarrow q$ we have $f(a,a) > f(1,a)$, and both pairs $[a,a]$ and $[1,a]$ are interior points, it follows from Claim 3 that the formula $p \rightarrow q$ is not equivalent to any formula $A(p,q)$ not containing $\rightarrow$. ■

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