On Purely Implicational Fragments of Intuitionistic Propositional Logic

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Outline

Fragments of IPL

are obtained from IPL by restricting the number of propositional atoms and/or the set of logical connectives.

Known facts

(a) (Statman, 1979) IPL is *PSPACE*-complete.
(b) $\mathrm{IPL}^{\{\rightarrow\}}$, the purely implicational fragment of IPL, is *PSPACE*-complete.
(c) (Rybakov, 2006) IPL(2), the fragment of IPL with two atoms only, is *PSPACE*-complete.
(d) (Rieger, 1949) IPL(1) is still infinite, but efficiently decidable (decidable in polynomial time).

Motivation

What happens if there are only two atoms and $\rightarrow$ is the only connective? I.e., how does $\mathrm{IPL}^{\{\rightarrow\}}(2)$ look like?

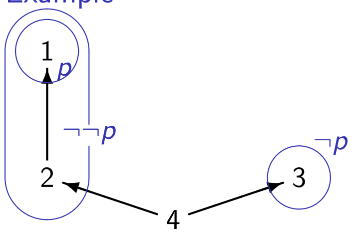
Kripke semantics for IPL

Definition

A Kripke model for intuitionistic logic is a triple $K = \langle W, \leq, \Vdash \rangle$ where $\leq$ is a transitive, reflexive, and weakly antisymmetric relation on the set $W \neq \emptyset$, and the relation $\Vdash$ satisfies:

– if $x \Vdash A$ and $x \leq y$ then $y \Vdash A$,
– $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$, and similarly for $A \,\&\, B$,
– $x \Vdash A \rightarrow B$ iff $\forall y {\geq} x (y \Vdash A \Rightarrow y \Vdash B)$, and similarly for $\neg A$.

Example



This model is a counter-example for the formula $\neg\neg p \vee (\neg\neg p \rightarrow p)$. It is simultaneously a counter-example for $\neg\neg p \rightarrow p$, for $p \vee \neg p$, and for $\neg p \vee \neg\neg p$.

Prime nodes

Let, in $\mathrm{IPL}^{\{\rightarrow\}}(n)$, the atoms be $p_1, \ldots, p_n$.

Definition

A node a of a Kripke model is prime if one of the atoms $p_1, \ldots, p_n$ is not satisfied in a but is satisfied in all successors of a.

Lemma 1

If a is not prime and B is satisfied in all successors of a then $a \Vdash B$.

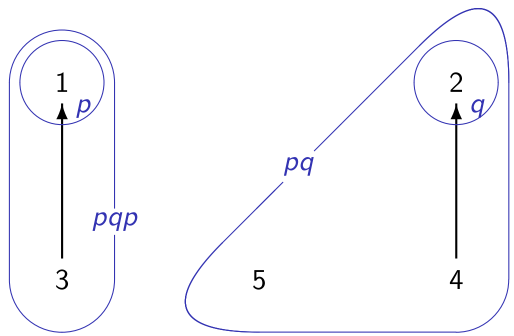
Lemma 2

If a is not prime and B is satisfied in all prime b's accessible from a then $a \Vdash B$.

Theorem

If a purely implicational formula built up from $p_1, \ldots, p_n$ has a counter-example, then it has a counter-example consisting of prime nodes only.

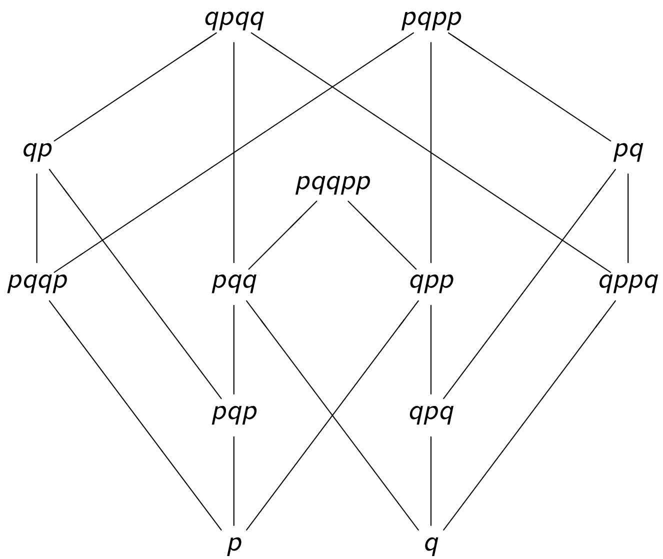
Model for atoms p and q, and definable sets in it



Theorem

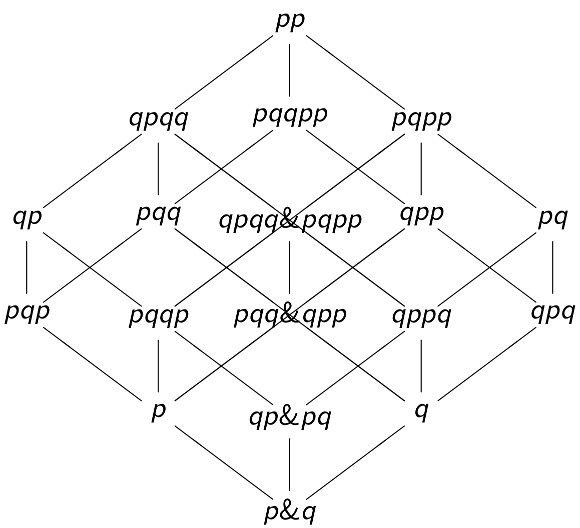
Every definable set containing 1 and 2 also contains 3 or 4. The sets $\emptyset$, $\{5\}$, $\{1,2\}$, $\{1,2,5\}$ are not definable. As the following figure shows, all of the remaining 14 sets are definable.

Formulas built up from p and q



Polish way of depicting the formulas built from two atoms

As it appears in papers by P. Krzystek and Z. Kostrzycka

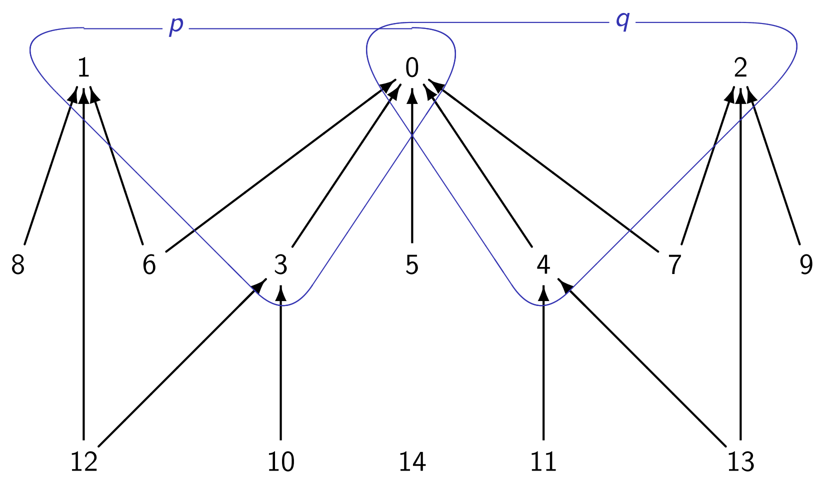


Some history of the two atoms case

Let $\mathcal{H}_n$ be the structure of purely implicational formulas built from n atoms. Let $\mathcal{J}_n$ be the structure of formulas built from n atoms using $\to$ and &. Then

- The method of prime nodes is elaborated in Blicha, 2010.
- The structures $\mathcal{H}_2$ and $\mathcal{J}_2$ are given in Kostrzycka, 2003.
- The fact that $|\mathcal{H}_2| = 14$ is in Hirokawa, 1995.
- The structure $\mathcal{J}_2$ appears in Krzystek, 1977.
- Urquhart, 1974 attributes the fact that all $\mathcal{H}_n$ are finite to Diego, and gives some upper and lower bounds on $|\mathcal{H}_n|$.

The model for two atoms and $\bot$



Theorem

(a) If 0 and not 3, then 2 and 4.

Formulas built up from p, q, $\bot$

The claims (a)–(f) again, shown only on the handout

(a) If 0 and not 3, then 2 and 4.
(b) If 1 and not 3, then 8.
(c) If $0, 1$ and not 3, then 6 and 8.
(d) If 0 and none of $1, 2$, then 3–5, 10, 11.
(e) If 3 and not 1, then 10.
(f) If 0–4 and none of $6, 8, 10, 12$, then $7, 9, 11, 13$.

Theorem

There exists *exactly* 518 non-equivalent formulas built up from p, q, and $\bot$.

Proof

The claims (a)–(f) allow only 518 formulas, and an sql script has generated that number of them.

Three atoms p, q, r

Urquhart mentions Diego's estimate 10^{27} for the number $|\mathcal{H}_3|$ of non-equivalent formulas, and improves it as follows:
$2^{23} < |\mathcal{H}_3| < 3 \cdot 2^{23}$.

The universal model has 61 nodes.

The lower bound can further be improved: $10\,684\,394 \leq |\mathcal{H}_3|$.

Krzystek, 1977 found the cardinality of $\mathcal{J}_3$:
$|\mathcal{J}_3| = 623\,662\,965\,552\,330$.

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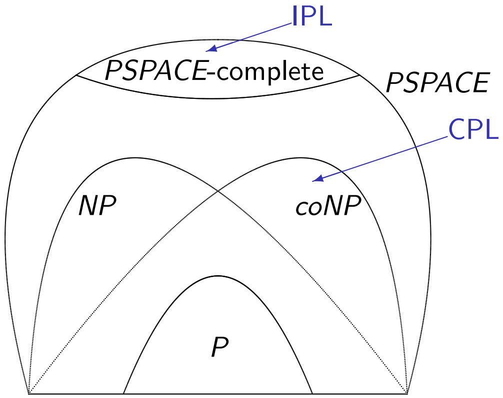
Appendix: IPL, CPL, and complexity classes



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