On the Polynomial-Space Completeness of Intuitionistic Propositional Logic

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Abstract

We present an alternative, purely semantical and relatively simple, proof of the Statman's result that both intuitionistic propositional logic and its implicational fragment are *PSPACE*-complete.

1 Introduction

R. Ladner in his 1977 paper [Lad77] presented a polynomial-space decision procedures for the most common modal logics like S4 and T, and proved that the decision problems in question are *PSPACE*-complete. The proofs in [Lad77] are purely semantical in the sense that modal logics are defined via their Kripke semantics; no properties of logical calculi are exploited or even mentioned. Later Statman [Sta79] showed that the intuitionistic propositional logic, along with its implicational fragment where all logical connectives except implication \rightarrow are forbidden, also has a *PSPACE*-complete decision problem. The proofs in [Sta79] use proof-theoretic methods.

The purpose of this paper is to present proofs of Statman's results which are in the spirit of Ladner's [Lad77] and which may be found a little bit simpler than those in [Sta79]. We will not use any particular property of intuitionistic logic, like finite model property, soundness, completeness, or cut-eliminability. However, we will concentrate on only the construction of the reduction from a known *PSPACE*-complete problem, namely the QBF problem. The positive part of the story, i.e. the fact that intuitionistic propositional logic *is* decidable in *PSPACE*, we take as granted. It can be proved using methods from [Lad77].

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2 Preliminaries

Propositional formulas are built up from propositional atoms and the nulary symbol \perp for falsity using the usual binary connectives \rightarrow , &, and \vee . Formulas $\neg A$ and $A \equiv B$ are shorthands for $A \rightarrow \perp$ and $(A \rightarrow B) \& (B \rightarrow A)$ respectively. In syntax analysis, implication \rightarrow has lower priority than conjunction & and disjunction \vee , but higher than equivalence \equiv .

A Kripke frame for intuitionistic logic is a pair $\langle W, \leq \rangle$ where $W \neq \emptyset$ and \leq is a reflexive and transitive relation on W. The elements of W are nodes; if $a \leq b$ then the node b is said to be accessible from a. A relation \parallel - between nodes of a Kripke frame $\langle W, \leq \rangle$ and propositional formulas is a truth relation on $\langle W, \leq \rangle$ if, for any two nodes $a, b \in W$, any propositional atom p and any two propositional formulas A and B, it satisfies the following conditions:

- if $a \leq b$ and $a \parallel -p$ then $b \parallel -p$,
- $a \parallel \neq \perp, a \parallel -A \& B \text{ iff } a \parallel -A \text{ and } a \parallel -B, a \parallel -A \lor B \text{ iff } a \parallel -A \text{ or } a \parallel -B,$
- $a \parallel A \rightarrow B$ iff $\forall b \ge a(b \parallel A \Rightarrow b \parallel B)$.

A triple $\langle W, \leq, ||-\rangle$ where ||- is a truth relation on a Kripke frame $\langle W, \leq \rangle$ is called *Kripke model* for intuitionistic propositional logic. The first condition in the definition of truth relation is called *persistency condition*. A straightforward induction shows that the persistency condition holds for all formulas, not just for atoms. We read $a \parallel - A$ as "A is satisfied in a".

An example of a Kripke model is shown in Fig. 1. Its frame has three nodes a, b, and c, where b and c are accessible from a. We have $b \parallel -p$ and $c \parallel -q$. It is understood that p is not satisfied in a and c, that q is not satisfied in a and b, and that each of the nodes a, b, and c is accessible from itself. One can easily verify that $a \parallel \neq p \rightarrow q, a \parallel \neq q \rightarrow p$, and thus $a \parallel \neq (p \rightarrow q) \lor (q \rightarrow p)$.

A formula A is valid in a model $K = \langle W, \leq, ||-\rangle$ if it is satisfied in all nodes $a \in W$. A model K is a (Kripke) counter-example to a formula A if A is not valid in K. A formula A is an intuitionistic tautology if A is valid in all Kripke models, i.e. if A has no counter-example. The set of all intuitionistic tautologies is denoted INTTAUT. Since classical tautologies are exactly those formulas which are valid in all one-element Kripke models, we have INTTAUT \subseteq TAUT, where TAUT is the set of all classical tautologies. Examples



Figure 1: Example Kripke model for intuitionistic logic

of formulas in TAUT – INTTAUT are $(p \rightarrow q) \lor (q \rightarrow p), \neg \neg p \rightarrow p$, or $p \lor \neg p$. Examples of formulas in INTTAUT are all instances of schemes $A \rightarrow \neg \neg A, \neg \neg \neg A \rightarrow \neg A$, or $A \lor B \rightarrow D \equiv (A \rightarrow D) \& (B \rightarrow D)$.

We shall call the least element of a model K (if it exists) a root of K. If $K = \langle W, \leq, \|-\rangle$ and $a_0 \in W$ then submodel generated by a_0 is the model $K_0 = \langle W_0, \leq_0, \|-_0\rangle$ where $W_0 = \{x \in W ; a_0 \leq x\}$ and \leq_0 and $\|-_0$ are the restrictions of \leq and $\|-$ to W_0 . It can be easily shown that if A is a propositional formula and $a \in W_0$ then $a \parallel - A \Leftrightarrow a \parallel -_0 A$. So in the sequel we can assume that if K is a counter-example to A then K has a root a and that it is the root a where $a \parallel \neq A$.

More about Kripke models can be found in various sources, e.g. in [vD86], [dJV88], or in [Tak75]. For the notions from theoretical computer science, like QBF, I recommend e.g. [Pap94].

3 The reduction

A key step in *PSPACE*-completeness proofs in [Lad77] is the construction of a sequence of propositional formulas such that the size of the formulas grows only polynomially, all have Kripke counter-example, but the size of the minimal counter-example grows exponentially. One can easily check that if the formulas D_n are defined by $D_0 = \bot$, $D_{n+1} = (p_{n+1} \rightarrow D_n) \lor (\neg p_{n+1} \rightarrow D_n)$ then each formula D_n has a Kripke counter-example and that each counter-example to D_{n+1} contains two disjoint counter-examples to D_n : one in which p_{n+1} is everywhere positive and another in which it is everywhere negative. So indeed the size of the minimal counter-example to D_n grows exponentially with n. This construction, however, does not work because, due to *two* occurrences of D_n in D_{n+1} , the size of D_n also grows exponentially. What works is this construction of E_n by recursion:

$$E_0 = \bot, \qquad E_{n+1} = (E_n \to q_{n+1}) \to (p_{n+1} \to q_{n+1}) \lor (\neg p_{n+1} \to q_{n+1})$$

where the intended meaning of the atom q_{n+1} is to be a shorthand for E_n . This is an explanation of the role of atoms q_j in our construction below. We will employ further auxiliary atoms s_j whose role is to avoid the use of disjunction in our formulas.

Let a quantified Boolean formula A be given. We may assume that A has the form $Q_m p_m \ldots Q_1 p_1 B(p_1, \ldots, p_m)$ where B contains no propositional quantifiers and no atoms except p_1, \ldots, p_m . We construct the formulas A_0^*, \ldots, A_m^* by recursion. Let A_0^* be $B(\underline{p})$ where \underline{p} stands for the *m*-tuple p_1, \ldots, p_m . If j > 0and $Q_j = \exists$ then A_j^* is

$$(A_{j-1}^* \to q_j) \& ((p_j \to q_j) \to s_j) \& ((\neg p_j \to q_j) \to s_j) \to s_j, s_j \to s_j \to$$



Figure 2: Amalgamating two models into one

whereas if $Q_j = \forall$ then A_j^* is

$$(A_{j-1}^* \to q_j) \& ((p_j \to q_j) \& (\neg p_j \to q_j) \to q_j) \to q_j.$$

Note that A_j^* is built up from $p_1, \ldots, p_m, q_1, \ldots, q_j$, and s_1, \ldots, s_j . More precisely, s_i for $i \leq j$ occurs in A_j^* if and only if Q_i is existential. Finally let A^* be A_m^* .

Lemma 1 Let $0 \leq j \leq m$ and let e be an evaluation of atoms p_{j+1}, \ldots, p_m . Then $e \not\models Q_j p_j \ldots Q_1 p_1 B(\underline{p})$ if and only if A_j^* has a Kripke counter-example in which each atom p_i , for i > j, is evaluated according to e (equally in all nodes).

Proof by induction on j. If e is an evaluation of p_1, \ldots, p_m and $e \not\models B(\underline{p})$ then the one-element model in which all p_1, \ldots, p_m are evaluated according to e is the required counter-example to A_0^* . Let, on the other hand, K be a counter-example to A_0^* , i.e. to $B(\underline{p})$, in which all p_1, \ldots, p_m are evaluated according to e. A straightforward induction shows that each subformula of $B(\underline{p})$ has the same value everywhere in K, namely the value assigned to it by e. So $e \not\models B(\underline{p})$.

Let j > 0 and $Q_j = \exists$ and assume $e \not\models \exists p_j Q_{j-1} p_{j-1} \dots Q_1 p_1 B(\underline{p})$. By the definition of propositional quantifiers, none of the two extensions $0 \neg e$ and $1 \neg e$ of e to atom p_j satisfies the formula $Q_{j-1} p_{j-1} \dots Q_1 p_1 B(\underline{p})$. So the induction hypothesis yields two Kripke counter-examples to A_{j-1}^* : K_0 with a root a_0 in which p_{j+1}, \dots, p_m are evaluated everywhere according to e and in which p_j is everywhere negative, and K_1 with a root a_1 in which p_{j+1}, \dots, p_m are also evaluated according to e and in which p_j is everywhere positive. Note that $a_0 \parallel \neg p_j$ and $a_1 \parallel \neg p_j$. Let K be the model depicted in Fig. 2, with a new root a. To complete the definition of K, we must specify the values of the new atoms q_j and s_j everywhere in K and also the values of all atoms $p_1, \dots, p_m, q_1, \dots, q_j$, and s_1, \dots, s_j in the new node a. This is done as follows:

- in a, atoms p_{j+1}, \ldots, p_m have the truth values assigned by e,
- $\circ p_1, \ldots, p_j, q_1, \ldots, q_{j-1}, \text{ and } s_1, \ldots, s_{j-1} \text{ are negative in } a,$
- q_j has everywhere the same truth value as the formula A_{j-1}^* ,
- s_j has everywhere the same truth value as $(p_j \rightarrow q_j) \lor (\neg p_j \rightarrow q_j)$.

Note that the only thing we had to ensure was the persistency condition, which we did. We have $a \parallel A_{j-1}^* \to q_j$. Since the formula

$$((p_j \to q_j) \to s_j) \& ((\neg p_j \to q_j) \to s_j)$$

is intuitionistically equivalent to $(p_j \to q_j) \lor (\neg p_j \to q_j) \to s_j$, it is everywhere satisfied. We also have $a_0 \parallel \not\vdash \neg p_j \to q_j$ and $a_1 \parallel \not\vdash p_j \to q_j$. Persistency yields $a \parallel \not\vdash (p_j \to q_j) \lor (\neg p_j \to q_j)$. So $a \parallel \not\vdash s_j$ and K is a counter-example to A_j^* .

Assume, on the other hand, that j > 0, $Q_j = \exists$ and K is a counter-example to A_j^* . We may assume that K has a root a and that $a \parallel \not= A_j^*$. So $a \parallel \not= s_j$. Since $a \parallel - (\neg p_j \rightarrow q_j) \rightarrow s_j$, we have $a \parallel \not= \neg p_j \rightarrow q_j$. So there exists a node $a_0 \ge a$ such that $a_0 \parallel - \neg p_j$ and $a_0 \parallel \not= q_j$. From $a \parallel - A_{j-1}^* \rightarrow q_j$ we have $a_0 \parallel \not= A_{j-1}^*$. So the submodel generated by a_0 is a counter-example to A_{j-1}^* in which p_j is everywhere negative. For analogical reasons, there exists a node a_1 such that the submodel generated by a_1 is a counter-example to A_{j-1}^* in which p_j is everywhere positive. The induction hypothesis says $0 \frown e \not\models Q_{j-1}p_{j-1} \dots Q_1p_1B(\underline{p})$ and $1 \frown e \not\models Q_{j-1}p_{j-1} \dots Q_1p_1B(\underline{p})$. So $e \not\models \exists p_j Q_{j-1}p_{j-1} \dots Q_1p_1B(\underline{p})$.

The reasoning in the case where j > 0 and $Q_j = \forall$ is similar. If K is a model with root a and a $\parallel -A_j^*$ then K has an node a_0 such that $a_0 \parallel -p_j \lor \neg p_j$ and the submodel K_0 generated by a_0 is a counter-example to A_{j-1}^* . Since p_j does not change value in K_0 , the induction hypothesis is applicable to K_0 . Details and the proof of the reverse implication are left to the reader.

One can check that if the possibility of avoiding the connectives &, \lor , \bot were not an issue, a simpler definition of A^* would do: A_j^* is $(A_{j-1}^* \to q_j) \to ((p_j \to q_j) \lor (\neg p_j \to q_j))$ or $p_j \lor \neg p_j \to A_{j-1}^*$ according to whether Q_j is \exists or \forall respectively.

Lemma 2 Let A be a formula and r an atom having no occurrences in A. Let further A^{\flat} be the result of substitution of r for all occurrence of \bot in A, and let $\Sigma(A)$ be the conjunction of all formulas $r \to p$ where p is an atom in A. Then A has a counter-example if and only if $\Sigma(A) \to A^{\flat}$ has a counter-example.

Proof If K is a counter-example to A then we can obtain a counter-example H to $\Sigma(A) \to A^{\flat}$ simply by evaluating the new atom r negatively everywhere in K. Assume that $H = \langle W, \leq, || - \rangle$ is a model with root a and a $|| \neq \Sigma(A) \to A^{\flat}$. An easy induction on complexity of B shows that each implication $r \to B^{\flat}$, where B is a subformula of A, is valid in H. Let K be $\langle W_1, \leq_1, || - 1 \rangle$, where $W_1 = \{x \in W; x \mid | \neq r\}$ and \leq_1 and || - 1 are the obvious restrictions of \leq and || -. From $a \mid | - r \to A^{\flat}$ and $a \mid | \neq A^{\flat}$ we have $a \in W_1$. We claim that for each $x \in W_1$ and each subformula B of A we have $x \mid | -B^{\flat} \Leftrightarrow x \mid | -1 B^{\flat}$. For if, for instance, $x \mid | -1 C^{\flat} \to D^{\flat}$ and $x \mid | \neq C^{\flat} \to D^{\flat}$ then for some $y \geq x$ where $y \in W - W_1$ we have $y \mid | -C^{\flat}$ and $y \mid | \neq D^{\flat}$. But $y \mid | \neq D^{\flat}$ and $y \mid | -r$ is a contradiction with the

statement typeset in italics above. Thus K is a counter-example to A^{\flat} in which r is everywhere negative. So indeed A has a counter-example.

Theorem 1 INTTAUT is a PSPACE-complete set. Its purely implicational fragment, i.e. the set of all intuitionistic tautologies built up from propositional atoms using implication as the only connective, also is PSPACE-complete.

Proof For j = m Lemma 1 says that $Q_m p_m ... Q_1 p_1 B(\underline{p})$ (i.e. A) is false in the sense of quantified Boolean formulae if and only if A_m^* (i.e. A^*) has a Kripke counter-example. So the function $A \mapsto A^*$ is a reduction from QBF to INTTAUT. This function is computable in polynomial time or even in logarithmic space. We agreed that the membership of INTTAUT in *PSPACE* we take as granted. To obtain a reduction to the implicational fragment, first replace the subformula $B(\underline{p})$ of A^* by a (classically) equivalent formula built up using only \rightarrow and \perp . Then use Lemma 2 to get rid of the symbol \perp . The resulting formula contains none of the symbols \vee and \perp and the statement of Lemma 2 and an inspection of our construction of formulas A_j^* make it clear that it contains conjunctions only in subformulas of the form $C_1 \& ... \& C_k \to D$. This last formula is intuitionistically equivalent to $C_1 \to (C_2 \to (... \to (C_k \to D)..)$.

Remark 1 The reader of Statman's proof in [Sta79] may be not quite sure whether the symbol \perp is also avoidable when constructing the *PSPACE*-reduction. So our theorem and Lemma 2 perhaps clarify this point.

Remark 2 Note that if a formula is satisfied in some node of some Kripke model then it is valid in some (one-element) Kripke model. This fact says that the set of all intuitionistically *satisfiable* formulas equals the set SAT of all classically satisfiable formulas. Or better, this fact shows that the satisfiability problem has not a good sense for intuitionistic propositional logic.

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