Weak Theories and Essential Incompleteness

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Outline

Essential Incompleteness and Essential Undecidability

Motivation

Which is the weakest axiomatic theory that is recursively axiomatizable and essentially incomplete?

Methods of essential incompleteness proofs

Essential incompleteness can be proved directly, or using interpretability.

Canonical source

The notions of essential incompleteness and essential undecidability, as well as the notion of interpretability, were introduced in [TMR53].

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Robinson's Arithmetic Q

Axioms

Q1: $\forall x \forall y (\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y)$,

Q2: $\forall x (\mathrm{S}(x) \neq 0)$,

Q3: $\forall x (x \neq 0 \rightarrow \exists y (x = \mathrm{S}(y)))$,

Q4: $\forall x (x + 0 = x)$,

Q5: $\forall x \forall y (x + \mathrm{S}(y) = \mathrm{S}(x + y))$,

Q6: $\forall x (x \cdot 0 = 0)$,

Q7: $\forall x \forall y (x \cdot \mathrm{S}(y) = x \cdot y + x)$.

Extensions and properties

Ordering can be defined by $x \leq y$ iff $\exists v (v + x = y)$.

Numerals: 0, $\mathrm{S}(0)$, $\mathrm{S}(\mathrm{S}(0))$, ... are denoted $\overline{0}$, $\overline{1}$, $\overline{2}$, ...

General facts, like $\forall x \forall y (x + y = y + x)$, are mostly unprovable.

$\mathrm{Q} \vdash \forall x (x \leq \overline{n} \rightarrow x = \overline{0} \vee \ldots \vee x = \overline{n})$, $\mathrm{Q} \vdash \overline{n} + \overline{m} = \overline{n + m}$.

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Essential Incompleteness Proofs

Ingredients of essential incompleteness proofs

A proof of essential incompleteness of a theory like Q usually uses
(i) definability of r.e. sets by Σ-formulas,
(ii) Σ-completeness (every true Σ-sentence is provable in Q),
plus one of additional conditions like:

(1) For each pair A, B of recursively enumerable sets there exists a Σ-formula $\varphi(x)$ such that $Q \vdash \varphi(\overline{n})$ for $n \in A - B$, and $Q \vdash \neg\varphi(\overline{n})$ for $n \in B - A$.

(2) Weak representability of recursive functions.

(3) The self-reference theorem.

Note

Proofs of additional conditions (1)–(3) usually use Rosser trick.
None of these conditions is needed if incompleteness is to be proved only for all Σ-sound extensions of Q.

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A Structural Incompleteness Proof

Let T be a consistent recursively axiomatized extension of Q.

- Take a pair A, B of disjoint recursively inseparable r.e. sets:

- Let $\varphi(x)$ be a formula like in the condition (1) above.
- Put $X = \{\, n\,;\ T \vdash \varphi(\overline{n})\,\}$. We have $A \subseteq X$ and X is r.e. Put $Y = \{\, n\,;\ T \vdash \neg\varphi(\overline{n})\,\}$. Again $B \subseteq Y$ and Y is r.e. Also $X \cap Y = \emptyset$.
- Fix $n_0 \notin X \cup Y$. Such an n_0 must exist, otherwise X and Y would be mutually complementary, and so X would be a recursive superset of A that is disjoint with B. Then $T \nvdash \varphi(\overline{n_0})$ and $T \nvdash \neg\varphi(\overline{n_0})$. So T is incomplete.

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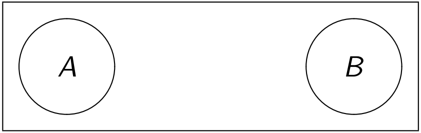

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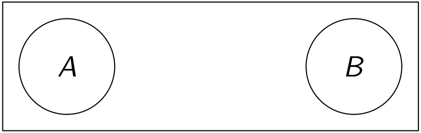

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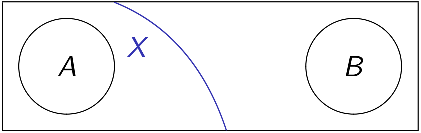

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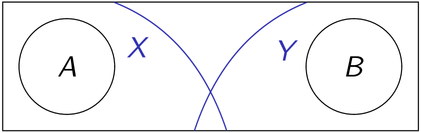

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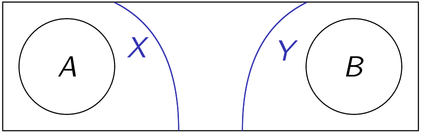

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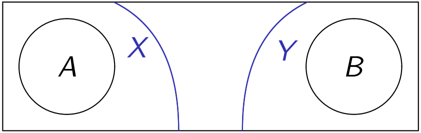

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The Grzegorczyk's Theory Q^-

The theory Q^-

has the language $\{0, \mathrm{S}, \mathrm{A}, \mathrm{M}\}$, where 0 and S play the same role as in Q, and A and M are ternary relation symbols for addition and multiplication. Axioms Q1–Q7 are replaced by variants saying that A and M are graphs of binary functions that satisfy some conditions but may be non-total. For example, axiom Q7 becomes if u is a product of x and y and w is a sum of u and x, then the product of x and $\mathrm{S}(y)$ exists and equals w.

Theorem

Q *is interpretable in* Q^-. *So* Q^- *is essentially incomplete.*

Proof

Using the Solovay's method of shortening of cuts.

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The Theory TC of Grzegorczyk and Zdanowski

The theory TC

has a binary symbol $\frown$ for concatenation, two constants a and b for two irreducible strings (i.e. one letter words) and some more or less obvious axioms like $\forall x \forall y \forall z (x \frown (y \frown z) = (x \frown y) \frown z)$.

History

Axioms were formulated by Tarski, some ideas go back to Quine.

Theorem ([GZ07])

TC *is essentially undecidable.*

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Is TC equi-interpretable with Q?

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$\Omega1$: $\overline{n} \neq \overline{m}$, for n different from m,
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R is the theory with schemata $\Omega1$–$\Omega5$, R_0 has only $\Omega1$–$\Omega4$.

Theorem

(a) Q is not interpretable in R (Hájek).
(b) R is interpretable in R_0 (Cobham, discussed in [JS83]).

Theorem

The self-reference theorem is true already for R_0.

Remarks

The schema $\Omega2$ can be omitted from R_0 ([Rob49]),
The connective $\equiv$ cannot be replaced by $\rightarrow$ in $\Omega4$.

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Ω2: $\overline{n} + \overline{m} = \overline{n+m}$,

Ω3: $\overline{n} \cdot \overline{m} = \overline{n \cdot m}$,

Ω4: $\forall x(x \leq \overline{n} \equiv x = \overline{0} \vee \;..\; \vee x = \overline{n})$,

Ω5: $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

R is the theory with schemata Ω1–Ω5, R_0 has only Ω1–Ω4.

Theorem

(a) Q is not interpretable in R (Hájek).

(b) R is interpretable in R_0 (Cobham, discussed in [JS83]).

Theorem

The self-reference theorem is true already for R_0.

Remarks

The schema Ω2 can be omitted from R_0 ([Rob49]),

The connective $\equiv$ cannot be replaced by $\rightarrow$ in Ω4.

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