Relatives of Robinson Arithmetic

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Outline

Introduction: the importance and properties of Robinson arithmetic

TC, the theory of concatenation

The theory F, its mutual interpretability with TC

Properties of Robinson arithmetic Q

Robinson arithmetic Q was defined in [TMR53] as an axiomatic theory with the language $\{0, \mathrm{S}, +, \cdot\}$ and with seven simple axioms like $\forall x \forall y(x + \mathrm{S}(y) = \mathrm{S}(x + y))$. Main properties:

- It is quite weak, e.g. $\mathrm{Q} \nvdash \forall x \forall y(x + y = y + x)$.
- Gödel 1st incompleteness theorem is true for it.
- It is finitely axiomatizable.
- It interprets some stronger theories, like $\mathrm{I}\Delta_0$.
- Gödel 2nd theorem is also true for it (after its meaning for such a weak theory is clarified).

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TC, the theory of concatenation

Constants ε, $\mathtt{a}$, $\mathtt{b}$, binary function symbol $\frown$ (usually omitted), and axioms:

TC1: $\forall x(x\varepsilon = \varepsilon x = x)$,

TC2: $\forall x \forall y \forall z((xy)z = x(yz))$,

TC3: $\forall x \forall y \forall u \forall v(xy = uv \rightarrow$

$\rightarrow \exists w((xw = u \ \& \ wv = y) \vee (uw = x \ \& \ wy = v)))$,

TC4: $\mathtt{a} \neq \varepsilon \ \& \ \forall x \forall y(xy = \mathtt{a} \rightarrow x = \varepsilon \vee y = \varepsilon)$,

TC5: $\mathtt{b} \neq \varepsilon \ \& \ \forall x \forall y(xy = \mathtt{b} \rightarrow x = \varepsilon \vee y = \varepsilon)$,

TC6: $\mathtt{a} \neq \mathtt{b}$.

Example proof of $\forall x(x\mathtt{a} \neq \varepsilon)$:

For, if $x\mathtt{a} = \varepsilon$, then $\mathtt{b}x\mathtt{a} = \mathtt{b}$. By TC5, $\mathtt{b}x = \varepsilon$ or $\mathtt{a} = \varepsilon$.
So $\mathtt{b}x = \varepsilon$, a contradiction with TC6.

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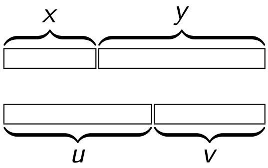

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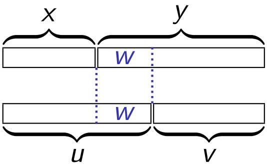

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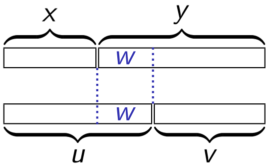

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Some properties of TC

Further examples

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Substrings and (no good notion of) occurrences

If $uxv = y$ for some u and v then one can say that x is a *substring* of y and write $x \sqsubseteq y$. Then one can prove e.g. $\forall x \forall y (\mathrm{a} \sqsubseteq xy \rightarrow \mathrm{a} \sqsubseteq x \vee \mathrm{a} \sqsubseteq y)$. In the same situation where $uxv = y$, one might be tempted to say that u is an occurrence of x in y. However, u is not uniquely determined.

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First ideas can be traced back to Tarski and Quine [Qui46]. Grzegorczyk proved undecidability of TC in [Grz05]. Grzegorczyk & Zdanowski proved essential undecidability of TC in [GZ08].

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Connection between Q and TC

Theorem 1 (Visser, V.Š., Ganea, Sterken, 2007)

TC interprets Q; in symbols, TC $\rhd$ Q.
So Q and TC are mutually interpretable.

Proof

Show TC $\rhd$ Q^-, where Q^- is a weaker variant of Q in which addition $+$ and multiplication $\cdot$ are possibly non-total.
Then use $Q^- \rhd Q$, and transitivity of $\rhd$.

Remark

$Q^- \rhd Q$ is proved in [Šve07b] using the (never published!) Solovay method of shortening of cuts, see [Sol76].

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Some example sentences

$F \vdash \forall x(x\mathtt{a} \neq \varepsilon)$, $F \vdash \mathtt{a} \neq \varepsilon$,
$F \vdash \forall x \forall y(xy = \varepsilon \rightarrow x = \varepsilon \ \& \ y = \varepsilon)$,
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so a letter can be created *ex nihilo*, as Albert Visser puts it.

Historical problem

Szmielew and Tarski claim in [TMR53] that F interprets Q, but give no proof.

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Mutual interpretability of F and TC

Theorem 2 (Ganea)

F interprets TC, i.e. F $\rhd$ TC. So from TC $\rhd$ Q we have F $\rhd$ Q.

Proof (a simplification of Ganea's proof)

In F (or in TC), write $x \sqsupset y$ for $\exists v(vx = y)$, i.e. for "x is an end segment of y". Then define *tame* strings as follows

$$\mathrm{Tame}(x) \equiv \forall v \forall z(z \sqsupset vx \rightarrow z \sqsupset x \vee x \sqsupset z).$$

One can verify that tame strings include ε, a, and b, are closed under concatenation, and satisfy the editor axiom TC3.

Theorem 3

TC interprets F.

Proof

Now in TC, work with *radical* strings, where

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Radical strings include ε, a, and b, etc.

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Mutual interpretability of F and TC

Theorem 2 (Ganea)

F interprets TC, i.e. $\mathsf{F} \rhd \mathsf{TC}$. So from $\mathsf{TC} \rhd \mathsf{Q}$ we have $\mathsf{F} \rhd \mathsf{Q}$.

Proof (a simplification of Ganea's proof)

In F (or in TC), write $x \sqsupset y$ for $\exists v(vx = y)$, i.e. for "x is an end segment of y". Then define *tame* strings as follows

$$\mathrm{Tame}(x) \equiv \forall v \forall z(z \sqsupset vx \rightarrow z \sqsupset x \vee x \sqsupset z).$$

One can verify that tame strings include ε, $\mathtt{a}$, and $\mathtt{b}$, are closed under concatenation, and satisfy the editor axiom TC3.

Theorem 3

TC interprets F.

Proof

Now in TC, work with *radical* strings, where

$$\mathrm{Rad}(x) \equiv \forall y \forall z(yx = zx \rightarrow y = z).$$

Radical strings include ε, $\mathtt{a}$, and $\mathtt{b}$, etc.

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