On Purely Implicational Fragments of Intuitionistic Propositional Logic

Vítězslav Švejdar

Dept. of Logic, College of Arts, Charles University,
http://www.cuni.cz/~svejdar/

Logica 2011
Hejnice, June 2011

Outline

Introduction: IPL, its fragments, algorithmical complexity

Kripke semantics, prime nodes, and the two atoms case

More than two atoms, and computer aided research

Fragments of IPL

are obtained from IPL by restricting the number of propositional atoms and/or the set of logical connectives.

Known facts

(a) (Statman, 1979) IPL is *PSPACE*-complete.
(b) $\mathrm{IPL}^{\{\to\}}$, the purely implicational fragment of IPL, is *PSPACE*-complete.
(c) (Rybakov, 2006) IPL(2), the fragment of IPL with two atoms only, is *PSPACE*-complete.
(d) (Rieger, 1949) IPL(1) is still infinite, but efficiently decidable (decidable in polynomial time).

Motivation

What happens if there are only two atoms and $\to$ is the only connective? I.e., how does $\mathrm{IPL}^{\{\to\}}(2)$ look like?

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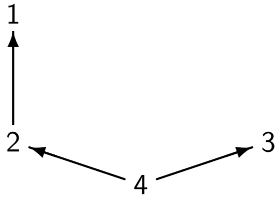
Kripke semantics for IPL

Definition

A Kripke model for intuitionistic logic is a triple $K = \langle W, \leq, \Vdash \rangle$ where $\leq$ is a transitive, reflexive, and weakly antisymmetric relation on the set $W \neq \emptyset$, and the relation $\Vdash$ satisfies:

- if $x \Vdash A$ and $x \leq y$ then $y \Vdash A$,
- $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$, and similarly for $A \,\&\, B$,
- $x \Vdash A \rightarrow B$ iff $\forall y {\geq} x (y \Vdash A \Rightarrow y \Vdash B)$, and similarly for $\neg A$.

Example



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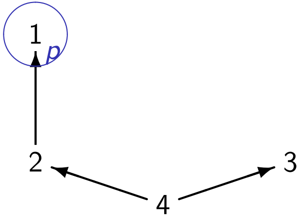

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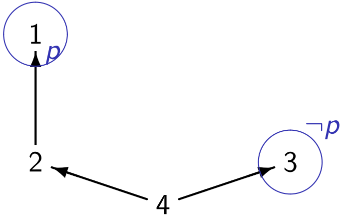

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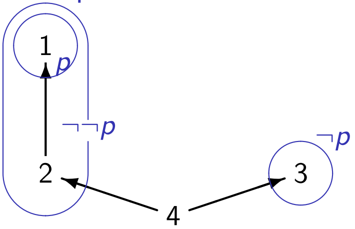

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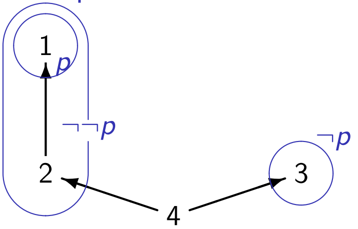

This model is a counter-example for the formula $\neg\neg p \vee (\neg\neg p \rightarrow p)$. It is simultaneously a counter-example for $\neg\neg p \rightarrow p$, for $p \vee \neg p$, and for $\neg p \vee \neg\neg p$.

Prime nodes

Let, in $\mathrm{IPL}^{\{\rightarrow\}}(n)$, the atoms be $p_1, \ldots, p_n$.

Definition

A node a of a Kripke model is prime if one of the atoms $p_1, \ldots, p_n$ is not satisfied in a but is satisfied in all successors of a.

Lemma 1

If a is not prime and B is satisfied in all successors of a then $a \Vdash B$.

Lemma 2

If a is not prime and B is satisfied in all prime b's accessible from a then $a \Vdash B$.

Theorem

If a purely implicational formula built up from $p_1, \ldots, p_n$ has a counter-example, then it has a counter-example consisting of prime nodes only.

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Model for atoms p and q, and definable sets in it

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Every definable set containing 1 and 2 also contains 3 or 4. The sets $\emptyset$, $\{5\}$, $\{1,2\}$, $\{1,2,5\}$ are not definable. As the following figure shows, all of the remaining 14 sets are definable.

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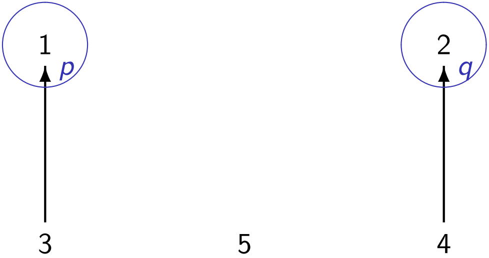

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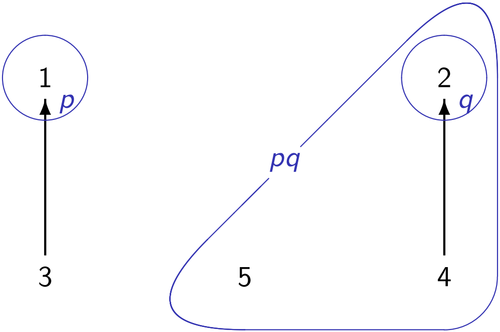

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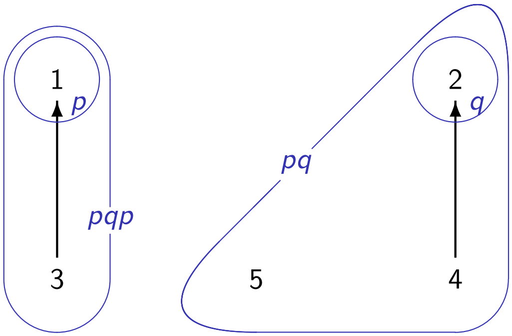

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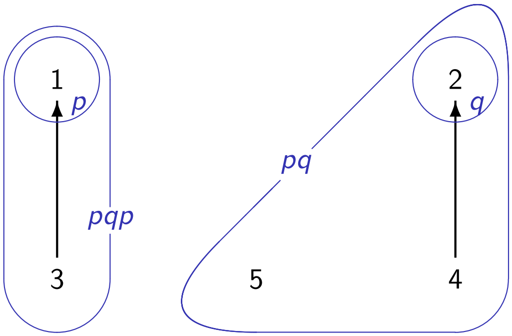

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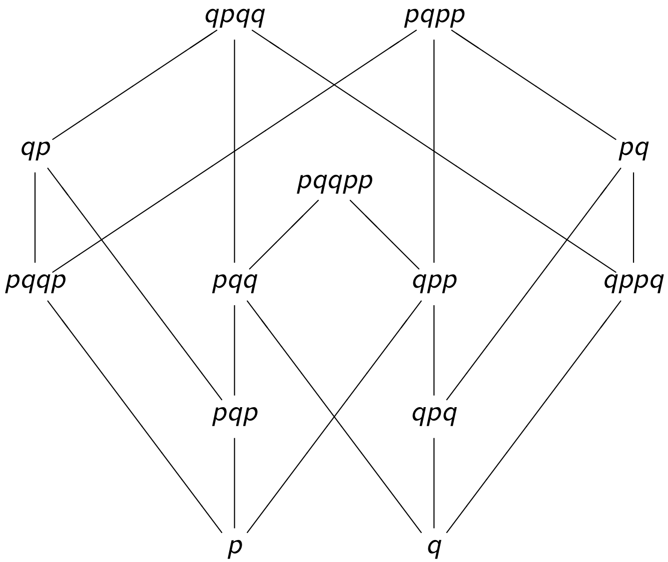
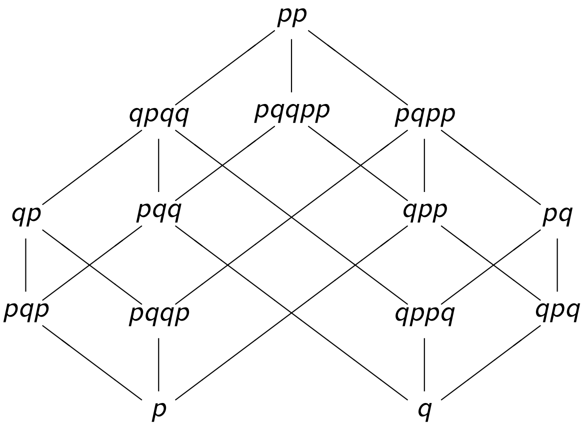
Formulas built up from p and q



Polish way of depicting the formulas built from two atoms

As it appears in papers by P. Krzystek and Z. Kostrzycka



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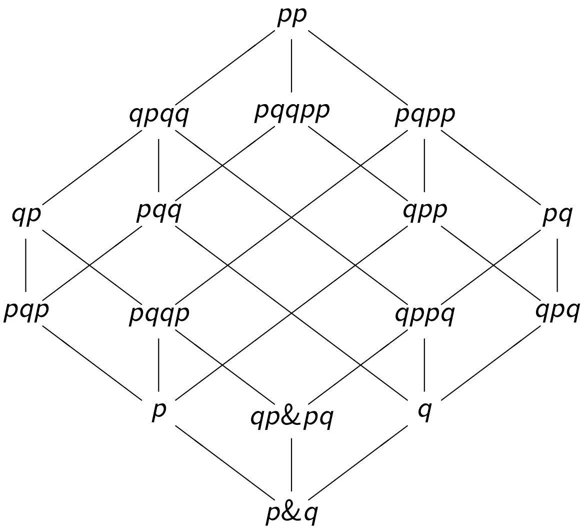

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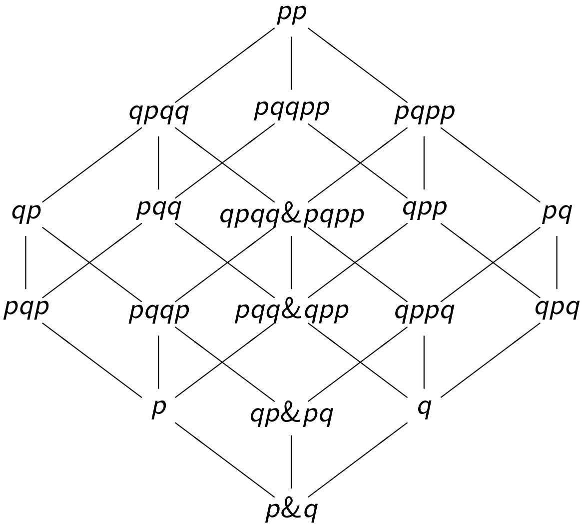

Some history of the two atoms case

Let $\mathcal{H}_n$ be the structure of purely implicational formulas built from n atoms. Let $\mathcal{J}_n$ be the structure of formulas built from n atoms using $\rightarrow$ and &. Then

- The method of prime nodes is elaborated in Blicha, 2010.
- The structures $\mathcal{H}_2$ and $\mathcal{J}_2$ are given in Kostrzycka, 2003.
- The fact that $|\mathcal{H}_2| = 14$ is in Hirokawa, 1995.
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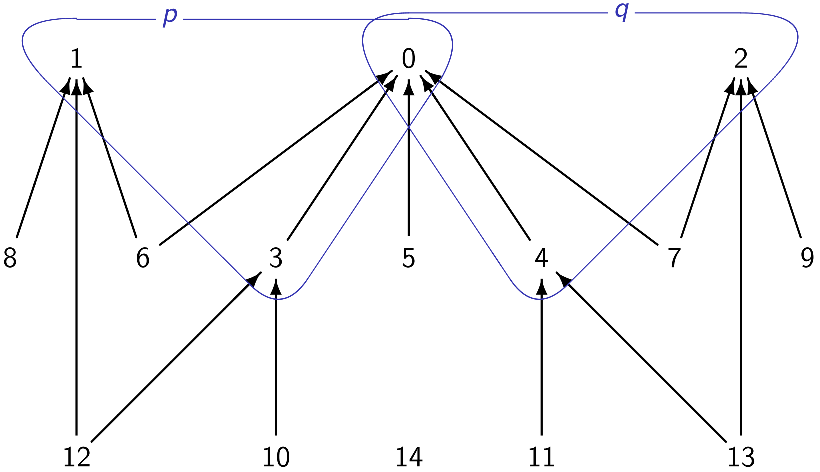
The model for two atoms and ⊥



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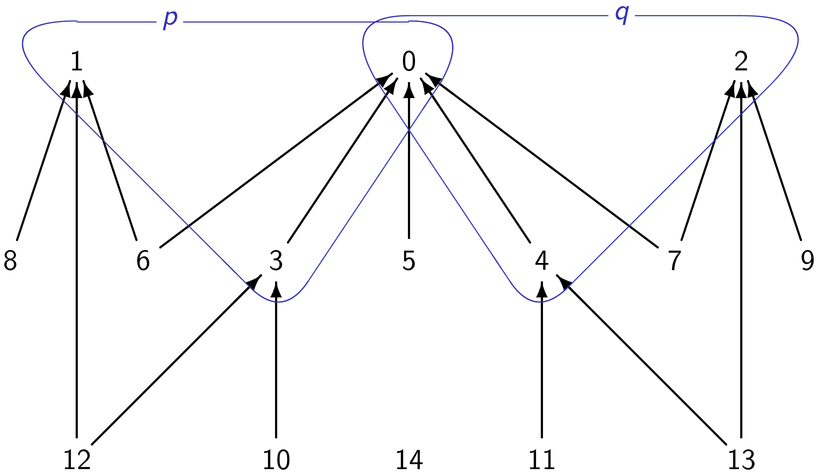
The model for two atoms and $\bot$



Theorem

(a) If 0 and not 3, then 2 and 4.

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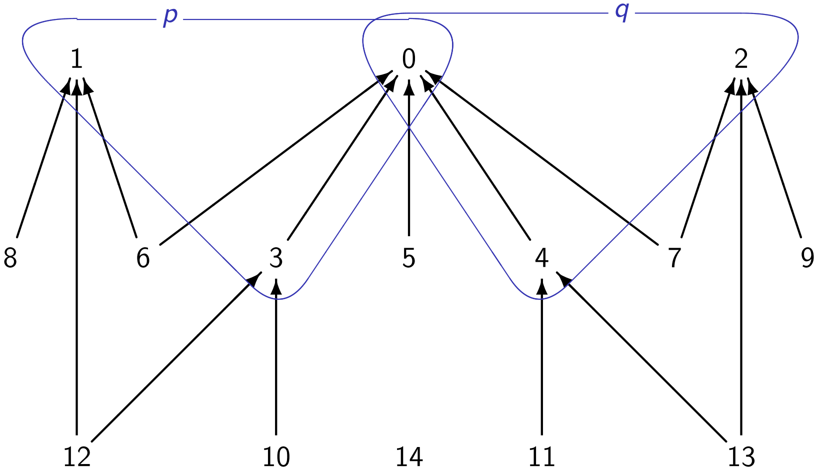

Theorem

(b) If 1 and not 3, then 8.

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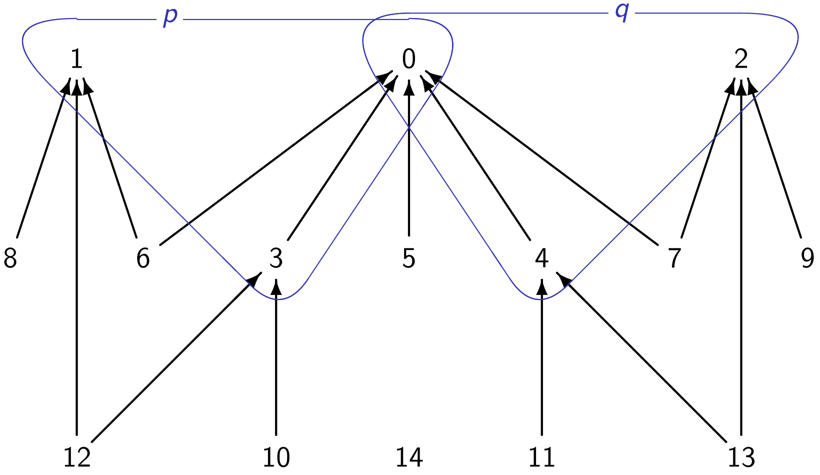

Theorem

(c) If $0, 1$ and not 3, then 6 and 8.

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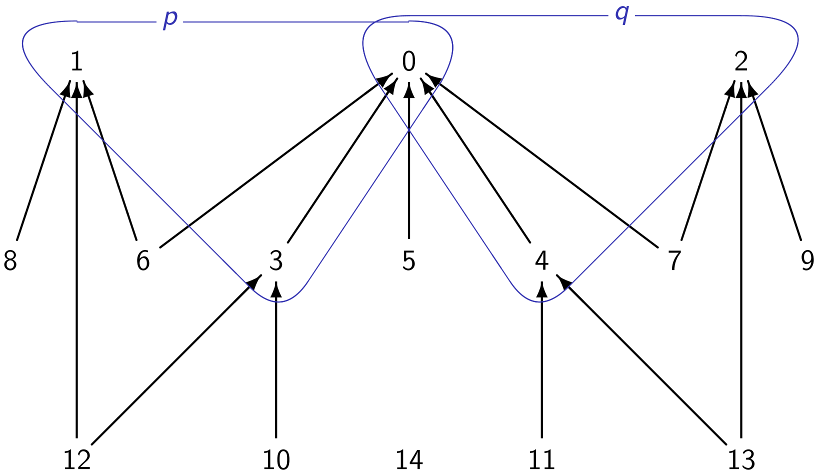

Theorem

(d) If 0 and none of 1, 2, then 3–5, 10, 11.

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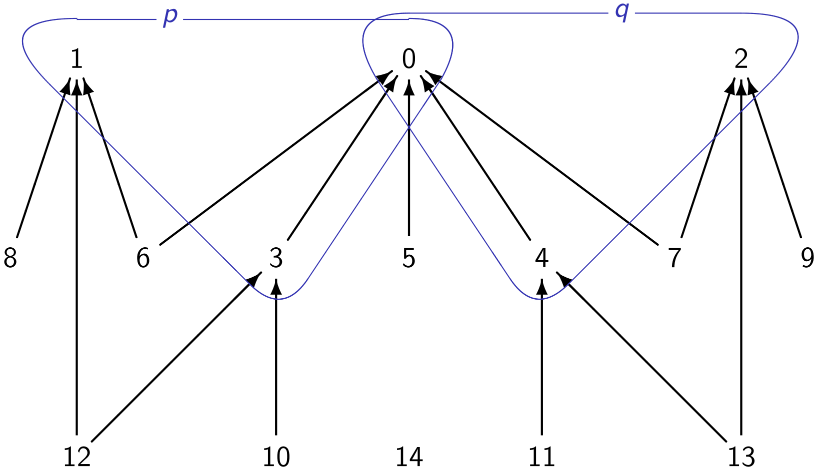

Theorem

(e) If 3 and not 1, then 10.

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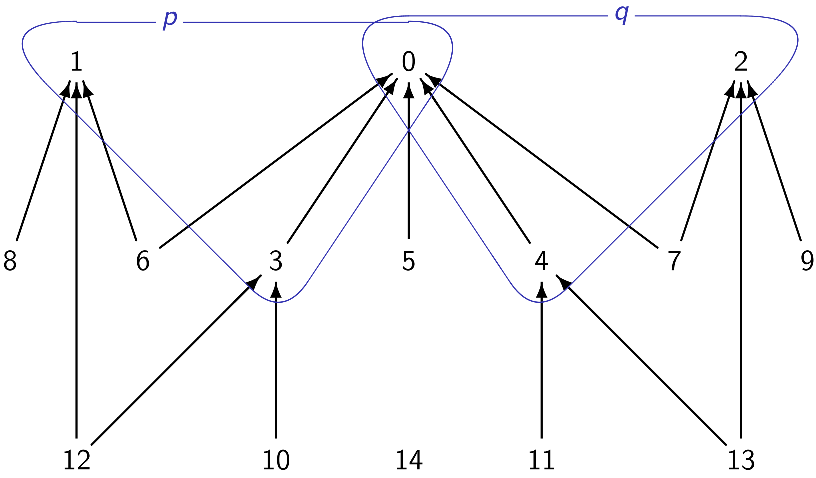

Theorem

(f) If 0–4 and none of $6, 8, 10, 12$, then $7, 9, 11, 13$.

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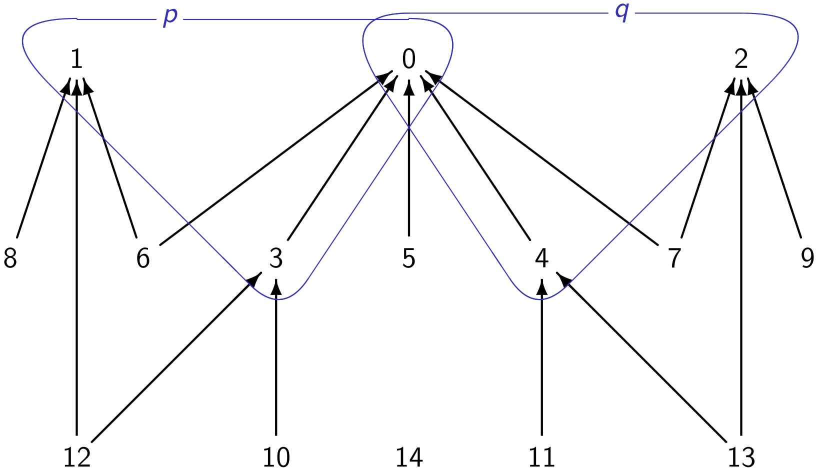
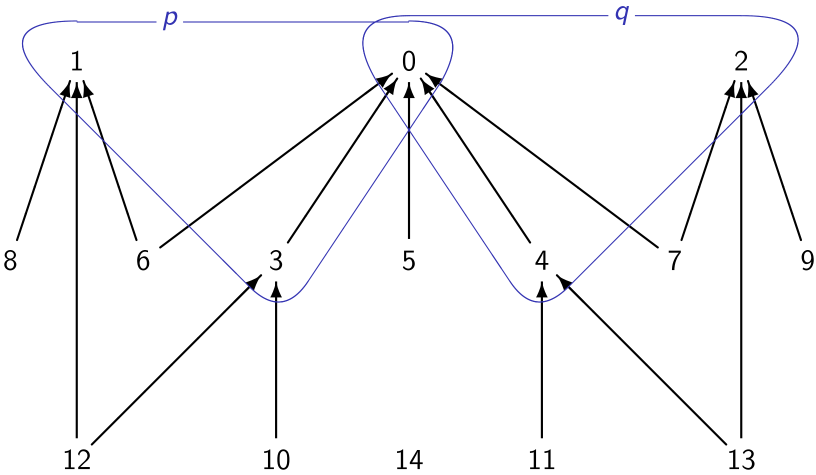


Theorem

Claims (a)–(f) allow at most 259 subsets of $\{0, \ldots, 13\}$.

The model for two atoms and $\bot$



Theorem

Thus there are *at most* 518 non-equivalent formulas.

Formulas built up from p, q, $\perp$

Formulas built up from p, q, $\bot$

Theorem

There exists *exactly* 518 non-equivalent formulas built up from p, q, and $\bot$.

Three atoms p, q, r

Urquhart mentions Diego's estimate 10^{27} for the number $|\mathcal{H}_3|$ of non-equivalent formulas, and improves it as follows:
$2^{23} < |\mathcal{H}_3| < 3 \cdot 2^{23}$.

The universal model has 61 nodes.

The lower bound can further be improved: $10\,684\,394 \leq |\mathcal{H}_3|$.

Krzystek, 1977 found the cardinality of $\mathcal{J}_3$:
$|\mathcal{J}_3| = 623\,662\,965\,552\,330$.

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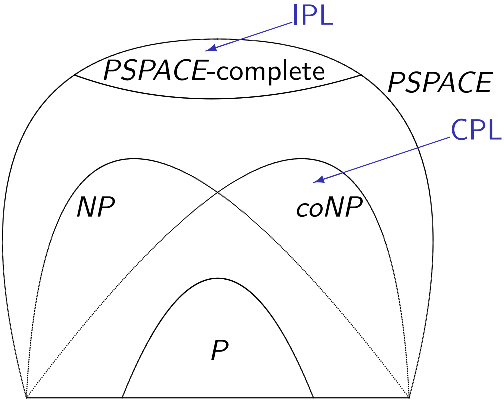
Appendix: IPL, CPL, and complexity classes



Back to Introduction