On Modal Systems with Rosser Modalities

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Outline

Prominent self-referential sentences

Gödel sentence

Gödel sentence of a theory T is a self-referential sentence ν saying I am not provable in T, i.e. satisfying $T \vdash \nu \equiv \neg\mathrm{Pr}(\overline{\nu})$.

Rosser sentence

of a theory T is a sentence ρ saying there exists a proof of my negation in T which is less that or equal to any possible proof of myself, i.e. satisfying $T \vdash \rho \equiv \exists y(\mathrm{Prf}(\overline{\neg\rho}, y) \mathbin{\&} \forall v{<}y\, \neg\mathrm{Prf}(\overline{\rho}, v))$.

Notation

$\mathrm{Prf}(x, y)$ is a proof predicate, i.e. an arithmetical formula saying y is a proof of x in T.
$\mathrm{Pr}(x)$ is a provability predicate; defined as $\exists y \mathrm{Prf}(x, y)$ and saying x is provable in T.

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The importance of provability logic

Important difference

$T \vdash \mathrm{Con}(T) \rightarrow \neg\mathrm{Pr}(\overline{\rho})\ \&\ \neg\mathrm{Pr}(\overline{\neg\rho})$.

$T \vdash \mathrm{Con}(T) \rightarrow \neg\mathrm{Pr}(\overline{\nu})$, but $T \nvdash \mathrm{Con}(T) \rightarrow \neg\mathrm{Pr}(\overline{\neg\nu})$.

Provability logic GL

$\mathrm{GL} \vdash \Box(p \equiv \neg\Box p)\ \&\ \neg\Box\bot \rightarrow \neg\Box p$.

Important remark (and a definition)

The arithmetical interpretation of the modal formula $\Box A$, i.e. an arithmetical sentence of the form $\mathrm{Pr}(..)$, is a Σ-sentence.
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The theory R of Guaspari and Solovay

Language

The usual modal language with propositional atoms, logical connectives, logical constants $\top$ and $\bot$, and the modality $\Box$, plus two additional binary modalities $\preceq$ and $\prec$ which are applicable only to formulas starting with $\Box$.

Example

$A \,\&\, \Box A \preceq \Box B \to \Box B$ is a shorthand for $(A \,\&\, (\Box A \preceq \Box B)) \to \Box B$.
$(A \vee \Box B) \preceq \Box B$ is not a formula.

Arithmetical interpretation

The interpretation (and reading) of $\Box A \preceq \Box B$ and $\Box A \prec \Box B$ is A has a proof which is less than or equal to (or less than, respectively) any proof of B.

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Axioms and rules of R are the axioms and rules of provability logic:

A1: all propositional tautologies,

A2: $\Box(A \to B) \to (\Box A \to \Box B)$,

A3: $\Box A \to \Box\Box A$,

A4: $\Box(\Box A \to A) \to \Box A$,

MP: $A \to B, A \,/\, B$,

Nec: $A \,/\, \Box A$.

plus $\Box A \,/\, A$, plus the basic axioms about witness comparison:

B1: $\Box A \preceq \Box B \to \Box A$,

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Example proof: $\mathsf{R} \vdash \Box(p \equiv \Box\neg p \preceq \Box p) \mathbin{\&} \neg\Box\bot \rightarrow \neg\Box p \mathbin{\&} \neg\Box\neg p$

Proof

Assume $\Box p$ or $\Box\neg p$. Then $\Box\neg p \preceq \Box p$ or $\Box p \prec \Box\neg p$ by B4.

$$
\begin{aligned}
\Box\neg p \preceq \Box p &\rightarrow \Box\neg p, && \text{by B1}\\
&\rightarrow \Box(\Box\neg p \preceq \Box p), && \text{by P}\\
&\rightarrow \Box p, && \text{since } \Box(\Box\neg p \preceq \Box p \rightarrow p)\\
&\rightarrow \Box\bot,\\
\Box p \prec \Box\neg p &\rightarrow \Box p, && \text{by B1}\\
&\rightarrow \Box(\Box p \prec \Box\neg p), && \text{by P}\\
&\rightarrow \Box\neg(\Box\neg p \preceq \Box p), && \text{by B3}\\
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&\rightarrow \Box\neg p, && \text{since } \Box(p \rightarrow \Box p \preceq \Box\neg p)\\
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Generalized proof predicate of PA

$\mathrm{Prf}^{\mathrm{h}}(x, y) \equiv$ the axioms of PA (with numerical codes) less than y are sufficient to prove (in the usual sense) the sentence x.

Fact

If the formalized proof predicate is used to interpret the modalities $\preceq$ and $\prec$ then $\Box A \preceq \Box B$ and $\Box A \prec \Box B$ are not Σ-sentences, and so the persistency axioms P are not valid.

The theory WR

has the axiom and the rule

W: $\Box A \rightarrow \Box(\neg B \rightarrow \Box A \prec \Box B)$, $\quad \Box A \mathbin{/} \neg B \rightarrow \Box A \prec \Box B$

instead of the axiom P and the rule $\Box A \mathbin{/} A$ of the theory R.

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The alternative theory SR

Language

The language of SR has two sorts of propositional atoms, normal atoms p, q, ... and Σ-atoms s, t, ...
Σ-formulas are formulas built up from $\top$, $\bot$, Σ-atoms, and formulas starting with $\Box$ using & and $\vee$ only.

Axioms

are as in WR, but W and the corresponding rule are replaced by stronger versions:

S: $\Box(E \to A) \to \Box(E \,\&\, \neg B \to \Box A \prec \Box B)$, $E \in \Sigma$,

$E \to A \,/\, E \,\&\, \neg B \to \Box A \prec \Box B$, $E \in \Sigma$.

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Example modal formula provable in SR . . .

If s is a Σ-atom then
$\mathrm{SR} \vdash \Box(p \equiv \Box\neg p \preceq \Box p) \rightarrow (\Box(s \rightarrow p) \vee \Box(s \rightarrow \neg p) \rightarrow \Box\neg s)$.

. . . and its arithmetical significance

If φ is a Rosser sentence constructed from the generalized proof predicate then neither φ nor $\neg\varphi$ is provable from any consistent Σ-sentence.

Put otherwise, both φ and $\neg\varphi$ are Π_1-conservative: each Π_1-sentence (i.e. negated Σ-sentence) provable from φ or $\neg\varphi$ is provable.

For Peano arithmetic, Π_1-conservativity is the same as interpretability.

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Some reading on Rosser constructions and Rosser logics

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