On Strong Fragments of Peano Arithmetic

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Outline

Introduction: Peano arithmetic and the induction schemas

The hierarchy of strong fragments of Peano arithmetic

The collection schema

Weak pigeon hole principle
Axiom schemas

Peano arithmetic \( PA \) is an axiomatic theory formulated in the arithmetical language, containing the symbols \( +, \cdot \), 0, \( S \) (plus possibly \( \leq \) and \( < \), but \( no \) such thing like \textit{exponentiation}).
Axiom schemas

Peano arithmetic PA is an axiomatic theory formulated in the arithmetical language, containing the symbols $+$, and $\cdot$, 0, $\mathcal{S}$ (plus possibly $\leq$ and $<$, but no such thing like exponentiation). It has seven (nine) simple axioms like $\forall x \forall y (x + S(y) = S(x + y))$, plus an axiom schema:
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**Ind:** $\varphi(0) \& \forall x (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall z \varphi(z)$,

**CoV:** $\forall x (\forall v < x \varphi(v) \rightarrow \varphi(x)) \rightarrow \forall z \varphi(z)$. 

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Note that LNP is the contraposition of CoV and vice versa, Ind follows from CoV, while Ind applied on $\forall v < x \varphi(v)$ yields CoV.
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**Example**

To show $\forall x \forall y (\exists v \leq y (v + x = y) \lor \exists v \leq x (v + y = x))$, one can either apply induction on $\forall y (\ldots \lor \ldots)$, or think of $y$ as parameter and apply induction on $\exists v \leq y (v + x = y) \lor \exists v \leq x (v + y = x)$, where $y$ is fixed.
What happens if all quantifiers are bounded

**Definition**

*Bounded quantifiers* are quantifiers of the form $\forall v \leq x$, $\forall v < x$, $\exists v \leq x$, $\exists v < x$. 
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properties of operations; properties of the divisibility relation including the fact that a number is prime iff it is irreducible; properties of the exponential function $x \mapsto 2^x$, but not its totality.
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$I\Delta_0 \not\vdash \forall x \exists w \neq 0 \forall v \leq x (v \neq 0 \rightarrow v \mid x)$,
$I\Delta_0 \not\vdash \text{there exist infinitely many primes}$,
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**Base theory:** $I\Delta_0 + \text{Exp}$ is $I\Delta_0$ plus the axiom $\forall x \exists y (y = 2^x)$. 
Strong fragments of PA

Definition
A $\Sigma_n$ formula ($\Pi_n$ formula) is a formula having a prefix of $n$ alternating quantifiers, the first of which is $\exists$ (or $\forall$, respectively), followed by a $\Delta_0$ formula.
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Basic facts
\( \Sigma_n \) is a stable theory: the schemas \( \text{Ind}(\Sigma_n) \), \( \text{CoV}(\Sigma_n) \), \( \text{LNP}(\Sigma_n) \), \( \text{Ind}(\Pi_n) \), \( \text{CoV}(\Pi_n) \), \( \text{LNP}(\Pi_n) \) are equivalent over \( \text{I} \Delta_0 + \text{Exp} \).
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The collection schema (bounding schema)
Coll: $\forall u < z \exists v \varphi(u, v, z) \rightarrow \exists w \forall u < z \exists v < w \varphi(u, v, z)$. 
An unbounded relation

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An unbounded relation

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The collection schema

The schema $\text{Coll}(\Gamma)$ prevents the existence of an unbounded $\Gamma$ relation $R$ with $\text{Dom}(R) = \{0, 1, \ldots, z - 1\}$.

Definition

$\mathsf{B}\Sigma_n$ and $\mathsf{B}\Pi_n$ are the theories obtained by adding the schema $\text{Coll}(\Sigma_n)$, or $\text{Coll}(\Pi_n)$ respectively, to $\text{I} \Delta_0 + \text{Exp}$. 
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The schema Coll(Γ) prevents the existence of an unbounded Γ relation \( R \) with \( \text{Dom}(R) = \{0, 1, \ldots, z - 1\} \).

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The schemas Coll(\( \Sigma_{n+1} \)) and Coll(\( \Pi_n \)) are equivalent over \( I\Delta_0 + \text{Exp} \). Thus \( I\Delta_0 + \text{Exp} \subseteq B\Pi_0 \iff B\Sigma_1 \subseteq I\Sigma_1 \subseteq B\Pi_1 \subseteq \ldots \).
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Question

What about WPHP($\Sigma_{n+1}$), saying that there can be no $\Sigma_{n+1}$ one-one mapping from the entire universe to $\{0, 1, \ldots, z - 1\}$?
Weak pigeon hole principle

\[ \text{WPHP}(\Sigma_{n+1}): \text{a one-one } \Sigma_{n+1} \text{ function must be unbounded.} \]

Obviously \( \mathcal{B}\Sigma_{n+1} \vdash \text{WPHP}(\Sigma_{n+1}). \)
Weak pigeon hole principle

\( \text{WPHP}(\Sigma_{n+1}) \): a one-one \( \Sigma_{n+1} \) function must be unbounded. Obviously \( \text{B}_{\Sigma_{n+1}} \vdash \text{WPHP}(\Sigma_{n+1}) \).

**Theorem**
\[ \text{I}_{\Sigma_n} \nvdash \text{WPHP}(\Sigma_{n+1}). \]
Weak pigeon hole principle

$\text{WPHP}(\Sigma_{n+1})$: a one-one $\Sigma_{n+1}$ function must be unbounded. Obviously $\text{B}\Sigma_{n+1} \vdash \text{WPHP}(\Sigma_{n+1})$.

**Theorem**
$I\Sigma_n \not\vdash \text{WPHP}(\Sigma_{n+1})$.

**Proof**
Can be extracted from a proof of $I\Sigma_n \not\vdash \text{Coll}(\Sigma_{n+1})$ in Paris-Kirby.
Weak pigeon hole principle

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**Theorem (unfinished)**
\(I\Sigma_n + \text{WPHP}(\Sigma_{n+1}) \not\vdash \text{Coll}(\Sigma_{n+1}).\)
**Weak pigeon hole principle**

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Obviously \( B\Sigma_{n+1} \vdash \text{WPHP}(\Sigma_{n+1}). \)

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**Theorem (unfinished)**

\( I\Sigma_n + \text{WPHP}(\Sigma_{n+1}) \not\vdash \text{Coll}(\Sigma_{n+1}). \)

**Theorem (Paris)**

If there exists a one-one \( \Sigma_{n+1} \) function bounded by \( z \) then there exists a one-one \( \Sigma_{n+1} \) function \( f \) with \( \text{Rng}(f) = \{0, \ldots, z - 1\} \).
Proof via sparse relations

If there exists a one-one $\Sigma_{n+1}$ function bounded by $z$ then there exists a $\Pi_n$ relation $R$ which is sparse in the following sense:

$\forall x \exists y R(x, y)$, but $\forall x(|R \cap (\{0, \ldots, x - 1\} \times \{0, \ldots, x - 1\})| < z)$.
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Vitek Svejdar, Charles U. in Prague

On Strong Fragments of Peano Arithmetic

9/10
Proof via sparse relations

If there exists a one-one $\Sigma_{n+1}$ function bounded by $z$ then there exists a $\Pi_n$ relation $R$ which is sparse in the following sense:

$$\forall x \exists y R(x, y), \text{ but } \forall x (|R \cap (\{0, \ldots, x - 1\} \times \{0, \ldots, x - 1\})| < z).$$

The numbers 2, 4, 5, 1 appear in stages 4, 5, 6, and 9 respectively.
Proof (continuation)

The function $f$ which is one-one and with $\text{Rng}(f) = \{0, \ldots, z - 1\}$ is obtained as a union of $f_0 \subseteq f_1 \subseteq \ldots$.
Proof (continuation)

The function $f$ which is one-one and with $\text{Rng}(f) = \{0, \ldots, z - 1\}$ is obtained as a union of $f_0 \subseteq f_1 \subseteq \ldots$ The functions $f_i$ are constructed by recursion, $f_0 = \emptyset$. Each $f_i$ is one-one and finite, and $\text{Rng}(f_i)$ is a proper subset of $\{0, \ldots, z - 1\}$.
Proof (continuation)

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- If no number appears in stage $i + 1$ then $f_{i+1} = f_i$. 
Proof (continuation)

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- If no number appears in stage $i + 1$ then $f_{i+1} = f_i$.
- If $x \geq z$ appears in stage $i + 1$ then $f_{i+1} = f_i \cup [x, y]$ where $y = \min(\{0, \ldots, z - 1\} - \text{Rng}(f_i))$. 
Proof (continuation)

The function $f$ which is one-one and with $\text{Rng}(f) = \{0, \ldots, z - 1\}$ is obtained as a union of $f_0 \subseteq f_1 \subseteq \ldots$. The functions $f_i$ are constructed by recursion, $f_0 = \emptyset$. Each $f_i$ is one-one and finite, and $\text{Rng}(f_i)$ is a proper subset of $\{0, \ldots, z - 1\}$.

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- If $x < z$ appears in stage $i + 1$ and $x \in \text{Rng}(f_i)$ then $f_{i+1} = f_i \cup [x, y]$ where $y = \min(\{0, \ldots, z - 1\} - \text{Rng}(f_i))$. 
The function \( f \) which is one-one and with \( \text{Rng}(f) = \{0, \ldots, z - 1\} \) is obtained as a union of \( f_0 \subseteq f_1 \subseteq \ldots \). The functions \( f_i \) are constructed by recursion, \( f_0 = \emptyset \). Each \( f_i \) is one-one and finite, and \( \text{Rng}(f_i) \) is a proper subset of \( \{0, \ldots, z - 1\} \).

- If no number appears in stage \( i + 1 \) then \( f_{i+1} = f_i \).
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- If \( x < z \) appears in stage \( i + 1 \) and \( x \notin \text{Rng}(f_i) \) then \( f_{i+1} = f_i \cup [x, x] \).