Decision Problems of Some Intermediate Logics and Their Fragments

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Outline

Introduction: propositional logics and algorithmical complexity

Restricting connectives and/or atoms in intuitionistic logic

Complexity of some intermediate logics
CPL, IPL, and computational complexity

Both CPL, the classical propositional logic, and IPL, the intuitionistic propositional logic, are decidable. None of them has an efficient decision procedure. However, CPL is coNP-complete, while

Theorem (Statman, 1979) IPL is PSPACE-complete.

Where: coNP is the class of problems $A$ such that non-membership to $A$ can be efficiently witnessed, PSPACE are problems solvable in polynomial space. Recall that $\text{coNP} \subseteq \text{PSPACE}$ and read “complete” as “no better classification is possible”.

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Where is the border between the somewhat simpler problems that are in coNP and the more difficult problems that are PSPACE-complete?
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Where is the border between the somewhat simpler problems that are in coNP and the more difficult problems that are PSPACE-complete?
A PSPACE-completeness proof

Take a sequence \( \{ D_n ; n \in \mathbb{N} \} \), where

\[
D_0 = \bot, \quad D_{n+1} = (p_n \rightarrow D_n) \lor (\neg p_n \rightarrow D_n),
\]

and consider a Kripke counter-example to \( D_{n+1} \):

\[
\not \models p_n \rightarrow D_n, \not \models \neg p_n \rightarrow D_n
\]

It *must* contain two disjoint copies of a counter-example to \( D_n \). So the size of the smallest counter-example to \( D_n \) grows exponentially with \( n \).

**Better:** Take \( D_{n+1} = (D_n \rightarrow q_n) \rightarrow (p_n \rightarrow q_n) \lor (\neg p_n \rightarrow q_n) \). Then it is still the case, but the size of \( D_n \) itself grows only polynomially.
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Possible restrictions

What happens to PSPACE-completeness, if

- the number of atoms is restricted, or
- the use of some logical connectives is forbidden, or
- IPL is replaced by some stronger (intermediate) logic?

**Theorem (Rybakov, 2006)**
IPL remains PSPACE-complete even if the number of propositional atoms is restricted to two.

**Rieger-Nishimura:**
With only one atom, IPL is efficiently decidable.

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IPL remains PSPACE-complete even if $\rightarrow$ (implication) is the only logical connective.
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Implicational fragments with finite number of atoms

\[(p \rightarrow q) \rightarrow p \]

\[(q \rightarrow p) \rightarrow p \quad \rightarrow \quad p \rightarrow q \]

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Example argument

If \( q \) then \( p \rightarrow q \). So if \( (p \rightarrow q) \rightarrow p \) then \( q \rightarrow p \).
Thus if \( (q \rightarrow p) \rightarrow p \) then \( ((p \rightarrow q) \rightarrow p) \rightarrow p \).

Theorem (Urquhart, 1974)

For each \( n \), the fragment of IPL built up using \( n \) atoms only and implication \( \rightarrow \) as the only connective is finite. It is thus efficiently decidable.
Implicational fragments with finite number of atoms

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$q$

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For each $n$, the fragment of IPL built up using $n$ atoms only and implication $\rightarrow$ as the only connective is finite. It is thus efficiently decidable.
Some intermediate logics

Gödel-Dummett logic $G$ (LG, BG)  
$IPL$ plus $(A \rightarrow B) \lor (B \rightarrow A)$.

Testability logic $KC$  
$IPL$ plus $\neg A \lor \neg \neg A$. This logic is also known as logic of weak excluded middle, or Jankov’s logic, or De Morgan logic. It is weaker than $G$: If $\neg \neg A \rightarrow \neg A$, then $\neg A$. If $\neg A \rightarrow \neg \neg A$, then $\neg \neg A$. It is complete w.r.t. Kripke models having a greatest element:

\[
p, q
\]

Theorem  
$KC$ is conservative over $IPL$ w.r.t. purely implicational formulas. Thus it is PSPACE-complete.
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![Diagram](attachment:image.png)

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![Diagram of Kripke model]

Theorem
KC is conservative over IPL w.r.t. purely implicational formulas. Thus it is PSPACE-complete.
Proof

Take an intuitionistic counter-model to a purely implicational formula $A$

Remarks

- Other popular intermediate logic (Kreisel-Putnam, Scott, Smetanich) are either weaker than KC, or stronger than G.
- KC is the weakest reflexive logic.
Proof
Take an intuitionistic counter-model to a purely impicational formula \( A \), and add a new node accessible from everywhere:

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\begin{aligned}
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