On Purely Implicational Fragments of Intuitionistic Propositional Logic

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Outline

Introduction: IPL, its fragments, algorithmical complexity

Kripke semantics, prime nodes, and the two atoms case

More than two atoms, and computer aided research
Fragments of IPL

are obtained from IPL by restricting the number of propositional atoms and/or the set of logical connectives.

Known facts
(a) (Statman, 1979) IPL is PSPACE-complete.
(b) IPL(→), the purely implicational fragment of IPL, is PSPACE-complete.
(c) (Rybakov, 2006) IPL(2), the fragment of IPL with two atoms only, is PSPACE-complete.
(d) (Rieger, 1949) IPL(1) is still infinite, but efficiently decidable (decidable in polynomial time).

Motivation
What happens if there are only two atoms and → is the only connective? I.e., how does IPL(→)(2) look like?

Kripke semantics for IPL

Definition
A Kripke model for intuitionistic logic is a triple \( K = (W, \leq, \models) \) where \( \leq \) is a transitive, reflexive, and weakly antisymmetric relation on the set \( W \neq \emptyset \), and the relation \( \models \) satisfies:
- if \( x \models A \) and \( x \leq y \) then \( y \models A \),
- \( x \models A \lor B \) iff \( x \models A \) or \( x \models B \), and similarly for \( A \land B \),
- \( x \models A \rightarrow B \) iff \( \forall y \geq x(y \models A \Rightarrow y \models B) \), and similarly for \( \neg A \).

Example
This model is a counter-example for the formula \( \neg \neg p \lor (\neg \neg p \rightarrow p) \). It is simultaneously a counter-example for \( \neg \neg p \rightarrow p \), for \( p \lor \neg p \), and for \( \neg p \lor \neg \neg p \).

Prime nodes

Let, in IPL(→)(n), the atoms be \( p_1, \ldots, p_n \).

Definition
A node \( a \) of a Kripke model is prime if one of the atoms \( p_1, \ldots, p_n \) is not satisfied in \( a \) but is satisfied in all successors of \( a \).

Lemma 1
If \( a \) is not prime and \( B \) is satisfied in all successors of \( a \) then \( a \models B \).

Lemma 2
If \( a \) is not prime and \( B \) is satisfied in all prime \( b \)'s accessible from \( a \) then \( a \models B \).

Theorem
If a purely implicational formula built up from \( p_1, \ldots, p_n \) has a counter-example, then it has a counter-example consisting of prime nodes only.

Model for atoms \( p \) and \( q \), and definable sets in it

Example
Theorem
Every definable set containing 1 and 2 also contains 3 or 4. The sets \( \emptyset, \{5\}, \{1,2\}, \{1,2,5\} \) are not definable. As the following figure shows, all of the remaining 14 sets are definable.
Formulas built up from \( p \) and \( q \)

Some history of the two atoms case

Let \( \mathcal{H}_n \) be the structure of purely implicational formulas built from \( n \) atoms. Let \( \mathcal{J}_n \) be the structure of formulas built from \( n \) atoms using \( \rightarrow \) and \&. Then

- The method of prime nodes is elaborated in Blicha, 2010.
- The structures \( \mathcal{H}_2 \) and \( \mathcal{J}_2 \) are given in Kostrzycka, 2003.
- The fact that \( |\mathcal{H}_2| = 14 \) is in Hirokawa, 1995.
- The structure \( \mathcal{J}_2 \) appears in Krzystek, 1977.
- Urquhart, 1974 attributes the fact that all \( \mathcal{H}_n \) are finite to Diego, and gives some upper and lower bounds on \( |\mathcal{H}_n| \).

Polish way of depicting the formulas built from two atoms

As it appears in papers by P. Krzystek and Z. Kostrzycka

The model for two atoms and \( \bot \)

(a) If 0 and not 3, then 2 and 4.
The claims (a)–(f) again, shown only on the handout
(a) If 0 and not 3, then 2 and 4.
(b) If 1 and not 3, then 8.
(c) If 0, 1 and not 3, then 6 and 8.
(d) If 0 and none of 1, 2, then 3–5, 10, 11.
(e) If 3 and not 1, then 10.
(f) If 0–4 and none of 6, 8, 10, 12, then 7, 9, 11, 13.

**Theorem**
There exists exactly 518 non-equivalent formulas built up from \( p, q, \perp \).

**Proof**
The claims (a)–(f) allow only 518 formulas, and an sql script has generated that number of them.

Urquhart mentions Diego’s estimate \( 10^{27} \) for the number \(|\mathcal{H}_3|\) of non-equivalent formulas, and improves it as follows:

\[ 2^{23} < |\mathcal{H}_3| < 3 \cdot 2^{23}. \]

The universal model has 61 nodes.

The lower bound can further be improved: \( 10,684,394 \leq |\mathcal{H}_3| \).

Krzystek, 1977 found the cardinality of \( \mathcal{J}_3 \):

\[ |\mathcal{J}_3| = 623,662,965,552,330. \]
Appendix: IPL, CPL, and complexity classes

The diagram illustrates the relationships between different complexity classes, including IPL, CPL, PSPACE-complete, NP, coNP, and P. IPL is shown to be PSPACE-complete, which is a significant result in computational complexity theory.