Decision Problems of Some Intermediate Logics and Their Fragments

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Logica 09, Hejnice, June 2009

Outline

Introduction: propositional logics and algorithmical complexity

Restricting connectives and/or atoms in intuitionistic logic

Complexity of some intermediate logics

CPL, IPL, and computational complexity

Both CPL, the classical propositional logic, and IPL, the intuitionistic propositional logic, are decidable. None of them has an efficient decision procedure. However, CPL is coNP-complete, while

Theorem (Statman, 1979) IPL is PSPACE-complete.

Where: coNP is the class of problems A such that *non*-membership to A can be efficiently witnessed, PSPACE are problems solvable in polynomial space. Recall that coNP $\subseteq$ PSPACE and read "complete" as "no better classification is possible".

Question

Where is the border between the somewhat simpler problems that are in coNP and the more difficult problems that are PSPACE-complete?

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A PSPACE-completeness proof

Take a sequence $\{ D_n \, ; \; n \in \mathrm{N} \}$, where

$$D_0 = \bot, \qquad D_{n+1} = (p_n \to D_n) \vee (\neg p_n \to D_n),$$

and consider a Kripke counter-example to D_{n+1}:

$\nVdash p_n \to D_n, \bullet \nVdash \neg p_n \to D_n$

It *must* contain two disjoint copies of a counter-example to D_n. So the size of the smallest counter-example to D_n grows exponentially with n.

Better: Take $D_{n+1} = (D_n \to q_n) \to (p_n \to q_n) \vee (\neg p_n \to q_n)$. Then it is still the case, but the size of D_n itself grows only polynomially.

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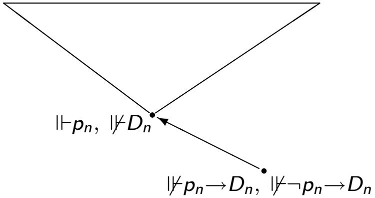

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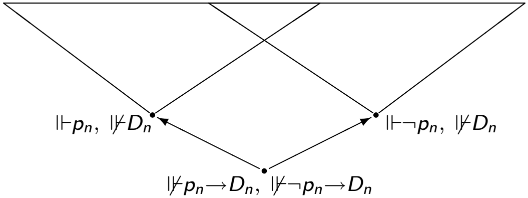

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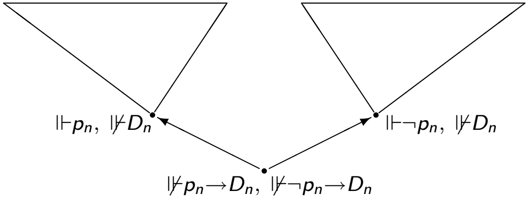

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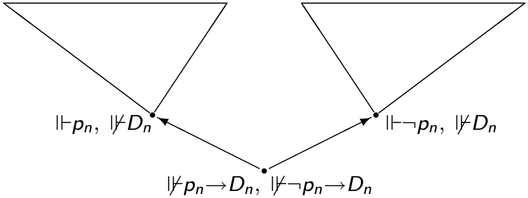

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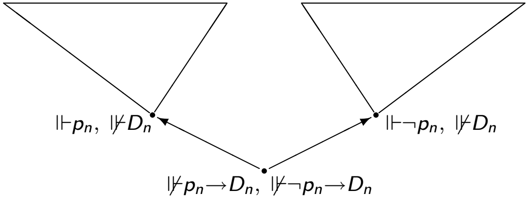

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Possible restrictions

What happens to PSPACE-completeness, if

- the number of atoms is restricted, or
- the use of some logical connectives is forbidden, or
- IPL is replaced by some stronger (intermediate) logic?

Theorem (Rybakov, 2006)

IPL remains PSPACE-complete even if the number of propositional atoms is restricted to two.

Rieger-Nishimura:

With only *one* atom, IPL is efficiently decidable.

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Implicational fragments with finite number of atoms

$$\begin{tikzcd}
 & ((p \to q) \to p) \to p \arrow[dl, no head] \arrow[dr, no head] & \\
(q \to p) \to p \arrow[dr, no head] & & p \to q \arrow[dl, no head] \\
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 & q &
\end{tikzcd}$$

Example argument

If q then $p \to q$. So if $(p \to q) \to p$ then $q \to p$.
Thus if $(q \to p) \to p$ then $((p \to q) \to p) \to p$.

Theorem (Urquhart, 1974)

For each n, the fragment of IPL built up using n atoms only and implication $\to$ as the only connective is finite. It is thus efficiently decidable.

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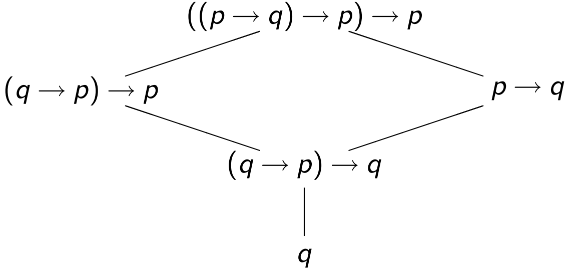

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Some intermediate logics

Gödel-Dummett logic G (LG, BG) IPL plus $(A \to B) \vee (B \to A)$.

Testability logic KC

IPL plus $\neg A \vee \neg\neg A$. This logic is also known as *logic of weak excluded middle*, or *Jankov's logic*, or *De Morgan logic*. It is weaker than G: If $\neg\neg A \to \neg A$, then $\neg A$. If $\neg A \to \neg\neg A$, then $\neg\neg A$. It is complete w.r.t. Kripke models having a greatest element:

Theorem

KC is conservative over IPL w.r.t. purely implicational formulas. Thus it is PSPACE-complete.

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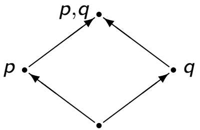

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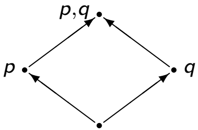

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Proof, final remarks

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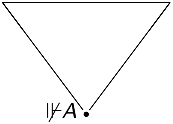
Take an intuitionistic counter-model to a purely implicational formula A



Remarks

- Other popular intermediate logic (Kreisel-Putnam, Scott, Smetanich) are either weaker than KC, or stronger than G.
- KC is the weakest *reflexive* logic.

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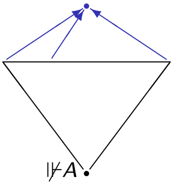
Take an intuitionistic counter-model to a purely implicational formula A, and add a new node accessible from everywhere:



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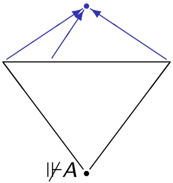

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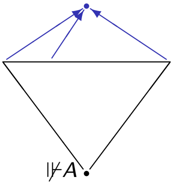

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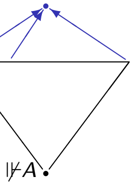

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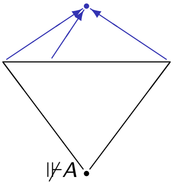

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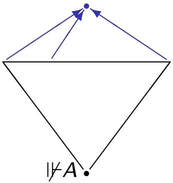

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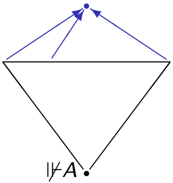

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