Relatives of Robinson Arithmetic

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Abstract

We consider two theories of concatenation, F and TC, proposed by Grzegorczyk, and proofs of their mutual interpretability. We also discuss the Grzegorczyk's project of replacing, in foundational studies, the Robinson's Q by some variant of the theory of concatenation.

1 Introduction: numbers, or strings?

Robinson arithmetic Q was introduced in Tarski, Mostowski, and Robinson (1953) as a base axiomatic theory for investigating incompleteness and undecidability. It is very weak, but all its recursively axiomatizable consistent extensions are both incomplete and undecidable. In logic textbooks, it often plays the role of the weakest reasonable theory with this property.

A. Grzegorczyk recently proposed to study the *theory of concatenation* as a possible alternative theory for studying incompleteness and undecidability. Unlike Robinson (or Peano) arithmetic, where the individuals are numbers that can be added or multiplied, in the theory of concatenation one has *strings* (or *texts*) that can be concatenated. So in the language of the theory of concatenation there is a binary function symbol $\frown$ for laying two strings end to end to form a new string. Axioms of the theory of concatenation postulate e.g. associativity of the operation $\frown$, or the existence of irreducible, i.e.

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single-letter, strings. Particular variants of the theory of concatenation may differ in the number of irreducible strings (with two as the most obvious choice), or in the existence of the empty string.

Before Grzegorczyk, some aspects of concatenation were considered and some axioms were formulated by Quine (1946) and Tarski. One variant of the theory, called theory F, appears already in the book Tarski et al. (1953), where it is claimed but not proved that F is essentially undecidable.

Grzegorczyk's motivation to study the theory of concatenation is philosophical. When reasoning or when performing a computation, we deal with texts. Our human capacity to perform these intellectual tasks depends on our ability to discern texts. Then it is natural to define notions like undecidability directly in terms of texts, without reference to natural numbers. When proving Gödel 1st incompleteness theorem, choosing the theory of concatenation as the base theory could be preferable to choosing Peano or Robinson arithmetic, because then one of the essential steps in the incompleteness proof, formalization of logical syntax, would be practically effortless.

We will discuss properties of two theories of concatenation, theory F defined in Tarski et al. (1953) and theory TC proposed by Grzegorczyk. It appears that an appropriate method of showing undecidability of all consistent extensions is proving mutual interpretability of these theories with Robinson arithmetic Q. We will consider methods of constructing interpretations, one of these being the well known Solovay method of shortening of cuts. We will also discuss the Grzegorczyk's project of replacing Robinson's Q by some version of theory of concatenation in more details. The pros of the project are obvious, but there are also some cons.

2 Some preliminaries

For an axiomatic theory T, let $\mathrm{Thm}(T)$ be the set of all sentences *provable* in T, in symbols $\mathrm{Thm}(T) = \{\varphi; T \vdash \varphi\}$, and let $\mathrm{Ref}(T)$ be the set of all sentences *refutable* in T, in symbols $\mathrm{Ref}(T) = \{\, \varphi\,;\, T \vdash \neg\varphi \,\}$. A theory T is *consistent* if $\mathrm{Thm}(T) \cap \mathrm{Ref}(T) = \emptyset$, i.e. if no sentence of T is simultaneously provable and refutable in T. A theory T is *complete* if it is consistent and each sentence of T is either provable or refutable in T. A theory T is *recursively axiomatizable* if it is equivalent to a theory T' with an algorithmically decidable set of axioms (i.e. with T'

algorithmically decidable). A theory is *decidable* if there exists an algorithm that decides about its provability, i.e. if the set $\mathrm{Thm}(T)$ is algorithmically decidable.

A theory S is an *extension* of a theory T if the language of T (i.e. the set of all non-logical symbols of T) is a subset of the language of S, and each sentence of T provable in T is provable also in S. A theory T is *essentially incomplete* if no recursively axiomatizable extension of T is complete; T is *essentially undecidable* if no consistent extension of T is decidable. It is known that a theory is essentially incomplete if and only if it is essentially undecidable. Thus we use these notions interchangeably or, following Grzegorczyk, we preferably speak about essential undecidability.

An *interpretation* of a theory T in a theory S is a mapping from formulas of T to formulas of S that well-behaves w.r.t. logical symbols and maps all axioms of T to sentences provable in S. A theory T is *interpretable* in S if there exists an interpretation of T in S. The notion of interpretation, as well as the notion of essential undecidability, first appeared in Tarski et al. (1953). Important facts about interpretability are the following: (i) if T is interpretable in S and S is consistent then T is consistent, too; (ii) if T is interpretable in S and T is essentially undecidable then then S is essentially undecidable, too. The notion of interpretability can be used as a means to measure strength of axiomatic theories: if T is interpretable in S and vice versa, i.e. if T and S are *mutually interpretable,* then we can think that T and S represent the same expressive and deductive strength.

3 The importance of Robinson arithmetic

Robinson arithmetic Q is an axiomatic theory having seven simple axioms formulated in the language $\{+, \cdot, 0, \mathrm{S}\}$ with symbols for addition and multiplication (of natural numbers), a constant for the number zero, and a unary function symbol S for the successor function $x \mapsto x + 1$. *Peano arithmetic* PA is obtained from Q by adding the induction schema. The theory $\mathrm{I}\Delta_0$ is like Peano arithmetic, but with the induction schema restricted to Δ_0-formulas (*bounded* formulas) only. The theory $\mathrm{I}\Delta_0{+}\Omega_1$ is $\mathrm{I}\Delta_0$ enhanced by the axiom asserting the totality of the function $x \mapsto x^{\log x}$. For a non-expert, the properties of natural numbers expressible by Δ_0-formulas constitute a class that is a subclass of all algorithmically decidable properties. An example of a Δ_0-formula is the formula $\exists v(v \cdot x = y)$, i.e. the formula the number x is a divisor of the number y. Other two examples are the number x is prime and the number x is divisible by some prime. An example of a formula that is *not* Δ_0 is there exists a $y > x$ such that $y \neq 0$ and y is divisible by all v such that $v \neq 0$ and $v \leq x$; this formula speaks about a thing similar to the factorial of x. Another example of a non-Δ_0 formula is there exists a y such that $y > x$ and y is prime. In the theory $\mathrm{I}\Delta_0$, one cannot prove that a factorial of x exists for each number x, while provability of the sentence a prime $y > x$ exists for each x is a difficult open problem. Both sentences are easily proved by unrestricted induction, i.e. in Peano arithmetic.

Basic properties of natural numbers, like associativity and commutativity of addition and multiplication, are provable in $\mathrm{I}\Delta_0$, but unprovable in Q. Generally, universal sentences are seldom provable in Q. However, $\mathrm{I}\Delta_0{+}\Omega_1$ is interpretable in Q. Gödel 1st incompleteness theorem, or better, its Rosser generalization, says that any recursively axiomatizable extension of Q is incomplete. So Q is essentially incomplete (essentially undecidable). The meaning of Gödel 2nd incompleteness theorem is somewhat questionable for Q. However, its usual proof goes through in $\mathrm{I}\Delta_0{+}\Omega_1$ without any changes.

Thus Robinson arithmetic Q is a very weak but still essentially undecidable theory. It represents a rich "degree of interpretability" because a lot of stronger theories, like $\mathrm{I}\Delta_0{+}\Omega_1$, are interpretable in it. Since it is finitely axiomatizable, it can be used in a straightforward proof of undecidability of classical predicate logic.

4 The theory TC

The theory of concatenation TC has the language $\{{}^\frown, \varepsilon, \mathsf{a}, \mathsf{b}\}$ with a binary function symbol ${}^\frown$, a constant ε for the empty string, and two other constants a and b. We usually omit the symbol ${}^\frown$, i.e. write xy for the concatenation $x{}^\frown y$ of the strings x and y. The axioms of TC are the following:

TC1: $\forall x(x\varepsilon = \varepsilon x = x)$,

TC2: $\forall x \forall y \forall z(x(yz) = (xy)z)$,

TC3: $$\forall x \forall y \forall u \forall v(xy = uv \rightarrow \exists w((xw = u \;\&\; wv = y) \;\vee \\ \vee\; (uw = x \;\&\; wy = v))),$$

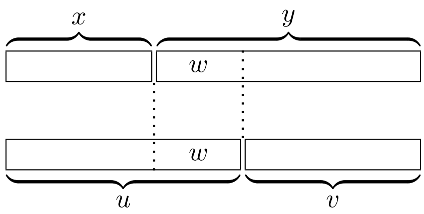

Figure 1: The editors axiom

TC4: $\mathtt{a} \neq \varepsilon \ \& \ \forall x \forall y (xy = \mathtt{a} \ \rightarrow \ x = \varepsilon \ \vee \ y = \varepsilon)$,

TC5: $\mathtt{b} \neq \varepsilon \ \& \ \forall x \forall y (xy = \mathtt{b} \ \rightarrow \ x = \varepsilon \ \vee \ y = \varepsilon)$,

TC6: $\mathtt{a} \neq \mathtt{b}$.

The axioms TC1 and TC2 can be described as axioms of semigroups; by TC2 we can omit parentheses in expressions whenever convenient. The axioms TC4-TC6 postulate that the strings a and b are different, and each of them is non-empty and irreducible (cannot be non-trivially decomposed into two strings). The axiom TC3 is called *editors axiom* in Grzegorczyk (2005). It describes what happens if two editors of a large work independently suggest splitting the text into two volumes. If their suggestions are x, y and u, v respectively, as shown in Fig. 1, then the first volume of one of the editors consists of two parts: the other editor's first volume, and a text w (possibly empty) that simultaneously occurs as a starting part of the other editor's second volume. In Ganea (2009) this text w is called an *interpolant* (of the equation $xy = uv$).

The theory TC was defined in Grzegorczyk (2005). However, the editors axiom is attributed to Tarski, and the idea about the importance of concatenation in incompleteness proofs can be traced back to Quine, who in Quine (1946) cites Tarski and Hermes and says: *Gödel's proof ... depended on constructing a model of concatenation theory within arithmetic.* Note that Quine does not list any axioms, and thus when he says "concatenation theory", he in fact means its standard model (defined below). Undecidability of TC is shown in Grzegorczyk (2005). Later Grzegorczyk and Zdanowski (2008) showed essential undecidability of TC. They also showed that by removing any of the axioms TC1, TC3–TC6 one obtains a theory that is not essentially undecidable. The question whether Robinson arithmetic is interpretable in TC is left open in Grzegorczyk and Zdanowski (2008). A. Visser and R. Sterken, see Visser (2009), M. Ganea in Ganea (2009), and the present author (Švejdar, 2009) independently gave a positive answer to this question. We give more information about interpretability in (and of) TC in Section 5 below.

The papers Grzegorczyk (2005) and Grzegorczyk and Zdanowski (2008) work with a variant of TC having no empty string. Then, for example, the axiom TC4 has the form $\forall x \forall y (xy \neq \mathsf{a})$. The paper Švejdar (2009) works with a variant of TC having three instead of two irreducible strings. The exact choice of variant of the theory is a matter of taste because, as shown in Grzegorczyk and Zdanowski (2008), all variants of the theory of concatenation are mutually interpretable, provided the irreducible strings are at least two in number.

Let A be the set $\{\mathsf{a}, \mathsf{b}\}^*$ of all strings in the two-letter alphabet $\{\mathsf{a}, \mathsf{b}\}$, and let $\mathbf{A}$ be the structure with A as a universe, with concatenation defined "normally" and with constants a and b realized by a and b respectively. Then $\mathbf{A}$ is the *standard model of* TC. The structure $\mathbf{B}$ having the set $B = \{\mathsf{a}, \mathsf{b}, \mathsf{e}\}^*$ as its universe and with all symbols also defined normally is another example of a model of TC. Let $x \sqsubseteq y$ mean $\exists u \exists v (uxv = y)$, and let $x \sqsupset y$ mean $\exists u (ux = y)$. The formulas $x \sqsubseteq y$ and $x \sqsupset y$ can be read the string x is a substring of y and the string y ends by x respectively. The model $\mathbf{B}$ above shows that the sentence $\forall x (x \neq \varepsilon \rightarrow \mathsf{a} \sqsubseteq x \vee \mathsf{b} \sqsubseteq x)$ is not provable in TC.

The following theorem gives some more examples of provable and unprovable sentences. Its purpose is to give the reader some feeling about provability in TC.

Theorem 1 *The following sentences (a)–(d) are provable in* TC*,*
(a) $\forall x (x\mathsf{a} \neq \varepsilon)$*,*
(b) $\forall x \forall y (xy = \varepsilon \rightarrow x = \varepsilon \ \& \ y = \varepsilon)$*,*
(c) $\forall x \forall y (x\mathsf{a} = y\mathsf{a} \rightarrow x = y)$*,*
(d) $\forall x \forall y (\mathsf{a} \sqsupset xy \rightarrow y = \varepsilon \vee \mathsf{a} \sqsupset y)$*,*
while the following sentence (e) is not *provable in* TC*:*
(e) $\forall x \forall y \forall z (xz = yz \rightarrow x = y)$*.*

Proof (a) Assume that $x\mathsf{a} = \varepsilon$. Then, using TC1 and TC2, we have $(\mathsf{b}x)\mathsf{a} = \mathsf{b}$. Irreducibility of b, i.e. TC5, yields $\mathsf{b}x = \varepsilon$ or $\mathsf{a} = \varepsilon$. The latter is excluded by TC4. Then from $\mathsf{b}x = \varepsilon$, $(\mathsf{b}x)\mathsf{a} = \mathsf{b}$, and TC1 we have $\mathsf{a} = \mathsf{b}$, a contradiction with TC6.

(b) If $xy = \varepsilon$ then $x(y\mathsf{a}) = \mathsf{a}$ using TC1 and TC2. So $x = \varepsilon$ or $y\mathsf{a} = \varepsilon$ by TC4. From (a) we have $x = \varepsilon$. Then $xy = \varepsilon$ yields $y = \varepsilon$.

(c) Let $x\mathsf{a} = y\mathsf{a}$. By the editors axiom TC3, there exists a w such that $xw = y$ & $w\mathsf{a} = \mathsf{a}$ or $yw = x$ & $w\mathsf{a} = \mathsf{a}$. Consider the first case, the second one is symmetric. From $w\mathsf{a} = \mathsf{a}$ and irreducibility of a we have $w = \varepsilon$. From that and $xw = y$ we indeed have $x = y$.

(d) Let $\mathsf{a} \sqsupset xy$, and let u be such that $u\mathsf{a} = xy$. The axiom TC3 yields a w satisfying $uw = x$ & $wy = \mathsf{a}$, or $xw = u$ & $w\mathsf{a} = y$. In the second case we obviously have $\mathsf{a} \sqsupset y$. So consider the first case. From $wy = \mathsf{a}$ we have $w = \varepsilon$ or $y = \varepsilon$. If $y = \varepsilon$ then we are done. If $w = \varepsilon$ then $y = \mathsf{a}$, and thus $\mathsf{a} \sqsupset y$.

(e) Let D be the set of all strings in $\{\mathsf{a}, \mathsf{b}, \mathsf{e}\}^*$ that have no occurrences of ae. Realize a and b by a and b respectively, and define $x + y$ accordingly: $x+y$ results from xy by repeating the substitution $\mathsf{ae} \to \mathsf{e}$ while possible. For example, $\mathsf{bab} + \mathsf{eb} = \mathsf{babeb}$, but $\mathsf{baa} + \mathsf{eb} = \mathsf{beb}$. One can check, in case of TC3 with a little effort, in case of the remaining axioms rather easily, that the structure $\mathbf{D} = \langle D, +, \varepsilon, \mathsf{a}, \mathsf{b}\rangle$ is a model of the theory TC. In $\mathbf{D}$ we have $\mathsf{a} + \mathsf{e} = \varepsilon + \mathsf{e}$. So the formula $x^\frown z = y^\frown z$ is not true in $\mathbf{D}$ if x, y, z are evaluated by a, the empty string, and e respectively, and thus the sentence (e) is not valid in $\mathbf{D}$. ∎

Another useful sentence is $\forall x \forall y(\mathsf{a} \sqsubseteq xy \to \mathsf{a} \sqsubseteq x \vee \mathsf{a} \sqsubseteq y)$. We leave its proof in TC as an exercise. More about the theory TC and about its models is in Visser (2009).

5 The theory F, interpretability

Theorem 2 *Robinson arithmetic* Q *is interpretable in* TC.

Proof We only give the basic idea of the proof given in Švejdar (2009). The full proof is rather technical.

When constructing an interpretation, one first has to specify its *domain*, which in our case means to work in TC and select strings that will play the role of natural numbers. It appears that the following definition works:

$$\mathrm{Num}(x) \equiv \forall u(u \sqsubseteq x \ \&\ u \neq \varepsilon \to \mathsf{a} \sqsupset u),$$

a string x is a number if each non-empty substring of x ends by $\mathtt{a}$. Note that, in the model $\mathbf{D}$ in the proof of Theorem 1, the string $\mathtt{e}$ starts by $\mathtt{a}$ (since $\mathtt{e} = \mathtt{a} + \mathtt{e}$). However, $\mathtt{e}$ is not a number because it is a non-empty substring of itself and cannot be written as $\mathtt{e} = z + \mathtt{a}$, i.e. does not end by $\mathtt{a}$.

Having numbers, addition is interpreted as concatenation, zero is interpreted as the empty string ε, and the successor function S is interpreted as the function $x \mapsto x\mathtt{a}$. These definitions work because in TC one can prove that ε and $\mathtt{a}$ are numbers and that numbers are closed under concatenation. All axioms of Q about 0, S, and + translate to sentences provable in TC under this interpretation.

To interpret *multiplication*, a straightforward idea is to first define the notion of a witnessing sequence. A sequence of pairs $[u_0, v_0], \ldots, [u_q, v_q]$ is a witnessing sequence for $x \cdot y$ if: $u_0 = v_0 = \varepsilon$, for each $i < q$ the pair $[u_{i+1}, v_{i+1}]$ equals $[u_i\mathtt{a}, v_i y]$, and $u_q = x$. Then one can define that $x \cdot y = z$ if there exists a witnessing sequence for $x \cdot y$ with $[x, z]$ as the last member. The problem here is that in TC it is not possible to prove that a witnessing sequence exists for each choice of x, y, and it is also not possible to prove that if it exists, it is uniquely determined. A way how to overcome this problem is interpreting not the full Robinson arithmetic Q, but rather its variant Q^- in which addition and multiplication are non-total functions. Then the result is obtained by combining the constructed interpretation of Q^- in TC with a fact known from Švejdar (2007) that Q is interpretable in Q^-.

■

The theory Q^- used in the proof of Theorem 2 was also introduced by Grzegorczyk. The interpretation of Q in Q^- in Švejdar (2007) is constructed using the *Solovay method of shortening of cuts.* This method is now widely known, but was never published: it is only explained in a letter to Petr Hájek (Solovay, 1976). M. Ganea in Ganea (2009) gives a different proof of interpretability of Q in TC, but he also uses the detour via Q^-. Sterken and Visser give a proof not using Q^-, see Visser (2009).

A consequence of the fact that Q is interpretable in TC is essential undecidability of TC. All proofs of interpretability of Q in TC are somewhat involved, but still simpler than the direct proof of essential undecidability of TC given in Grzegorczyk and Zdanowski (2008). These interpretability proofs might use some ideas developed by Grzegorczyk and Zdanowski: that is certainly true about the author's proof in Švejdar (2009).

Since TC is interpretable in $\mathrm{I}\Delta_0$, the theories TC and Q are mutually interpretable; thus they represent the same expressive and deductive power. This is a piece of information missing in Grzegorczyk and Zdanowski (2008).

An interesting alternative theory of concatenation is the *theory* F. It has the same language as TC, and its axioms are:

F1: $\forall x(x\varepsilon = \varepsilon x = x)$,

F2: $\forall x \forall y \forall z(x(yz) = (xy)z)$,

F3: $\forall x \forall y \forall z(yx = zx \lor xy = xz \to y = z)$,

F4: $\forall x \forall y(x\mathtt{a} \neq y\mathtt{b})$,

F5: $\forall x(x \neq \varepsilon \to \exists u(x = u\mathtt{a} \lor x = u\mathtt{b}))$.

Axioms F1 and F2 are the same as axioms TC1 and TC2 of TC. It is easy to verify that axiom F4 is provable in TC; axioms F3 and F5, as is evident from models **D** and **B** in the previous section, are not provable in TC. From the opposite point of view, axioms TC4–TC6 and sentences (a) and (b) in Theorem 1 are examples of sentences provable in F; we leave their proofs to the reader as an interesting exercise. Albert Visser, (Visser, 2009), has constructed a model $\mathbf{M}$ of F such that $\mathbf{M} \not\models \forall x \forall y(\mathtt{a} \sqsubseteq xy \to \mathtt{a} \sqsubseteq x \lor \mathtt{a} \sqsubseteq y)$. Thus in F, one can have strings w_1 and w_2 such that $\mathtt{a} \sqsubseteq w_1 w_2$, $\mathtt{a} \not\sqsubseteq w_1$, $\mathtt{a} \not\sqsubseteq w_2$; Albert Visser describes this situation as creating a letter *ex nihilo*. A consequence of these remarks is that Thm(TC) and Thm(F) are incomparable sets of sentences.

It is claimed in Tarski et al. (1953) that W. Szmielew and A. Tarski proved essential undecidability of F by interpreting Q in F; however, no proof is given. Ganea (2009) constructed an interpretation of TC in F. In conjunction with Theorem 2, this gives a proof of the theorem of Szmielew and Tarski. We give (a slight simplification of) Ganea's proof below in Theorem 3. Note however, that it is still an interesting historical problem what proof could Szmielew and Tarski have had in mind. Ours (Ganea's) proof implicitly uses the Solovay's shortening technique, formulated long after the book Tarski et al. (1953) was published. A. Visser has some possible explanation of this historical problem.

Theorem 3 (Ganea) TC *is interpretable in* F.

Proof Work in F and define *tame* strings as follows:

$$\mathrm{Tame}(x) \equiv \forall v \forall z(z \sqsupset vx \rightarrow z \sqsupset x \vee x \sqsupset z),$$

where $\sqsupset$ has the same meaning as in TC.

(i) We first show (prove within F) that tame strings are closed under concatenation. So assume that x and y are tame, and let v and z be such that $z \sqsupset vxy$. We need to show that $z \sqsupset xy$ or $xy \sqsupset z$. Since y is tame, we have $z \sqsupset y$ or $y \sqsupset z$. If $z \sqsupset y$ then $z \sqsupset xy$ and we are done. So assume that $y \sqsupset z$ and take t such that $ty = z$. From $z \sqsupset vxy$ we have a u such that $uz = vxy$; thus $uty = vxy$. From axiom F3 we have $ut = vx$. Since x is tame, we have $t \sqsupset x$ or $x \sqsupset t$. Then $ty \sqsupset xy$ or $xy \sqsupset ty$. Since $ty = z$, we indeed have $z \sqsupset xy$ or $xy \sqsupset z$.

(ii) Next we show that if wy is tame, then also w is tame. So let v and z be such that $z \sqsupset vw$. We want to show that $z \sqsupset w$ or $w \sqsupset z$. From $z \sqsupset vw$ we have $zy \sqsupset vwy$. Since wy is tame, we have $zy \sqsupset wy$ or $wy \sqsupset zy$. Then a straightforward use of axiom F3 yields $z \sqsupset w$ or $w \sqsupset z$.

Now we are ready to verify that the domain of tame strings, together with the identical mapping of symbols ($\mathtt{a}$, $\mathtt{b}$, and ε to $\mathtt{a}$, $\mathtt{b}$, and ε respectively, concatenation to concatenation), defines an interpretation of TC in F. It is not difficult to verify that $\mathtt{a}$, $\mathtt{b}$, and ε are tame; this together with (i) means that the domain of tame strings is closed under all operations. The axiom TC1 translates to the sentence $\forall x(\mathrm{Tame}(x) \rightarrow x\varepsilon = \varepsilon x = x)$. This sentence is evidently provable in F. A similar argument shows that axioms TC2 and TC4–TC6 translate to sentences provable in F as well. This is so easy because TC2 and TC4–TC6 are universal sentences.

Thus it remains to prove the the translation of the editors axiom TC3 is provable in F. Note that TC3 is the only axiom of TC that is not a universal sentence; it contains an existential quantifier. Let x, y, u, v, be tame strings such that $xy = uv$. We have to show that there exists a *tame* w satisfying $xw = u \;\&\; wv = y$ or $uw = x \;\&\; wy = v$. Since y is tame, from $uv = xy$ we have $v \sqsupset y$ or $y \sqsupset v$. It is sufficient to consider the latter, the former is symmetric. We have a w such that $wy = v$. Then $uwy = uv$ and $uwy = xy$. From axiom F3 we have

$uw = x$. So w is an interpolant. Since v is tame, from $wy = v$ and (ii) above we know that w is tame. ■

Since F is easily interpretable in $\mathrm{I}\Delta_0$, from the other results mentioned in this paper we know that F and TC are deductively incomparable, but from interpretability point of view they represent the same degree of deductive strength. It may be of some interest to directly interpret F in TC.

Theorem 4 F *is interpretable in* TC*.*

Proof Now in TC, work with *radical* strings, where

$$\mathrm{Rad}(x) \equiv \forall y \forall z (yx = zx \rightarrow y = z).$$

It is not difficult to show that radical strings include ε, `a`, and `b`, and that the domain of all radical strings that are empty or end in either `a` or `b` is closed under concatenation and defines an interpretation of F. ■

6 On the Grzegorczyk's project

Let us repeat from the Introduction that Grzegorczyk's suggestion is to consider strings and concatenation on both formal and metamathematical level. On formal level, the theory of concatenation can serve as an alternative to Robinson arithmetic; on metamathematical level, dealing with texts is philosophically better justified because intellectual activities like reasoning and computing involve working with texts.

So Grzegorczyk's interest, purely philosophical as he puts it, consists in methodologically confronting two approaches: *on one hand the whole metalogical construction in which one uses traditional arithmetical methods and defines recursiveness as Gödel does, and on the other hand the whole metalogical construction in which one does not use numbers but speaks only about texts and concatenation and defines decidability as discernibility*, as done in Grzegorczyk (2005).

The motivations for accepting strings rather then numbers as the basic notion can briefly be summarized as follows:

- in Gödel's argument, the only use of numbers is coding of syntactical objects,
- then Gödel theorems are presented as a part of mathematics, but their significance is broader,

- when reasoning, communicating, or even computing, we deal with texts, not with numbers,
- on metamathematical level, the notion of computability can be defined without reference to numbers.

One could remark that mathematics in not necessarily identified by working with numbers; Gödel theorems could be presented as part of mathematics even if they were reformulated without numbers, and they transcede mathematics regardless whether their formulation involves strings or numbers. With this little remark in mind, one can say that the arguments for superiority of strings over numbers are clear and easily acceptable. The definition of recursiveness without using numbers, as done in Grzegorczyk (2005), is very interesting; knowing how all proofs of Kleene's normal form theorem are, it always makes sense to think about more transparent proofs.

However, it is also possible to find some arguments that speak *contra* strings, or at least for modifying or extending the "string approach". First, when reasoning or computing, we also *substitute*. Creating a grammatically correct sentence in a natural language can be described as substituting into patterns. In logic, we have substitution in formulation of predicate axioms. So one can think that the theory of concatenation, if enhanced by some notion of occurrence or substitution, could better serve its purpose. Second, when proving essential undecidability, one also needs an *order*. Known proofs usually (is it a mistake to say *always*?) contain some sort of *Rosser trick*, i.e. speak about an event that occurs *before* some other event. One can think that considering order is more natural in the environment of numbers than in the environment of strings. In fact, defining an order of strings is one of crucial and rather difficult steps in the essential undecidability proof of TC contained in Grzegorczyk and Zdanowski (2008).

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