On Strong Fragments of Peano Arithmetic

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Outline

Axiom schemas

Peano arithmetic PA is an axiomatic theory formulated in the arithmetical language, containing the symbols $+$, and $\cdot$, 0, S (plus possibly $\leq$ and $<$, but *no* such thing like *exponentiation*). It has seven (nine) simple axioms like $\forall x \forall y (x + \mathrm{S}(y) = \mathrm{S}(x + y))$, plus an axiom schema (possibly with parameters):

Ind: $\varphi(0) \mathbin{\&} \forall x(\varphi(x) \to \varphi(\mathrm{S}(x))) \to \forall z \varphi(z)$,

CoV: $\forall x(\forall v{<}x\, \varphi(v) \to \varphi(x)) \to \forall z \varphi(z)$,

LNP: $\exists z \varphi(z) \to \exists x(\varphi(x) \mathbin{\&} \forall v{<}x \neg\varphi(v))$,

Note that LNP is the contraposition of CoV and vice versa, Ind follows from CoV, while Ind applied on $\forall v{<}x\, \varphi(v)$ yields CoV.

Example

To show $\forall x \forall y (\exists v{\leq}y\, (v + x = y) \vee \exists v{\leq}x\, (v + y = x))$, one can either apply induction on $\forall y(\ldots \vee \ldots)$, or think of y as parameter and apply induction on $\exists v{\leq}y\, (v + x = y) \vee \exists v{\leq}x\, (v + y = x)$, where y is fixed.

What happens if all quantifiers are bounded

Definition

Bounded quantifers are quantifiers of the form $\forall v{\leq}x$, $\forall v{<}x$, $\exists v{\leq}x$, $\exists v{<}x$. A formula is *bounded*, or Δ_0, if all quantifiers in it are bounded. $\mathrm{I}\Delta_0$ is a theory like PA, but with the induction schema restricted to Δ_0 formulas.

$\mathrm{I}\Delta_0$ can prove

properties of operations; properties of the divisibility relation including the fact that a number is prime iff it is irreducible; properties of the exponential function $x \mapsto 2^x$, but *not its totality*.

$\mathrm{I}\Delta_0$ cannot prove

$\mathrm{I}\Delta_0 \nvdash \forall x \exists w{\neq}0 \forall v{\leq}x\, (v \neq 0 \to v \mid x)$,

$\mathrm{I}\Delta_0 \nvdash$ there exist infinitely many primes (?),

$\mathrm{I}\Delta_0 \nvdash \forall x \exists y (y = 2^x)$.

Base theory: $\mathrm{I}\Delta_0{+}\mathrm{Exp}$ is $\mathrm{I}\Delta_0$ plus the axiom $\forall x \exists y (y = 2^x)$.

Strong fragments of PA

Definition

A Σ_n formula (Π_n formula) is a formula having a prefix of n alternating quantifiers, the first of which is $\exists$ (or $\forall$, respectively), followed by a Δ_0 formula. $\mathrm{I}\Sigma_n$ is $\mathrm{I}\Delta_0{+}\mathrm{Exp}$ plus $\mathrm{Ind}(\Sigma_n)$, the induction schema restricted to Σ_n formulas.

Basic facts

$\mathrm{I}\Sigma_n$ is a stable theory: the schemas $\mathrm{Ind}(\Sigma_n)$, $\mathrm{CoV}(\Sigma_n)$, $\mathrm{LNP}(\Sigma_n)$, $\mathrm{Ind}(\Pi_n)$, $\mathrm{CoV}(\Pi_n)$, $\mathrm{LNP}(\Pi_n)$ are equivalent over $\mathrm{I}\Delta_0{+}\mathrm{Exp}$. The hierarchy of theories $\mathrm{I}\Sigma_1 \subseteq \mathrm{I}\Sigma_2 \subseteq \ldots$ does not collapse. Each $\mathrm{I}\Sigma_n$ for $n \geq 1$ is finitely axiomatizable.

The collection schema (bounding schema)

Coll: $\forall u{<}z \exists v \varphi(u, v, z) \to \exists w \forall u{<}z \exists v{<}w\, \varphi(u, v, z)$.

An unbounded relation

⋮		⋮		⋮		⋮		⋮		⋮		⋮	
0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0

0 ... ... z

The collection schema

The schema $\mathrm{Coll}(\Gamma)$ prevents the existence of an unbounded Γ relation R with $\mathrm{Dom}(R) = \{0, 1, .., z-1\}$.

Definition

$\mathrm{B}\Sigma_n$ and $\mathrm{B}\Pi_n$ are the theories obtained by adding the schema $\mathrm{Coll}(\Sigma_n)$, or $\mathrm{Coll}(\Pi_n)$ respectively, to $\mathrm{I}\Delta_0{+}\mathrm{Exp}$.

Basic facts

The schemas $\mathrm{Coll}(\Sigma_{n+1})$ and $\mathrm{Coll}(\Pi_n)$ are equivalent over $\mathrm{I}\Delta_0{+}\mathrm{Exp}$. Thus $\mathrm{I}\Delta_0{+}\mathrm{Exp} \subseteq \mathrm{B}\Pi_0 \Leftrightarrow \mathrm{B}\Sigma_1 \subseteq \mathrm{I}\Sigma_1 \subseteq \mathrm{B}\Pi_1 \subseteq \dots$ If there exists an unbounded Π_n relation on $\{0, 1, .., z-1\}$, then there exists an unbounded Π_n *function* on $\{0, 1, .., z-1\}$. $\mathrm{B}\Pi_n$ is also equivalent to $\mathrm{PHP}(\Sigma_{n+1})$, saying that no Σ_{n+1} function can be a one-one mapping from $\{0, 1, .., z\}$ to $\{0, 1, .., z-1\}$.

Question

What about $\mathrm{WPHP}(\Sigma_{n+1})$, saying that there can be no Σ_{n+1} one-one mapping from the entire universe to $\{0, 1, .., z-1\}$?

Weak pigeon hole principle

$\mathrm{WPHP}(\Sigma_{n+1})$: a one-one Σ_{n+1} function must be unbounded. Obviously $\mathrm{B}\Sigma_{n+1} \vdash \mathrm{WPHP}(\Sigma_{n+1})$.

Theorem

$\mathrm{I}\Sigma_n \nvdash \mathrm{WPHP}(\Sigma_{n+1})$.

Proof

Can be extracted from a proof of $\mathrm{I}\Sigma_n \nvdash \mathrm{Coll}(\Sigma_{n+1})$ in Paris-Kirby.

Theorem (unfinished)

$\mathrm{I}\Sigma_n{+}\mathrm{WPHP}(\Sigma_{n+1}) \nvdash \mathrm{Coll}(\Sigma_{n+1})$.

Theorem (Paris)

If there exists a one-one Σ_{n+1} function bounded by z then there exists a one-one Σ_{n+1} function f with $\mathrm{Rng}(f) = \{0, .., z-1\}$.

Proof via sparse relations

If there exists a one-one Σ_{n+1} function bounded by z then there exists a Π_n relation R which is sparse in the following sense: $\forall x \exists y R(x, y)$, but $\forall x(|R \cap (\{0, .., x-1\} \times \{0, .., x-1\})| < z)$.

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
	9	4		5	6							

The numbers 2, 4, 5, 1 appear in stages 4, 5, 6, and 9 respectively.

Proof (continuation)

The function f which is one-one and with $\mathrm{Rng}(f) = \{0, .., z-1\}$ is obtained as a union of $f_0 \subseteq f_1 \subseteq \dots$ The functions f_i are constructed by recursion, $f_0 = \emptyset$. Each f_i is one-one and finite, and $\mathrm{Rng}(f_i)$ is a proper subset of $\{0, .., z-1\}$.

- If no number appears in stage $i+1$ then $f_{i+1} = f_i$.
- If $x \geq z$ appears in stage $i+1$ then $f_{i+1} = f_i \cup [x, y]$ where $y = \min(\{0, .., z-1\} - \mathrm{Rng}(f_i))$.
- If $x < z$ appears in stage $i+1$ and $x \in \mathrm{Rng}(f_i)$ then $f_{i+1} = f_i \cup [x, y]$ where $y = \min(\{0, .., z-1\} - \mathrm{Rng}(f_i))$.
- If $x < z$ appears in stage $i+1$ and $x \notin \mathrm{Rng}(f_i)$ then $f_{i+1} = f_i \cup [x, x]$.