The Limit Lemma in Fragments of Arithmetic

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Abstract

The recursion theoretic limit lemma, saying that each function with a Σ\(_{n+2}\) graph is a limit of certain function with a Δ\(_{n+1}\) graph, is provable in BΣ\(_{n+1}\).

Keywords Limit lemma, fragments of arithmetic, collection scheme.

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Let N be the set of all natural numbers and let a function G : N\(^k\) \to N be such that for each \(x_1,..,x_k\) the function \(s \mapsto G(x, s)\), where \(x\) is a shorthand for \(x_1,..,x_k\), is eventually constant. Then we use \(\lim G(x, s)\) to denote the value the function \(s \mapsto G(x, s)\) assumes in each sufficiently large \(s\). The limit lemma says that for each set \(A \subseteq N^k\) such that \(A \in \Delta_2\) there exists a recursive function \(G : N^k \to N\) such that \(\lim G(x, s) = 1\) whenever \([x_1,..,x_k]\) \(\in A\), and \(\lim G(x, s) = 0\) whenever \([x_1,..,x_k]\) \(/\in A\). For the definition of Σ\(_n\), Π\(_n\), and Δ\(_n\), where \(n \geq 1\), see e.g. [5], and recall that a set is Δ\(_1\) if and only if it is recursive, and that Δ\(_n\) = Σ\(_n\) \cap Π\(_n\). The version of the limit lemma for functions says that for each function \(F : N^k \to N\) whose graph is Σ\(_2\) there exists a recursive \(G : N^k \to N\) such that \(F(x) = \lim G(x, s)\) for each \(k\)-tuple \([x_1,..,x_k]\). As can be seen e.g. from [4] and [2], the limit lemma is a useful tool in recursion theory.

Peano arithmetic PA is an axiomatic theory formulated in the arithmetical language \(\{+,-,0,\cdot,\leq,<\}\); its axioms can be described as a finite set of base axioms plus the induction scheme. For details see e.g. [3]. Bounded quantifiers are quantifiers of the form \(\forall v \leq x, \exists v \leq x, \forall v < x,\) and \(\exists v < x\). A bounded formula, or a Δ\(_0\)-formula, is a formula all quantifiers of which are bounded. A Σ\(_n\)-formula is a formula having the form \(\exists v_1\forall v_2\exists v_3\ldots v_n \varphi\), with \(n\) alternating quantifiers, where the first quantifier is existential and the matrix \(\varphi\) is a Δ\(_0\)-formula. A Π\(_n\)-formula is a formula of the form \(\forall v_1\exists v_2\forall v_3\ldots v_n \varphi\) where again \(\varphi \in \Delta_0\). So \(\Sigma_0 = \Pi_0 = \Delta_0\). The theory \(\Gamma^\prime\), where \(\Gamma = \Sigma_0\) or \(\Pi_0\), is PA with the induction scheme restricted to Π-formulas. The collection scheme is the scheme

\[\forall y\forall x(\forall v \leq x \exists z \varphi(v, z, y) \rightarrow \exists y \forall v \leq x \exists z \leq t \varphi(v, z, y)).\]

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The theory $B\Gamma$, where again $\Gamma$ is $\Sigma_n$ or $\Pi_n$, is $\Delta_0$ extended by the collection scheme restricted to $\Gamma$-formulas. It is known that for each $n$ the theories $\Sigma_n$ and $\Pi_n$ are equivalent, and both $B\Sigma_n$ and $B\Sigma_{n+1}$ are equivalent. $B\Sigma_{n+1}$ is a theory stronger than $\Sigma_n$, but weaker than $\Sigma_{n+1}$. For details and proofs, see again e.g. [3]. A useful property of $\Sigma_n$ is that it proves induction for $\Sigma_n(\Sigma_n)$-formulas, i.e. for formulas built up from $\Sigma_n$-formulas using logical connectives and bounded quantification. Also the least number principle for $\Sigma_n(\Sigma_n)$-formulas is provable in $\Sigma_n$. A useful property of $B\Sigma_{n+1}$ is that any formula obtained from $\Sigma_n$-formulas by bounded quantification is $B\Sigma_{n+1}$-equivalent to a $\Sigma_n$-formula. This fact can be used to verify that each $\Sigma_0(\Sigma_n)$-formula is $B\Sigma_{n+1}$-equivalent to a $\Sigma_n$-formula. We will also use the fact that $\Sigma_0(\Sigma_n)$-induction is provable in $B\Sigma_{n+1}$.

P. Hájek and A. Kučera show in [2] that the limit lemma for sets is provable in $I\Sigma_1$. P. Clote in an earlier paper [1] uses a version of the limit lemma for $\Sigma_n$-formulas, saying that any function having a $\Sigma_n$-2 graph is a limit of a function having a $\Delta_{n+1}$ graph, and proves this version in $B\Sigma_{n+2}$. I show that the results from [2] and [1] can be considerably improved: the limit lemma for $\Sigma_{n+2}$ functions is provable already in $B\Sigma_{n+1}$.

Note that speaking about sets definable in a model, in the formulation of Lemma 1 and Theorem 1 below, is a way to overcome the difficulty that one cannot directly speak about sets and functions in the arithmetical language. In proofs of Lemma 1 and Theorem 1 we are less careful and ignore this difficulty. Recall that if $n \geq 1$ then a set is $\Sigma_n$ if and only if it is $\Sigma_n$-definable in the standard model of arithmetic. A set $\Sigma_n$- and $\Pi_n$-definable in a model corresponds to a set which, on metamathematical level, is $\Delta_n$.

**Lemma 1** Let $M$ be a model of $B\Sigma_{n+1}$ with domain $M$ and let $A \subseteq M^k$ be simultaneously $\Sigma_n$- and $\Pi_n$-definable in $M$. Then there exists a function $G : M^k \rightarrow M$ with a graph $\Sigma_0(\Sigma_n)$-definable in $M$ such that $\lim_s G(x, s) = 1$ whenever $[x_1, \ldots, x_k] \in A$ and $\lim_s G(x, s) = 0$ whenever $[x_1, \ldots, x_k] \notin A$.

**Proof** Let the set $A$ be as specified and let $\varphi$ and $\psi$ be $\Sigma_n$-formulas such that $A = \{ [x_1, \ldots, x_k] : \exists u \forall v \varphi(x, u, v) \}$ and $A = \{ [x_1, \ldots, x_k] : \exists u \forall v \varphi(x, u, v) \}$, where $A$ is the complement of $A$. Think of the $k$-table $x$ as fixed and think of $\varphi$ and $\psi$ as zero-one tables unbounded in two directions, with $u$ running down and $v$ running to the right. One and only one of the two tables contains rows consisting entirely of ones. Let the function $H$ be defined as follows:

$$H(x, s) = \begin{cases} 1 & \text{if } \forall u \leq s (\forall v \leq s \varphi(x, u, v) \rightarrow \exists u' \leq u \forall v \leq s \varphi(x, u', v)) \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $[x_1, \ldots, x_k] \notin A$. Then $\exists u \forall v \psi(x, u, v)$ and $\forall u \exists v \neg \varphi(x, u, v)$. Let $u_0$ be some number satisfying $\forall v \psi(x, u_0, v)$; note that the existence of least such number is not guaranteed in $B\Sigma_{n+1}$. By $B\Sigma_{n+1}$ there exists a number $s_0$ such that $\forall u \leq u_0 \exists v \leq s_0 \neg \varphi(x, u, v)$. We can assume $s_0 \geq u_0$. If $s \geq s_0$ then there exists a number $u \leq s$, namely $u_0$, such that $\forall v \leq s \varphi(x, u, v)$ and simultaneously $\forall u' \leq u \exists v \leq s \varphi(x, u', v)$. So $H(x, s) = 0$ for all such $s$, i.e. $\lim_s H(x, s) = 0$. The proof that $\lim_s H(x, s) = 1$ whenever $[x_1, \ldots, x_n] \in A$ is similar. The graph of $H$ is $\Sigma_0(\Sigma_n)$. So the function $H$ is as desired. QED

**Theorem 1** Let $M$ be a model of $B\Sigma_{n+1}$ with domain $M$ and let $F : M^k \rightarrow M$ have a graph $\Sigma_0(\Sigma_n)$-definable in $M$. Then there exists a function $G : M^{k+1} \rightarrow M$ with a graph $\Sigma_0(\Sigma_n)$-definable in $M$ such that $F(x) \equiv \lim_s G(x, s)$ for each $x$.

**Proof** Let $F \in \Sigma_{n+2}$ with $k$ variables be given. It is clear that $F \in \Delta_{n+2}$ since for the complement of its graph we have $[x, y] \notin F \iff \exists y' \neq y \land [x, y'] \notin F$. By Lemma 1 applied to the graph of $F$ there exists a function $H \in \Sigma_0(\Sigma_n)$ such that $\lim_s H(x, y, t) = 1$ whenever $F(x) = y$ and $\lim_s H(x, y, t) = 0$ whenever $F(x) \neq y$.

As in the proof of Lemma 1, let $x$ be fixed and think of the function $H$ as a table with $t$ running down and $y$ running to the right. Let the score of a number $y$ at stage $s$ be defined as the length of maximal contiguous segment of ones which lies in column $y$, the bottom end of which is in row $s$ and the top end of which is in a row $t \geq y$. If $H$ is, for example, as in Fig. 1 then the scores of numbers 2, 3, and 5 at stage 5 are 2, 2, and 1, respectively, and the score of any other number at stage 5 is zero. The scores of numbers 1, 2, 3, and 5 at stage 8 are 2, 5, 4, and 4. Let $G(x, s)$ be defined as the least $y$ having maximal possible score at stage $s$. So in our example from Fig. 1 we have $G(\bar{x}, 5) = 2$ and $G(\bar{x}, 8) = 3$. It is evident that a score of a number $y \leq s$ at stage $s$ is a number not exceeding $s + 1 - y \leq s + 1$ and that all $y$’s greater than $s$ have zero score at stage $s$. The formula

$$\exists u \leq s + 1 (z + u = s + 1 \land y \leq u \land \forall t \leq s (u \leq t \rightarrow H(x, y, t) = 1)),$$

i.e. the formula the score of $y$ at stage $s$ is at least $z$, is a $\Sigma_0(\Sigma_n)$-formula. So $\Sigma_0(\Sigma_n)$-induction available in $B\Sigma_{n+1}$, there exists a greatest $z$ satisfying this formula, and the score of a number $y$ at stage $s$ is correctly defined. Also, the formulas the number $z$ is the maximal score at stage $s$ and the number $y$ is the least number having the maximal score at stage $s$ are $\Sigma_0(\Sigma_n)$-formulas. So

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Figure 1: Computing scores.
again by $\Sigma^0_0(\Sigma_n)$-induction, the maximal score exists, and the function $G$ is correctly defined. We have to verify that $\lim_s G(x,s) = F(x)$. Let $y_0 = F(x)$. We know that $\lim_t H(y_0, y_0, t) = 1$. So let the number $t_0$ be such that $t_0 \geq y_0$ and $\forall t (t \geq t_0 \rightarrow H(x, y_0, t) = 1)$. We also know that $\lim_t H(x, y, t) = 0$ for each $y \leq t_0$ such that $y \neq y_0$. Thus

$$\forall y \leq t_0 (y \neq y_0 \rightarrow \exists t (t \geq t_0 \& H(x, y, t) = 0)).$$

By $\Sigma_{n+1}$-collection (more precisely, by $\Sigma^0_0(\Sigma_n)$-collection available in $B\Sigma_{n+1}$) there exists an $s_0$ such that

$$\forall y \leq t_0 (y \neq y_0 \rightarrow \exists t \leq s_0 (t \geq t_0 \& H(x, y, t) = 0)).$$

This means that if $s \geq s_0$ then the score of all numbers $y \leq t_0$ such that $y \neq y_0$ at stage $s$ is lower than the score of $y_0$. Since ones occurring in column $y$ above the diagonal line do not count, the score of any $y > t_0$ at stage $s$ is automatically lower than the score of $y_0$. So $G(x, s) = y_0$ for each $s \geq s_0$, and thus $\lim_s G(x, s) = y_0$.

QED

References


