Gödel-Dummett Predicate Logics and Axioms of Prenexability

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1 Introduction

Gödel-Dummett logic in general is a multi-valued logic where a truth value of a formula can be any number from the real interval $[0,1]$ and where implication $\rightarrow$ is evaluated via the Gödel implication function. As to *truth values,* 0 (falsity) and 1 (truth) are the extremal truth values whereas the remaining truth values are called intermediate. *Gödel implication function* $\Rightarrow$ is defined as follows: $a \Rightarrow b = 1$ if $a \leq b$, and $a \Rightarrow b = b$ otherwise. The truth functions of the remaining propositional symbols conjunction & and disjunction $\vee$ are the functions min and max respectively. Negation $\neg A$ of a formula A is in Gödel-Dummett logic understood as $A \rightarrow \bot$ where $\bot$ is a constant for falsity with a truth value equal 0. Thus truth function of negation is the function $a \mapsto (a \Rightarrow 0)$; speaking exactly, $a \Rightarrow 0 = 1$ if $a = 0$ and $a \Rightarrow 0 = 0$ for all $a > 0$.

A particular Gödel-Dummett logic is obtained by restricting the range of possible truth values, i.e. by specifying a truth value set. More exactly, a *logic* T is *based on* a truth value set V where $\{0,1\} \subseteq V \subseteq [0,1]$ if only the elements of V can be chosen as truth values of propositional atoms. Then a propositional formula A is a *tautology of* that *logic* T or a *tautology of* the *set* V if $v(A) = 1$ for each truth evaluation v based on V, i.e. for each truth evaluation v (a function defined on all propositional atoms and extendible uniquely to all propositional formulas) whose range is a subset of V. One can easily verify that (i) each truth value set V such that $\{0,1\} \subseteq V \subseteq [0,1]$ is closed under all truth functions $\Rightarrow$, min, and max, (ii) if $V_1 \subseteq V_2$ then all tautologies of the Gödel-Dummett logic based on V_2 are simultaneously tautologies of the logic based on V_1, and (iii) if two truth value sets are order isomorphic then the logics based on them are the same (equivalent). Also, to

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show that a particular propositional formula A is not a tautology of a logic T, a finite number of truth values is always sufficient. Since in many considerations truth value sets correspond to Kripke frames, we call this simple fact a *finite model property* and denote FMP. As a result, (iv) all propositional Gödel-Dummett logics based on an infinite truth value set are equivalent.

Thus we can define BG, the basic Gödel-Dummett logic, as the logic based on the full real interval $[0,1]$ (or as the logic based on any infinite truth value set V). Furthermore, we can define the logic G_m as the logic based on (any) m-element truth value set, containing the two extremal values 0 and 1 and $m-2$ intermediate values. We have $\mathrm{BG} \subseteq \ldots \subseteq \mathrm{G}_4 \subseteq \mathrm{G}_3 \subseteq \mathrm{G}_2$, where inclusion $T_1 \subseteq T_2$ between logics indicates that each tautology of T_1 is simultaneously a tautology of T_2. It is evident that Gödel implication function restricted to two-element truth value set is exactly the classical truth function of implication, so G_2 is the classical logic.

An elegant *axiomatization* of the logic BG is obtained by adding the *prelinearity schema* $(A \to B) \vee (B \to A)$ to a Hilbert-style calculus for intuitionistic logic. So BG as well as all the logics G_m are extensions of intuitionistic logic. An example of a formula (schema) which is a tautology of BG is $\neg A \vee \neg\neg A$, while $A \vee \neg A$, the principle of excluded middle, is in general not a tautology either of BG or of any of the logics G_m for $m \geq 3$.

Gödel-Dummett logic is sometimes also called *Gödel logic* or *Gödel fuzzy logic*. It was originally invented by Gödel in connection with the question whether a finitely valued semantics can be developed for intuitionistic logic; nowadays it is mostly studied as one of the fuzzy logics, see e.g. Hájek (1998). *Dummett's* important *contribution* is the result that $A \vee B$ is in the logic BG equivalent to $((A \to B) \to B) \,\&\, ((B \to A) \to A)$, so disjunction is in Gödel-Dummett logic expressible in terms of the remaining connectives. Canonical references for Gödel-Dummett logic are the papers Gödel (1932) and Dummett (1959). My motivation to study these logics is probably close to Gödel's: they are interesting extensions of intuitionistic logic.

In this paper we consider Gödel-Dummett *predicate* logics with an emphasis on properties like prenexability and inter-expressibility of quantifiers. The paper overlaps with Kozlíková and Švejdar (2006) co-authored by my former student Blanka Kozlíková. In comparison with Kozlíková and Švejdar (2006), in the present paper we skip some results and most proofs, but we introduce the notion of characteristic class of a logic and we add some semantical considerations. We also borrow a lot of notions and ideas from Baaz, Preining, and Zach (2003).

2 Gödel-Dummett predicate logics

In Gödel-Dummett *predicate* logic we consider the same *formulas* as in classical logic, built up from atomic formulas using the propositional symbols $\to$,

&, $\vee$, and $\neg$, and quantifiers $\forall$ and $\exists$. As to *omitting parentheses,* we accept the more or less usual convention that implication $\rightarrow$ has higher priority than equivalence $\equiv$, but lower than & and $\vee$.

A *multi-valued structure* $\mathcal{J}$ *based on a truth value set* V, or a *multi-valued model based on* V, has a non-empty domain and a truth assignment that associates a truth value $\mathcal{J}(\varphi)[e]$ with every pair φ, e where φ is an atomic formula and e an evaluation of (free) variables. The truth assignment extends uniquely to all formulas using the truth functions of logical connectives defined above, and using the conditions $\mathcal{J}(\forall x\varphi)[e] = \inf_{a\in D} \mathcal{J}(\varphi)[e(x/a)]$ and $\mathcal{J}(\exists x\varphi)[e] = \sup_{a\in D} \mathcal{J}(\varphi)[e(x/a)]$, where D is the domain of the structure $\mathcal{J}$, inf and sup denote the least upper bound (infimum) and greatest lower bound (supremum) respectively, and $e(x/a)$ is the evaluation identical to e except that the variable x is evaluated by $a \in D$. To ensure the existence of suprema and infima, we define a truth value set as a (topologically) *closed* set V such that $\{0,1\} \subseteq V \subseteq [0,1]$. In full analogy with the classical case, a formula φ is a *logical truth* of a set V if it is valid in each structure $\mathcal{J}$ based on V, i.e. if $\mathcal{J}(\varphi)[e] = 1$ for each structure $\mathcal{J}$ based on V and each evaluation e of variables.

Example 1 Let $V = \{\frac{1}{2}, 1\} \cup \{\, \frac{1}{2} - \frac{1}{k}\, ;\, k \geq 2\,\}$ and consider a language $\{P\}$ with a single unary predicate P. Let the domain D be the set $\{d_2, d_3, d_4, \dots\}$ and let the truth assignment be defined by $\mathcal{J}(P(x))[e(x/d_k)] = \frac{1}{2} - \frac{1}{k}$. Note that the numbering of elements of D is chosen so that we have the same k on both sides of the latter equality. Then

$$\mathcal{J}(\exists y P(y))[e] = \sup_{k\geq 2} \mathcal{J}(P(y))[e(y/d_k)] = \tfrac{1}{2}$$

regardless of e, and $\mathcal{J}(\exists y P(y) \rightarrow P(x))[e(x/d_k)] = \frac{1}{2} - \frac{1}{k}$ by the definition of Gödel implication function. So $\mathcal{J}$ is a structure based on V in which the sentence $\exists x(\exists y P(y) \rightarrow P(x))$ is not valid because its truth value is $\frac{1}{2}$ under some (and also any) truth evaluation of variables. Thus that sentence is not a logical truth either of our V or of the full real interval $[0,1]$.

One can even think a little further and verify that the existence of a truth value $a < 1$ in V which is a limit of lower values is essential for Example 1 to work. The sentence $\exists x(\exists y P(y) \rightarrow P(x))$ is a logical truth of any truth value set containing no $a < 1$ which is a limit of lower values, and in particular it is a logical truth of any finite truth value set. So Example 1 also shows that finite model property is not true for predicate Gödel-Dummett logic.

The usual lemma saying that if e_1 and e_2 are two evaluations of variables that agree on all free variables of a formula φ then $\mathcal{J}(\varphi)[e_1] = \mathcal{J}(\varphi)[e_2]$ is true also for multi-valued structures. So if φ is a sentence then we can write only $\mathcal{J}(\varphi)$ without specifying the evaluation e. Also, we will write for example $\mathcal{J}(P(d))$ instead of the more correct $\mathcal{J}(P(x))[e(x/d)]$.

By a *logic* we mean any deductively closed set of formulas, i.e. any set of predicate formulas that is closed under the modus ponens and generalization rules. Let G_V, the *Gödel-Dummett logic based* on a truth value *set* V, or a logic *determined by* V, be the logic of all logical truths of V. The basic Gödel-Dummett logic BG is defined as the logic based on the real interval $[0,1]$, in symbols, $\mathrm{BG} = G_{[0,1]}$. The logic G_m for $m \geq 2$ is, as in the propositional case, the logic based on (any) m-element truth value set. In predicate logic it is not true that all infinite truth value sets determine the same logic; this can also be deduced from Example 1. If the properties (i)–(iv) from the second paragraph of Introduction are reformulated for predicate logic, (i)–(iii) remain true, but (iv) is false.

The logic BG is axiomatizable, see e.g. Takano (1987). Its axiomatization is obtained by taking the propositional calculus for BG mentioned above and by adding one quantifier schema

S_1: $\quad \forall x(\psi \vee \varphi(x)) \rightarrow \psi \vee \forall x \varphi(x),$

where x is not free in ψ (recall the convention for omitting parentheses above). Each of the logics G_m is axiomatizable as well, see Preining (2003). Baaz et al. (2003) define two more interesting logics $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$ as logics determined by the sets $\mathrm{V}_\downarrow = \{0\} \cup \{\, \frac{1}{k}\,;\, k \geq 1 \,\}$ and $\mathrm{V}_\uparrow = \{1\} \cup \{\, 1 - \frac{1}{k}\,;\, k \geq 1 \,\}$ respectively. The formula $\exists x(\exists y P(y) \rightarrow P(x))$ is a logical truth of both logics $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$. Baaz et al. (2003) also show that neither $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$ nor any logic based on a countable infinite truth value set is axiomatizable. Petr Hájek in Hájek (2005) recently obtained more accurate results about the position of the logics $\mathrm{G}_\downarrow$ and $\mathrm{G}_\uparrow$ in arithmetical hierarchy.

Recall that, in classical logic, prenex operations are formulated as eight equivalences, i.e. sixteen implications, and the schema S_1 is one of only three prenex implications that are not intuitionistically valid. The remaining two intuitionistically non-valid prenex implications are

S_2: $\quad (\psi \rightarrow \exists x \varphi(x)) \rightarrow \exists x(\psi \rightarrow \varphi(x)),$

S_3: $\quad (\forall x \varphi(x) \rightarrow \psi) \rightarrow \exists x(\varphi(x) \rightarrow \psi),$

where again x is not free in ψ. Since S_1 is so important in the axiomatization of the logic GB, it seems interesting to think also about S_2 and S_3 as potential axiom schemas. So we *define* S2G, S3G, and PG to be the logics obtained by adding S_2, or S_3, or both S_2 and S_3 respectively, as additional axiom schema(s) to the basic logic BG. Thus PG is the weakest extension of BG in which all the classical prenex operations are available. We will discuss some properties of the logics S2G, S3G, and PG, and we will relate them to the logics $\mathrm{G}_\downarrow$, $\mathrm{G}_\uparrow$, G_m known from literature.

The idea to study the extensions of the logic BG given by axioms of prenexability may look somewhat unusual because these logics are not determined by truth value sets. Our approach is that a schematical extension

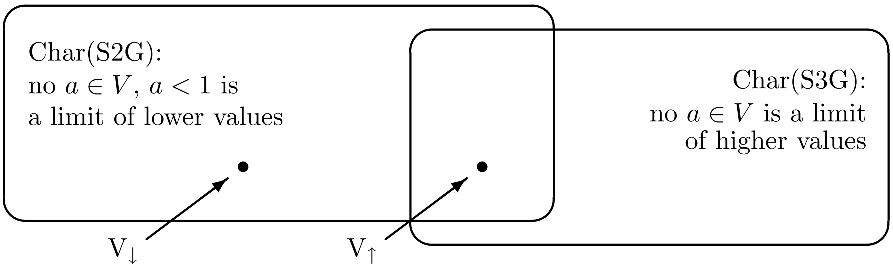

Figure 1: Characteristic classes of S2G, S3G, and PG

of a Gödel-Dummett logic can still be called Gödel-Dummett logic. This is, I suppose, fully in the spirit of Hájek (1998).

Let Char(T), the *characteristic class* of a logic T, be defined as the class of all truth value sets V such that all logical truths of T are valid in all multi-valued structures based on V.

Lemma 2 *(a) If $T_1 \subseteq T_2$, i.e. if each logical truth of a logic T_1 is simultaneously a logical truth of T_2, then* $\mathrm{Char}(T_2) \subseteq \mathrm{Char}(T_1)$.
(b) If V is a truth value set and T a logic, then $V \in \mathrm{Char}(T)$ if and only if $T \subseteq G_V$.

Proof If φ is a logical truth of T then φ is valid in any structure $\mathcal{J}$ based on any set in Char(T). If, in addition, $V \in \mathrm{Char}(T)$ then φ is valid in any structure based on V. So $\varphi \in G_V$. On the other hand, if $V \notin \mathrm{Char}(T)$ then there exists a structure $\mathcal{J}$ based on V and a sentence $\varphi \in T$ not valid in $\mathcal{J}$. Since $\varphi \notin G_V$, we have $T \nsubseteq G_V$. The proof of (a) is similar. ■

Theorem 3 *Over* BG*, the logic* S2G *is equivalently axiomatized by any of the schemas*

$C_\downarrow$: $\exists x(\exists y\varphi(y) \to \varphi(x))$,

E: $\forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x)) \to \exists x\varphi(x)$.

Its characteristic class is the class of all truth value sets where no value except possibly 1 *is a limit of lower values.*

Proof We show that $C_\downarrow$ and E are (already intuitionistically) equivalent. We omit the proof that S_2 is equivalent to $C_\downarrow$ because it is known or implicit in literature, i.e. in Baaz et al. (2003). We proceed informally, the reader should have no difficulty with formalizing the argument in the appropriate calculus.

$C_\downarrow \Rightarrow$ E: Assume that $\forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x))$ and let x_0 be such that $\exists y\varphi(y) \to \varphi(x_0)$. We have $\forall y(\varphi(y) \to \varphi(x_0)) \to \varphi(x_0)$. Since $\exists y\varphi(y) \to \varphi(x_0)$ is intuitionistically equivalent to $\forall y(\varphi(y) \to \varphi(x_0))$, we have $\varphi(x_0)$. So indeed, $\exists x\varphi(x)$.

$\mathrm{E} \Rightarrow \mathrm{C}_\downarrow$: To show that $\exists x(\exists y\varphi(y) \to \varphi(x))$, the schema E says that it is sufficient to verify that

$$\forall x(\forall z((\exists y\varphi(y) \to \varphi(z)) \to (\exists y\varphi(y) \to \varphi(x))) \to (\exists y\varphi(y) \to \varphi(x))).$$

So let x be given. Since $A \to (B \to C)$ is equivalent to $A \mathbin{\&} B \to C$, and $(A \to B) \mathbin{\&} A$ is equivalent to $A \mathbin{\&} B$, to verify that

$$\forall z((\exists y\varphi(y) \to \varphi(z)) \to (\exists y\varphi(y) \to \varphi(x))) \to (\exists y\varphi(y) \to \varphi(x))$$

it is sufficient to verify that

$$\forall z(\exists y\varphi(y) \mathbin{\&} \varphi(z) \to \varphi(x)) \mathbin{\&} \exists y\varphi(y) \to \varphi(x). \tag{$*$}$$

Taking y_0 such that $\varphi(y_0)$, which is possible by the right conjunct, and then applying the left conjunct to $z := y_0$ quickly shows that $(*)$ is true.

Assume now that V is a truth value set such that no its element, except possibly the element 1, is a limit of lower values. We have to verify that $\exists x(\exists y\varphi(y) \to \varphi(x))$ is valid in any structure $\mathcal{J}$ based on V. So let $\mathcal{J}$ with domain D be given and take $a_0 = \mathcal{J}(\exists y\varphi(y)) = \sup_{d\in D} \mathcal{J}(\varphi(d))$. If $a_0 = 1$ then $\mathcal{J}(\exists x(\exists y\varphi(y) \to \varphi(x))) = \sup_{d\in D} \mathcal{J}(\exists y\varphi(y) \to \varphi(d)) \geq \sup_{d\in D} \mathcal{J}(\varphi(d)) = 1$. If a least upper bound of a set is not a limit of lower values then it must be an element of that set. So, in the remaining case where $a_0 < 0$, there exists an element $d_0 \in D$ such that $a_0 = \sup_{d\in D} \mathcal{J}(\varphi(d)) = \mathcal{J}(\varphi(d_0))$. Then $\mathcal{J}(\exists x(\exists y\varphi(y) \to \varphi(x))) \geq \mathcal{J}(\exists y\varphi(y) \to \varphi(d_0)) = 1$. Note that in both cases the definition of the Gödel implication function $\Rightarrow$ played a role.

It remains to verify that if the truth value set V contains a value $a < 1$ which is a limit of lower values then there exists a structure $\mathcal{J}$ based on V such that some instance of the schema $\mathrm{C}_\downarrow$ is violated. This is however already clear from Example 1. ■

Since the following Theorem 4 does not involve a new schema (like the schema E above), we omit its proof. It is however similar to that of Theorem 3.

Theorem 4 S3G *is equivalently axiomatized by* $\exists x(\varphi(x) \to \forall y\varphi(y))$. *Its characteristic class is the class of all truth value sets where no value is a limit of higher values.*

Characteristic classes of logics S2G, S3G, and PG, and the membership of the prominent truth value sets $\mathrm{V}_\downarrow$ and $\mathrm{V}_\uparrow$, are depicted in Fig. 1; it is evident that $\mathrm{Char(PG)} = \mathrm{Char(S2G)} \cap \mathrm{Char(S3G)}$. It is important to observe that

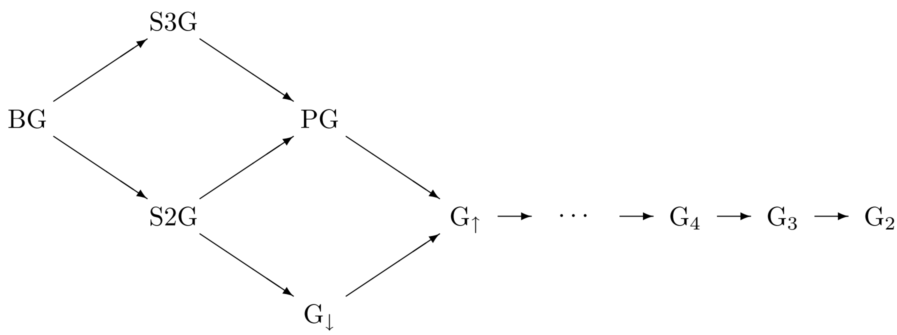

Figure 2: Relationships between Gödel-Dummett logics

Char(PG) is rather small: if $V \in$ Char(PG), i.e. if no element of V, except possibly the element 1, is a limit of other values, then V is finite or isomorphic to $V_\uparrow$.

It is easy to verify that the schema $\forall x(\forall y(\varphi(y) \to \varphi(x)) \to \varphi(x)) \equiv \exists x \varphi(x)$, resulting from replacing the outermost implication in the schema E by equivalence, is also provable in S2G. So we have the following Theorem.

Theorem 5 *In* S2G *and thus in all its extensions, the existential quantifier is expressible in terms of the remaining logical symbols.*

Theorem 6 *The relationships between the logics we consider are as shown in Fig. 2.*

Proof S2G $\subseteq$ PG and S3G $\subseteq$ PG is immediate. S2G $\subseteq G_\downarrow$ follows from Lemma 2(b), as well as PG $\subseteq G_\uparrow$. The inclusions $G_\downarrow \subseteq G_m$ and $G_\uparrow \subseteq G_m$, for each m, follow from property (ii) in the Introduction. Baaz et al. (2003) show that $G_\uparrow = \bigcap_{m \geq 2} G_m$. From this we have $G_\downarrow \subseteq G_\uparrow$.

As to non-inclusions, the fact that S3G $\not\subseteq G_\downarrow$ follows from $V_\downarrow \notin$ Char(S3G) and Lemma 2(b). Also, S2G $\not\subseteq$ S3G follows from Char(S3G) $\not\subseteq$ Char(S2G) and Lemma 2(a). For the more complicated proof of $G_\downarrow \not\subseteq$ PG see Kozlíková and Švejdar (2006); the proof is also outlined in Section 3 below. ∎

So, by Theorem 5, the quantifier $\exists$ is expressible in terms of $\forall$ and logical connectives in the logics S2G, PG, $G_\downarrow$, $G_\uparrow$, and all G_m. Petr Cintula verified that the schema E, with equivalence as the outermost symbol, is provable also in logics that we do not consider here, namely in all logics extending MTL+S_2, where the logic MTL is defined in Esteva and Godo (2001). So also in all these logics the existential quantifier is expressible in terms of the remaining logical symbols. Petr Cintula also remarked that the fact that the existential quantifier is expressible using *only* the symbols $\forall$ and $\rightarrow$ may be new even for the logic G_2, the classical two valued logic.

Further results in Kozlíková and Švejdar (2006) say that the quantifier $\exists$ is not expressible in terms of $\forall$ and logical connectives in S3G, and the quantifier $\forall$ is not expressible in terms of $\exists$ and logical connectives even in G_3. Also, for both logics S2G and S3G there exist formulas that are not equivalent to prenex formulas. To obtain these results, Kripke semantics is sometimes used as well. It is important to realize that one can work with a semantics—multi-valued or Kripke—even in the absence of completeness theorem: for some results, the soundness theorem is sufficient.

While PG is the weakest extension of the basic logic BG in which all the classical prenex operations are valid, it still seems to be an interesting problem whether PG is the weakest extension of BG in which any formula is equivalent to a prenex formula.

3 Remarks on semantics and completeness

The non-inclusion $G_\downarrow \not\subseteq$ PG asserts the existence of a sentence $\varphi \in G_\downarrow$ such that $\varphi \notin$ PG. However, if V is a set in Char(PG), i.e. if V is finite or isomorphic to $V_\uparrow$ then, by $G_\downarrow \subseteq G_\uparrow$, the sentence φ is valid in any structure based on V. So we conclude that $\varphi \notin$ PG cannot be shown by taking a truth value set from the logic's characteristic class and defining a structure $\mathcal{J}$ based on V such that $\mathcal{J}(\varphi) < 1$. The logic PG is *incomplete* with respect to its characteristic class.

The problem whether PG (or S2G, or S3G) is complete with respect to some semantics is left open in Kozlíková and Švejdar (2006). Hájek and Cintula (2006) offer a solution: the logic PG is complete with respect to witnessed structures. Their result can probably be generalized also for S2G and S3G. A structure $\mathcal{J}$ with a domain D is *witnessed* if, whenever $\varphi(x, y_1, .., y_n)$ is a formula and the variables $y_1, .., y_n$ are evaluated by elements $d_1, .., d_n \in D$, the set $\{\ \mathcal{J}(\varphi(d, d_1, .., d_n))\ ;\ d \in D\ \}$ of truth values has both maximal and minimal element.

Without using the notion of witnessed structure, a structure $\mathcal{J}$ satisfying the definition is constructed in Kozlíková and Švejdar (2006) to show that $G_\downarrow \not\subseteq$ PG. The structure $\mathcal{J}$ looks as follows. The truth value set V contains a value $a_0 < 1$ which is a limit of lower values. There are only finitely many values greater than a_0 and all values in V except a_0 are isolated. Let Q be a function from V to V defined by $Q(a) = a$ for $a \leq a_0$ and $Q(a) = a_0$ for $a \geq a_0$. Importantly, the function $[a, b] \mapsto Q(a \Rightarrow b)$, from V^2 to V, is continuous. The structure $\mathcal{J}$ is chosen so that its domain D is equipped with a compact topology and so that for each atomic formula $\varphi(x_1, .., x_n)$ the function $[d_1, .., d_n] \mapsto Q(\mathcal{J}(\varphi(d_1, .., d_n)))$ is continuous as a function from D^n to V. Then using some topological knowledge and equations like

$Q(\min\{a,b\}) = \min\{Q(a), Q(b)\}$ and $Q(a \Rightarrow b) = Q(Q(a) \Rightarrow Q(b))$ one can show that the function $[d_1,..,d_n] \mapsto Q(\mathcal{J}(\varphi(d_1,..,d_n)))$ is continuous for *every* formula φ. So every set of the form $\{\, Q(\mathcal{J}(\varphi(d,d_1,..,d_n)))\,;\, d \in D\,\}$ is topologically closed, and as such it must have both maximal and minimal element. The set $\{\, \mathcal{J}(\varphi(d,d_1,..,d_n))\,;\, d \in D\,\}$ may be not closed, but one can conclude that it must have both maximal and minimal element, too.

The construction described in the previous paragraph suggests that, in particular case, it may not be so easy to verify that a given structure is witnessed.

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