Weak Theories and Essential Incompleteness

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1 Introduction: essential incompleteness and essential undecidability

This paper is motivated by the following question: what is the weakest theory that is essentially incomplete or essentially undecidable?

An axiomatic theory T is *complete* if it is consistent and for every sentence φ in its language it is the case that either $T \vdash \varphi$ (the sentence φ is *provable* in T) or $T \vdash \neg\varphi$ (φ is *refutable* in T). So if T is consistent and incomplete then there exist sentences *independent* of T, i.e. sentences φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$. A theory is *recursively axiomatizable* if it (is equivalent to a theory that) has a recursive, i.e. algorithmically decidable, set of axioms. A theory S is an *extension* of a theory T if the language of T is a subset of that of S and all axioms of T are provable also in S. *Gödel 1st incompleteness theorem*, or better, *Rosser generalization* of Gödel incompleteness theorem, says that all recursively axiomatizable extensions of certain weak base theory are incomplete.

The incompleteness theorem applies also to Peano arithmetic, a theory whose incompleteness is rather difficult to prove using elementary methods. But still, there is something more to say about incompleteness theorems. Before they were discovered, incompleteness of a theory could have been seen as an imperfection in formulation of its axioms: a theory is incomplete because some axioms are missing. However, if T is a theory to which incompleteness theorem is applicable and if φ is any sentence independent of T, the incompleteness theorem is applicable also to the two (consistent) extensions T, φ and $T, \neg\varphi$ of T. So there is no such thing as missing axiom; the theory T is not only incomplete but also *incompletable.* Thus the incompleteness theorem urges us to reconsider the notion of incompleteness and leads us to

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essential incompleteness (Tarski, Mostowski, & Robinson, 1953): a theory is *essentially incomplete* if all its recursively axiomatizable extensions are incomplete. Then Gödel (Rosser) theorem in fact says that a certain weak base theory (which is recursively axiomatizable and of which Peano arithmetic is an extension) is essentially incomplete.

A theory T is *decidable* if the set of all its theorems, i.e. the set of all sentences provable in T, is recursive. If T is not decidable then it is *undecidable*. Trivially, a decidable theory is recursively axiomatizable, and an inconsistent theory is decidable. A theory T is *essentially undecidable* if all its consistent extensions are undecidable. It is known that a recursively axiomatizable complete theory is decidable, and that a decidable consistent theory has a decidable complete extension (formulated in the same language). Knowing these two not so trivial facts, it is an interesting exercise to show that the two notions, essential incompleteness and essential undecidability, coincide. So we will use them interchangeably. Note that an essentially incomplete theory may be complete. It is decidable if and only if it is inconsistent.

An *interpretation* of a theory T in a theory S is a mapping from formulas of T to formulas of S that satisfies certain conditions (e.g. preserves logical connectives and the number of free variables) and maps all axioms of T to sentences provable in S. Precise definition of the notion of interpretation is (again) in Tarski et al. (1953). Basic facts about interpretations are the following. If T is *interpretable* in S, i.e. if there exists an interpretation $*$ of T in S, then all sentences provable (refutable) in T are mapped, by the function $*$, to sentences provable (refutable) in S; if S is consistent then T is consistent, too; and if T is essentially undecidable then S is essentially undecidable, too. If S is an extension of T then T is trivially interpretable in S. The incompleteness theorem can be generalized using of interpretability: there exists a weak recursively axiomatizable consistent base theory T such that each recursively axiomatizable theory S in which T is interpretable is incomplete. Interpretability can be accepted as a *measure of strength* of axiomatic theories: if T is interpretable in S but *not* vice versa then T can be considered weaker than S; if T is interpretable in S *and* vice versa, i.e. if T and S are *mutually interpretable,* then T and S are equally strong.

2 Essential incompleteness of Robinson arithmetic

It is usually *Robinson arithmetic* Q that is taken as the weak base theory, i.e. the theory for which essential incompleteness is stated and proved. It is defined in Tarski et al. (1953) as a theory with the language $\{+, \cdot, 0, \mathrm{S}\}$ containing two binary function symbols, a constant, and a unary function symbol, and with the following axioms:

Q1: $\forall x \forall y(\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y)$,

Q2: $\forall x(\mathrm{S}(x) \neq 0)$,
Q3: $\forall x(x \neq 0 \rightarrow \exists y(x = \mathrm{S}(y)))$,
Q4: $\forall x(x + 0 = x)$,
Q5: $\forall x \forall y(x + \mathrm{S}(y) = \mathrm{S}(x + y))$,
Q6: $\forall x(x \cdot 0 = 0)$,
Q7: $\forall x \forall y(x \cdot \mathrm{S}(y) = x \cdot y + x)$.

Nowadays it is common to add also the symbols $\leq$ and $<$ to the language of Robinson arithmetic, and two axioms about them:

Q8: $\forall x \forall y(x \leq y \equiv \exists z(z + x = y))$,
Q9: $\forall x \forall y(x < y \equiv \exists z(\mathrm{S}(z) + x = y))$.

This definitional extension makes it easier to define bounded quantification and the notion of Σ-formulas (see below).

Evidently the structure $\mathbf{N} = \langle \mathrm{N}, +^{\mathbf{N}}, \cdot^{\mathbf{N}}, 0^{\mathbf{N}}, \mathrm{s}\rangle$, i.e. the *structure of natural numbers* with "normal" operations, "normal" number zero, and the successor function s where $\mathrm{s}(a) = a + 1$, is a model of Q. *Peano arithmetic* PA is obtained by adding the induction scheme to the axioms of Q. Without induction, i.e. in Q itself, general statements are usually unprovable. Examples of sentences unprovable in Q are $\forall y(0 + y = 0)$ and $\forall x(\mathrm{S}(x) \neq x)$. However, some general statements can be proved in Q; an example is $\forall x \forall y(x + y = 0 \rightarrow x = 0 \ \& \ y = 0)$. Indeed, if $y \neq 0$ then $y = \mathrm{S}(z)$ for some z by Q3. Then $x + \mathrm{S}(z) = 0$, and Q5 yields $\mathrm{S}(x + z) = 0$, a contradiction with Q2. So $y = 0$, and from $x + 0 = 0$ and Q4 we have $x = 0$.

The closed terms 0, $\mathrm{S}(0)$, $\mathrm{S}(\mathrm{S}(0))$, ... are called *numerals* and denoted $\overline{0}$, $\overline{1}$, $\overline{2}$, ... Thus 0 and $\overline{0}$ represent the same closed term; its value in $\mathbf{N}$ is $0^{\mathbf{N}}$, the number zero. Numerals make it possible to speak, in the language of Q, about particular numbers.

We write $\forall v{\leq}x\,\varphi$ and $\exists v{\leq}x\,\varphi$ for $\forall v(v \leq x \rightarrow \varphi)$ and $\exists v(v \leq x \ \& \ \varphi)$, where v and x are different variables. $\forall v{<}x\,\varphi$ and $\exists v{<}x\,\varphi$ are defined analogically. The expressions $\forall v{\leq}x$, $\exists v{\leq}x$, $\forall v{<}x$, and $\exists v{<}x$ are called *bounded quantifiers.* A Δ_0-*formula,* or a *bounded formula,* is a formula whose all quantifiers are bounded. A Σ_1-*formula* is a formula having the form $\exists y \theta$, where $\theta \in \Delta_0$, whereas a Σ-*formula* is any formula obtained from Δ_0-formulas using conjunctions, disjunctions, existential quantification, and bounded quantification. Any Σ_1-formula is simultaneously a Σ-formula.

There are two important facts about Σ_1- and Σ-formulas: Σ-*completeness theorem* and *definability theorem.* These are also *basic ingredients* of essential incompleteness proofs. Σ-completeness theorem says that each true (i.e. valid in the structure of natural numbers) Σ-sentence is provable in Q. Definability theorem says that for each r.e. set $A \subseteq \mathrm{N}^k$ there exists a Σ-formula

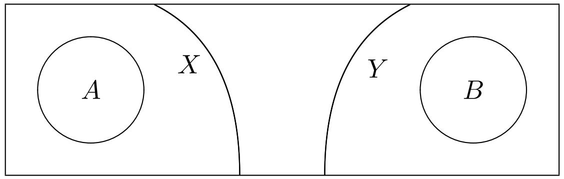

Figure 1: An essential incompleteness proof

$\varphi(x_1,\ldots,x_k)$ that defines A, i.e. satisfies $[n_1,\ldots,n_k] \in A \Leftrightarrow \mathbf{N} \models \varphi(\overline{n_1},\ldots,\overline{n_k})$ for each k-tuple $[n_1,\ldots,n_k] \in \mathrm{N}^k$. We omit proofs of both theorems as too laborious (and also known).

The two basic ingredients, in fact definability alone, are sufficient to show incompleteness of any Σ-*sound* extension of Q, i.e. of any extension that does not prove any false Σ-sentence, see e.g. Švejdar (2003). They are also sufficient to show undecidability of Q. To show essential incompleteness of Q, one usually needs one of additional conditions like the following:

(i) For each pair A, B of recursively enumerable sets there exists a formula $\varphi(x)$ such that $\mathrm{Q} \vdash \varphi(\overline{n})$ for $n \in A - B$, and $\mathrm{Q} \vdash \neg\varphi(\overline{n})$ for $n \in B - A$.

(ii) Weak representability of recursive functions: for each recursive function $f : \mathrm{N} \to \mathrm{N}$ there exists a formula $\varphi(x, y)$ such that, for each number n, $\mathrm{Q} \vdash \forall y(\varphi(\overline{n}, y) \equiv y = \overline{f(n)})$.

(iii) The self-reference theorem: for each formula $\psi(x)$ there exists a sentence φ satisfying $\mathrm{Q} \vdash \varphi \equiv \psi(\overline{\varphi})$.

A proof of essential incompleteness of Q using (iii) and a proof of (iii) using (ii) are well known, the reader may consult e.g. Feferman (1960) or Smoryński (1985). Below in Theorem 3.3 we give a proof of (ii), for a theory weaker than Q. Proofs of (i) and (ii) usually use some version of *Rosser's trick:* if two or more events may but should not be compatible, they can be made incompatible by considering *which of them occurs first.*

A proof of incompleteness that uses properties of recursively enumerable and recursive sets rather than self-reference can be called *structural.* We now show such a structural proof; more exactly, we show essential incompleteness of Q using the condition (i). The proof is now folklore, I know it probably from a manuscript by Smoryński. Recall that two sets $A, B \subseteq \mathrm{N}$ are *recursively inseparable* if there is no recursive $D \supseteq A$ such that $D \cap B = \emptyset$; it is known that pairs of disjoint recursively inseparable r.e. sets exist.

So let S be a recursively axiomatizable extension of Q. We may assume that S is consistent because otherwise it is incomplete by definition. Let A and B be disjoint recursively inseparable r.e. sets of natural numbers. Let $\varphi(x)$ be a formula such that $\mathrm{Q} \vdash \varphi(\overline{n})$ for $n \in A - B = A$, and $\mathrm{Q} \vdash \neg\varphi(\overline{n})$ for

$n \in B - A = B$. Put $X = \{\, n\,;\, S \vdash \varphi(\overline{n}) \,\}$ and $Y = \{\, n\,;\, S \vdash \neg\varphi(\overline{n}) \,\}$. Since S is an extension of Q, we have $A \subseteq X$ and $B \subseteq Y$. The inclusions may be strict. The sets X and Y are r.e. since S is recursively axiomatizable, and they are disjoint since S is consistent. So their relationship is as depicted in Fig. 1. If X and Y were mutually complementary then, by Post's theorem, they would both be recursive; recursiveness of X would contradict recursive inseparability of A and B. So X and Y are not complementary, and we can take $n_0 \notin X \cup Y$. Then $S \nvdash \varphi(\overline{n_0})$ and $S \nvdash \neg\varphi(\overline{n_0})$. So $\varphi(\overline{n_0})$ is an independent sentence and thus S is incomplete.

3 Weak alternatives to Robinson arithmetic

3.1 Grzegorczyk's theory Q^-

In connection with a project to base explanation of incompleteness theorems on a theory different from and perhaps more natural than Q, Andrzej Grzegorczyk considered a theory Q^- in which addition and multiplication do satisfy natural reformulations of axioms of Q but are possibly non-total functions. More exactly, the language of Q^- is $\{0, \mathrm{S}, \mathrm{A}, \mathrm{M}\}$, where A and M are ternary relations, and the axioms of Q^- are the axioms Q1–Q3 of Q plus the following six axioms about A and M:

A: $\forall x \forall y \forall z_1 \forall z_2(\mathrm{A}(x, y, z_1) \;\&\; \mathrm{A}(x, y, z_2) \to z_1 = z_2)$,

M: $\forall x \forall y \forall z_1 \forall z_2(\mathrm{M}(x, y, z_1) \;\&\; \mathrm{M}(x, y, z_2) \to z_1 = z_2)$,

G4: $\forall x \mathrm{A}(x, 0, x)$,

G5: $\forall x \forall y \forall z(\exists u(\mathrm{A}(x, y, u) \;\&\; z = \mathrm{S}(u)) \to \mathrm{A}(x, \mathrm{S}(y), z))$,

G6: $\forall x \mathrm{M}(x, 0, 0)$,

G7: $\forall x \forall y \forall z(\exists u(\mathrm{M}(x, y, u) \;\&\; \mathrm{A}(u, x, z)) \to \mathrm{M}(x, \mathrm{S}(y), z))$.

A. Grzegorczyk asked whether Q^- was essentially undecidable. Petr Hájek considered a somewhat stronger theory, with axioms

H5: $\forall x \forall y \forall z(\exists u(\mathrm{A}(x, y, u) \;\&\; z = \mathrm{S}(u)) \equiv \mathrm{A}(x, \mathrm{S}(y), z))$,

H7: $\forall x \forall y \forall z(\exists u(\mathrm{M}(x, y, u) \;\&\; \mathrm{A}(u, x, z)) \equiv \mathrm{M}(x, \mathrm{S}(y), z))$.

instead of G5 and G7. He showed that this stronger variant of Q^- is essentially undecidable, and also that it is essentially undecidable if the underlying logic (i.e. the classical first-order predicate logic) is replaced by a weak fuzzy logic, see Hájek (2007). The following Theorem, proved in Švejdar (2007a), yields a positive answer to Grzegorczyk's original question.

Theorem 1 Q *is interpretable in* Q^-*. So* Q^- *is essentially incomplete.*

Recall the sentence $\forall x \forall y(x+y=0 \rightarrow x=0 \ \& \ y=0)$; its proof in Q was shown above. If $\leq$ is introduced using axiom Q8, the meaning of this sentence is $\forall y{\leq}0\,(y=0)$. A simple model of Q^- can be constructed to show that this sentence, as well as the (weaker) sentence $\forall y(0+y=0 \rightarrow y=0)$, is unprovable in Q^-. Since $\forall y{\leq}0\,(y=0)$ is a Σ-sentence, Σ-completeness theorem is (in this sense) false for Q^-. On the other hand, one can easily verify that Σ-completeness theorem is true for Hájek's variant; that is in fact a step in the essential incompleteness proof in Hájek (2007).

3.2 The theory TC

Besides the theory Q^-, A. Grzegorczyk considered another weak theory, the *theory of concatenation*. It has the language $\{{}^\frown, \alpha, \beta\}$ with a binary function symbol and two constants, and the following axioms:

TC1: $\forall x \forall y \forall z(x^\frown(y^\frown z) = (x^\frown y)^\frown z)$,

TC2: $\forall x \forall y \forall u \forall v(x^\frown y = u^\frown v \rightarrow ((x=u \ \& \ y=v) \vee \exists w((u = x^\frown w \ \& \ w^\frown v = y) \vee (x = u^\frown w \ \& \ w^\frown y = v))))$,

TC3: $\forall x \forall y \neg(\alpha = x^\frown y)$,

TC4: $\forall x \forall y \neg(\beta = x^\frown y)$,

TC5: $\alpha \neq \beta$.

The objects of the theory TC can be called *texts* or *strings*. The axioms TC3–TC5 say that α and β are irreducible, i.e. they are one letter strings that are mutually different. The axiom TC2 is called *editor axiom*; it describes what happens if two editors independently suggest splitting a large text into two volumes: if their suggestions are not identical then the first volume of one of the editors consists of two parts, the other editor's first volume and a text that appears as a starting part of the other editor's second volume.

According to the paper by Grzegorczyk and Zdanowski (2008), the theory TC was first considered—but in a different context—by Quine, see Quine (1946), and the editor axiom was formulated by Tarski.

Andrzej Grzegorczyk proved (mere) undecidability of the theory TC in Grzegorczyk (2005). Later, essential undecidability of TC was proved in Grzegorczyk and Zdanowski (2008); in fact two different (and both rather technically involved) proofs of essential undecidability of TC are given in that paper. The paper Grzegorczyk and Zdanowski (2008) formulates but leaves unanswered an interesting problem: are TC and Q mutually interpretable?[1]

[1] Added in proof, March 2008: this problem has a positive solution, see (Visser, 2007; Ganea, 2007; Švejdar, 2007b).

3.3 The theory R

A first step in a full essential incompleteness proof of a theory T (like the proof that is sketched in Theorem 2 above) usually consists in verifying that the following five schemes

$\Omega1$: $\overline{n} + \overline{m} = \overline{n+m}$,

$\Omega2$: $\overline{n} \cdot \overline{m} = \overline{n \cdot m}$,

$\Omega3$: $\overline{n} \neq \overline{m}$, for n different from m,

$\Omega4$: $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \,..\, \vee x = \overline{n})$,

$\Omega5$: $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$

are provable in T. Petr Hájek sometimes called their provability in Q *Mrs. Karp's lemma.* It makes sense to think about these schemes as of a yet another interesting theory; this theory is called *theory* R in Tarski et al. (1953). It makes no difference whether numerals are considered primitive constants or defined in terms of the successor function S. It however can make some difference whether $\leq$ is a primitive symbol or defined in terms of $+$. To speak unambiguously, let the *language of* R be $\{+, \cdot, \overline{0}, \overline{1}, \overline{2}, \dots, \leq\}$ with infinitely many constants as names for natural numbers and with $\leq$ as primitive symbol.

An example of a sentence provable in theory R is $\overline{n} \leq \overline{n}$: indeed, $\Omega5$ says $\overline{n} \leq \overline{n} \vee \overline{n} \leq \overline{n}$. Another example of a provable sentence is $\overline{k} \leq \overline{n}$ for $k < n$: indeed, inside R one may reason that if $\overline{k} \not\leq \overline{n}$, then $\overline{n} \leq \overline{k}$ by $\Omega5$; then $\overline{n}$ is one of the numbers $\overline{0}, .., \overline{k}$ by $\Omega4$; that is however impossible by $\Omega3$. These two examples show that a scheme similar to $\Omega4$, namely

$\Omega4'$: $\forall x(x \leq \overline{n} \equiv x = \overline{0} \vee \,..\, \vee x = \overline{n})$,

is provable in R. The sentence $\forall y(0+y=0 \rightarrow y=0)$ is unprovable in R, as can again be easily verified by constructing an appropriate model.

The theory R is considerably weaker than Q in the sense that Q is not interpretable in R. This fact can be proved by the following argument, due to Petr Hájek: if Q were interpretable in R, then it would also be interpretable in some finite fragment of R; it is however easy to verify that each finite fragment of R has a finite model.

If the scheme $\Omega5$ is removed from R, the resulting theory with only $\Omega1$–$\Omega4$ is *not* essentially undecidable. This can be seen by mapping the symbols $+$ and $\cdot$ to addition and multiplication of real numbers, mapping numerals to (real) numbers 0, 1, 2, ... and by mapping $\leq$ to *empty* relation; the resulting model is a decidable structure by Tarski's theorem on decidability of reals.

However, an interesting theory, named *theory* R_0 in Jones and Shepherdson (1983), is obtained by dropping $\Omega5$ and by replacing $\Omega4$ by $\Omega4'$. So axioms of R_0 are $\Omega1$–$\Omega3$ and $\Omega4'$.

Using, in R_0, the implication $\leftarrow$ in $\Omega 4'$, one can easily prove $\overline{k} \leq \overline{n}$ for $k \leq n$. Also, $\overline{k} \not\leq \overline{n}$ for $k > n$ can be proved using the implication $\rightarrow$ in $\Omega 4'$, and then $\Omega 3$. An example of a sentence provable in R but unprovable in R_0 is $\forall y(0 \leq y)$.

A thing to notice is that the Σ-completeness theorem is provable in R_0 and thus R_0 is undecidable, whereas the scheme $\Omega 5$ usually plays a role when proving some of the additional conditions like (i)–(iii) in section 2. Long ago, this fact led the author to a conjecture that R_0 was undecidable but not essentially undecidable, and that the scheme $\Omega 5$ was intimately connected to the Rosser trick and to essential undecidability in general. However, a result of Cobham, mentioned in Vaught (1962) and in Jones and Shepherdson (1983), throws a doubt (better, *ruins*) this conjecture: R is interpretable in R_0.

Interpretability of R in R_0 directly implies essential undecidability of R_0. However, neither essential undecidability nor interpretability of R in R_0 seem to automatically imply that the self-reference theorem is valid for R_0. Nevertheless, we can adopt the Cobham's result, as presented in Jones and Shepherdson (1983), to show that it *is* valid.

We first define, inside R_0, three auxiliary notions. Since the sentence $\forall x(x \leq \overline{n} \;\&\; x \neq \overline{n} \equiv x \leq \overline{n} \;\&\; \overline{n} \not\leq x)$ is provable in R_0, there are two reasonable ways of defining *strict order*. Let us opt for the first and say that $x < y$ if $x \leq y \;\&\; x \neq y$. Let a number y be *regular* if $0 \leq y$ and $\forall v(v < y \rightarrow v+\overline{1} \leq y)$. Then a binary relation $\lessdot$ is defined as follows: $x \lessdot y$ iff $x \leq y$ or y is not regular.

Lemma 2 *The following facts are provable in* R_0 *for each* n*:*
(a) the number $\overline{n}$ *is regular,*
(b) $\forall x(x \leq \overline{n} \equiv x \lessdot \overline{n})$,
(c) $\forall x(x \lessdot \overline{n} \vee \overline{n} \lessdot x)$.

Proof (a) $\overline{0} \leq \overline{n}$ follows from $\leftarrow$ in $\Omega 4'$. Assume $v < \overline{n}$, i.e. $v \leq \overline{n}$ and $v \neq \overline{n}$. We have $v = \overline{0} \vee \ldots \vee v = \overline{n}$ and simultaneously $v \neq \overline{n}$. So v is one of the numbers $0, \ldots, \overline{n-1}$. Then $\Omega 1$ says that $v + \overline{1}$ equals some of the numbers $\overline{1}, \ldots, \overline{n}$. All these numbers are known to be $\leq \overline{n}$.

(b) Follows directly from (a).

(c) Assume, for example, that $n = 3$ and reason in R_0 again: Let x be given. We may assume that x is regular because otherwise $\overline{3} \lessdot x$. So $\overline{0} \leq x$. If $\overline{0} = x$ then we are done since $\overline{0} \lessdot \overline{3}$. Otherwise, i.e. if $\overline{0} < x$, we can apply the condition in the definition of regular number to $v := \overline{0}$ and obtain $\overline{1} \leq x$. If $\overline{1} = x$ then $x \lessdot \overline{3}$. If $\overline{1} < x$ then we can take $v := \overline{1}$ and obtain $\overline{2} \leq x$. Once again, if $\overline{2} = x$ then $x \lessdot \overline{3}$, and if not then $\overline{3} \lessdot x$. ∎

The previous Lemma shows how the Cobham's result is obtained: if we take the formula $x = x$ for the domain and if we map $\leq$ to $\lessdot$ and map the remaining symbols of R_0 to themselves, we have an interpretation of R in R_0.

Theorem 3 *The self-reference theorem is provable already in* R_0.

Proof The proof has two parts, first verifying that the condition (ii) above, weak representability of recursive functions, is true already for R_0, and then proving the self-reference theorem itself with the help of this condition (ii). We omit the second part as standard. The first part is also standard, the only change being that $\lessdot$ is used instead of $\leq$. To keep the paper self-contained, we give the proof of the first part.

So let a recursive function f of one variable be given. Let $\lambda(x, y, v)$ be a Δ_0-formula such that $\exists v \lambda(x, y, v)$ defines the graph of f in $\mathbf{N}$:

$$m = f(n) \Leftrightarrow \mathbf{N} \models \exists v \lambda(\overline{n}, \overline{m}, v). \tag{1}$$

Let $\gamma(x, y)$ be the following formula:

$$\begin{aligned} &\exists w(y \lessdot w \;\&\; \exists v(v \lessdot w \;\&\; \lambda(x, y, v)) \;\&\\ &\qquad \& \; \forall y' \forall u(y' \lessdot w \;\&\; u \lessdot w \;\&\; \lambda(x, y', u) \rightarrow y = y')). \end{aligned} \tag{2}$$

We claim and verify that $\gamma(x, y)$ weakly represents f, i.e. that

$$\mathrm{R}_0 \vdash \forall y(\gamma(\overline{n}, y) \equiv y = \overline{f(n)}) \tag{3}$$

for each n. So let n_0 be given. Let $m_0 = f(n_0)$. Using (1) and the Σ-completeness theorem, we may take k_0 such that

$$\mathrm{R}_0 \vdash \lambda(\overline{n_0}, \overline{m_0}, \overline{k_0}). \tag{4}$$

We also know from (1) that

$$\mathrm{R}_0 \vdash \neg\lambda(\overline{n_0}, \overline{m}, \overline{k}) \quad \text{for each } m \neq m_0 \text{ and each } k. \tag{5}$$

Put $q = \max\{m_0, k_0\}$. Reason in R_0.

We know $\overline{k_0} \leq \overline{q}$. From Lemma 2(b) we have $\overline{k_0} \lessdot \overline{q}$, and thus (4) yields $\exists v(v \lessdot \overline{q} \;\&\; \lambda(\overline{n_0}, \overline{m_0}, v))$. Similarly, $\overline{m_0} \lessdot \overline{q}$. So the first and second conjunct in parenthesis in (2) are true for $y := \overline{m_0}$ and $w := \overline{q}$.

To verify the third conjunct, let y' and u be such that $y' \lessdot \overline{q}$ and $u \lessdot \overline{q}$ and $\lambda(\overline{n_0}, y', u)$. By Lemma 2(b) and $\Omega 4'$, both y' and u must be one of the numbers $\overline{0}, \ldots, \overline{q}$. However, (5) yields $\overline{m_0} = y'$. So $\gamma(\overline{n_0}, \overline{m_0})$.

Thus we know that the implication $\leftarrow$ in (2) is true. To verify $\rightarrow$, reason in R_0 again.

Let y be such that $\gamma(\overline{n_0}, y)$. So there exist w and v satisfying conditions: $y \lessdot w$, $v \lessdot w$, $\lambda(\overline{n_0}, y, v)$, and $\forall y' \forall u(y' \lessdot w \;\&\; u \lessdot w \;\&\; \lambda(\overline{n_0}, y', u) \rightarrow y = y')$. By Lemma 2(c), it is sufficient to distinguish cases $w \lessdot \overline{q}$ and $\overline{q} \lessdot w$. Assume first that $w \lessdot \overline{q}$. Then w must be one of the numbers $\overline{0}, \ldots, \overline{q}$. Since $v \lessdot w$

and $y \lessdot w$, also v and y must be one of these numbers. However, (5) says that $\lambda(\overline{n_0}, y, v)$ can hold only for such pair $[y, v]$ where $y = \overline{m_0}$. So $y = \overline{m_0}$. Assume now that $\overline{q} \lessdot w$. Then we also have $\overline{k_0} \lessdot w$ and $\overline{m_0} \lessdot w$. Since we know that $\lambda(\overline{n_0}, \overline{m_0}, \overline{k_0})$, we can apply the condition $\forall y' \forall u(\dots)$ to $y' := \overline{m_0}$ and $u := \overline{k_0}$. Then $y = \overline{m_0}$ follows. ∎

Let me finally remark that all axioms of R_0 are (can be rewritten as) Σ-sentences, and thus must be provable in all theories satisfying the Σ-completeness theorem. So besides essential undecidability, R_0 is the weakest theory satisfying Σ-completeness (in its language). It is also known, see Jones and Shepherdson (1983), that if the symbol + and the scheme Ω1 is dropped from R_0 then the resulting theory is still essentially undecidable.

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