Decision Problems of some Intermediate Logics and Their Fragments

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Abstract

We review known facts about computational complexity of intermediate logics and their fragments. We show that the Testability logic KC is, but Gödel-Dummett logic G is *not* conservative over intuitionistic propositional logic IPL w.r.t. purely implicational formulas. The former implies *PSPACE*-completeness of IPL. At least three atoms are needed to construct an example corresponding to the latter.

1 Tautologies and complexity classes

One of the basic notions in logic is that of *tautology*: a propositional formula A is a (classical) tautology if $\forall v(v(A) = 1)$, i.e. if it has the value 1 (true) under every truth evaluation v. Let, e.g., A be the formula $p\&q\rightarrow(\neg r\rightarrow\neg p)$, where some parentheses are omitted because we assume that conjunction & and disjunction $\vee$ have higher priority than implication $\rightarrow$ (and equivalence $\equiv$). This formula A is not a tautology because, for the evaluation v such that $v(p) = v(q) = 1$, $v(r) = 0$, we have $v(A) = 0$. The procedure of finding out whether A is a tautology by going through all relevant truth evaluation is known as the *truth table method*.

As an *algorithm*, the truth table method can be analysed from the point of view of its efficiency, i.e. by estimating its time and space

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requirements. Here time is the time needed to process the given input (given instance of the decision problem), and space is the size of memory needed for axiliary data when processing the given input. Define *length of a formula* A as the number of occurrences of logical connectives and atoms in A. So parentheses do not count; note however that further computational-complexity considerations show that the exact definition of length does not make much difference. Under our definition of length, a formula A of length n can contain as much as $(n+1)/2$ different atoms, and thus there are $2^{(n+1)/2}$ truth evaluations that the truth table method has to take into account when processing the formula A. Likewise, the difference between the functions $n \mapsto 2^{(n+1)/2}$ and $n \mapsto 2^n$ (and $n \mapsto 10^n$, etc.) does not make much difference; these are functions of exponential growth.

Examples of tautologies in elementary logic textbooks usually contain two or three atoms. One of the reasons is that, with four or more atoms, the truth table is unpleasantly long. A formula having 20 different atoms can still fit one single line; the 2^{20} lines of the corresponding truth table is much more than an average (or any) book.

Thus both time and space requirements of the truth table method grow exponentially with the size of the input formula, i.e. the truth table method is an algorithm working in exponential time and in exponential space. While no (essential) improvement in time is known, an improvement in space is possible. This is because rather than writing down all the truth evaluations for the given formula A at once, it is possible to consider only one of them at a time, and reuse the same memory to cycle through all of them. Such improved algorithm works in polynomial space (and exponential time). Since the truth evaluation of a formula A (can be written down so that it) has size not exceeding that of the formula A, it in fact works in linear space. The distinction between linear and polynomial is however not essential for our purpose: both linear and polynomial functions grow slowly in comparison with the exponential function.

The classes of all problems decidable in polynomial time, polynomial space, and exponential time are denoted *P*, *PSPACE*, and *EXPTIME* respectively. The class *P* is often considered to be the class of all efficiently decidable problems. It is clear from the cosiderations above that CPL, the problem to decide whether a given formula is a classical tautology (the letters stand for "classical propositional logic"), is a problem both in *PSPACE* and in *EXPTIME*. In fact,

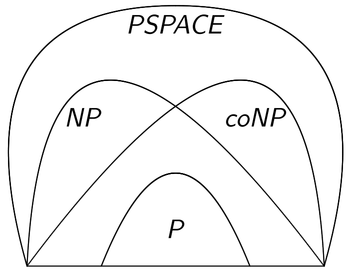

Figure 1: Relations between the classes *P*, *NP*, *coNP* and *PSPACE*

PSPACE is a subclass of *EXPTIME*. It is however *not known* whether CPL is in *P*: no algorithm considerably better than the truth table method has been invented, but a proof that no such algorithm is possible also has not been exhibited.

If $A \notin$ CPL, i.e. if a formula A is not a tautology, it might be difficult to find the truth evaluation v such that $v(A) = 0$. However, once the truth evaluation v is given (or guessed), one can verify in polynomial time that $v(A) = 0$. The class of all decision problems with efficient verifiability of positive instances is denoted *NP*. These are problems efficiently decidable by an algorithm that can guess (proceed non-deterministically). Note that such non-deterministic algorithms have no applications in say software development; the notion of non-deterministic algorithm is a theoretical tool for discriminating between problems that are not efficiently decidable. As to tautologies, CPL is a member of *coNP*, the problems with efficiently verifiable *negative* instances, since it is not tautologies but non-tautologies that can be recognized using a non-deterministic algorithm. The relationship between the classes *P*, *NP*, *coNP*, and *PSPACE* is shown in Fig. 1. It is believed but not proved that all these classes are different. Even $P \neq PSPACE$ is an open problem.

A problem D in some complexity class is *complete* in that class if every member of that class is, in a well defined sense, reducible to D. Problems complete in a class are the most complex (most difficult) problems in that class. Each of the classes *NP*, *coNP*, and *PSPACE* have complete problems in them; indeed, CPL is complete in *coNP*. Proving completenss of a problem can be seen as establishing fully its algorithmic complexity. Completeness of a problem in one class

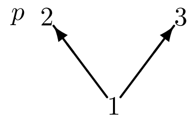

Figure 2: A Kripke model for intuitionistic logic

can be taken as an evidence that the problem is not a member of any smaller class: a *coNP*-complete problem is not in *P*, and a *PSPACE*-complete problem is neither in *NP* nor in *coNP*, in both cases unless two or more classes in Fig. 1 coincide. Such a collapse is not proved to be impossible, but is considered highly unexpected.

Besides CPL, in this paper we will also consider IPL, the decision problem of intuitionistic propositional logic. We will survey results, mostly known, about complexity of IPL and also of some of its subproblems.

2 IPL and its complexity

In intuitionistic propositional logic, we deal with the same formulas as in classical propositional logic. They are built from atoms and the symbol $\bot$ for falsity using the symbols $\rightarrow$, &, $\vee$, $\neg$.

A *Kripke model* (for intuitionistic logic) is a triple $K = \langle W, \leq, \Vdash \rangle$, where W is a non-empty set, $\leq$ is a transitive and reflexive relation on the set W, and $\Vdash$, the *truth relation* of K, is a relation between elements of W and propositional atoms satisfying the *persistency condition*: if $x \Vdash p$ and $x \leq y$ then $y \Vdash p$. The truth relation uniquely extends to a relation (still denoted $\Vdash$) between elements of W and *all* propositional formulas satisfying the following conditions: $x \not\Vdash \bot$, $x \Vdash A \,\&\, B$ iff $x \Vdash A$ and $x \Vdash B$, $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$, $x \Vdash A \rightarrow B$ iff there is no $v \geq x$ such that $v \Vdash A$ and $v \not\Vdash B$, and $x \Vdash \neg A$ iff there is no $v \geq x$ such that $v \Vdash A$. We read $x \leq y$ as "y is accessible from x", and we read $x \Vdash A$ as "x satisfies A" or "A is satisfied in x". The elements of W are called *nodes* (sometimes *possible worlds*). One can easily verify that the persistency condition is true for all formulas, not just atoms.

An example Kripke model is in Fig. 2. In this model the atom p is satisfied in the node 2 and not satisfied in nodes 1 and 3, all remaining atoms are nowhere satisfied. In this model we have $2 \not\Vdash \neg p$, since

there is a node v accessible from 2, namely 2 itself, such that $v \Vdash p$. Similarly, we have $2 \not\Vdash p \rightarrow q$. For still similar reasons, or by the persistency condition, we have $1 \not\Vdash p \rightarrow q$. On the other hand, since $3 \not\Vdash p$ and 3 is the only element accessible from itself, we have $3 \Vdash \neg p$. And similarly, $3 \Vdash p \rightarrow q$.

A model $\langle W, \leq, \Vdash \rangle$ is a *counter-model* of a formula A if there exists an $x \in W$ such that $x \not\Vdash A$. A formula A is an *intuitionistic tautology* if there is no counter-model of A. Let IPL be the set of all intuitionistic tautologies.

An example of an intuitionistic tautology is any formula of the form $\neg\neg A \rightarrow A$. In the model in Fig. 2 we have $1 \not\Vdash \neg\neg p \vee (p \rightarrow q)$. Since this formula is a classical tautology, we have IPL $\neq$ CPL. One can easily show that IPL $\subseteq$ CPL and that any formula A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology. The latter fact is known as *Kolmogorov theorem*. This theorem, i.e. the equivalence $A \in$ CPL $\Leftrightarrow$ $\neg\neg A \in$ IPL, says that CPL is reducible to IPL. Put otherwise, IPL is not algorithmically simpler than CPL. The following theorem says that it is in fact strictly more complicated: *PSPACE*-completeness of IPL implies that IPL $\notin$ *coNP* (unless *PSPACE* = *NP* = *coNP*).

Theorem 1 (Statman, 1979) IPL *is PSPACE-complete.*

For the proof see (Statman, 1979); an alternative later proof in (Švejdar, 2003) might be even simpler. We are not giving the proof in this paper. However, we do give the essential step. It consists in constructing formulas that have Kripke counter-model, but have no small counter-model. Consider the sequence $\{\, D_i \,;\, i \in \mathrm{N} \,\}$ of formulas defined recursively as follows:

$$D_0 = \bot, \qquad D_{n+1} = (D_n \rightarrow q_n) \rightarrow (p_n \rightarrow q_n) \vee (\neg p_n \rightarrow q_n).$$

The formula D_0 contains no atoms and, of course, is not a classical tautology. Each of the remaining formulas D_{n+1} contains atoms $p_0, .., p_n$ and $q_0, .., q_n$ (only) and is a classical tautology: the subformula $(p_n \rightarrow q_n) \vee (\neg p_n \rightarrow q_n)$ itself is a classical tautology. None of the formulas D_n is an intuitionistic tautology. This is proved by the following induction. Assume that K_0 is a counter-model of D_n, i.e. a model with an element a_0 such that $a_0 \not\Vdash D_n$. One can assume that a_0 is the least element of K_0 (its root). Then a counter-model of D_{n+1} can be constructed from K_0, its disjoint copy K_1 with root a_1, and one

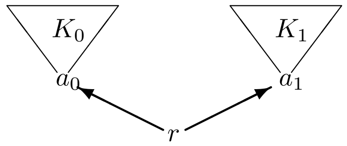

Figure 3: Constructing counter-model of D_{n+1}

additional element r (new root) as shown in Fig. 3. In K_0 and K_1, the atoms $p_0, .., p_{n-1}$ and $q_0, .., q_{n-1}$ have their original values, and they are evaluated negatively in r. The atom q_n is everywhere negative. The atom p_n is positive in a_1 (and hence everywhere in K_1) and negative in all remaining nodes, i.e. inside K_0 and in r. From $a_0 \not\Vdash D_n$ and the persistency condition we have $r \not\Vdash D_n$. Thus D_n is nowhere satisfied, and $r \Vdash D_n \to q_n$. There is a node v accessible from r, namely a_1, such that $v \Vdash p_n$ and $v \not\Vdash q_n$; so $r \not\Vdash p_n \to q_n$. Similarly, there is a node v accessible from r, namely a_0, such that $v \Vdash \neg p_n$ and $v \not\Vdash q_n$; so $r \not\Vdash \neg p_n \to q_n$. Thus indeed, $r \not\Vdash (p_n \to q_n) \vee (\neg p_n \to q_n)$, and $r \not\Vdash D_{n+1}$.

Some more thinking shows that a counter-model of D_{n+1} cannot be much different from the model in Fig. 3: it *must* contain two disjoint copies of a counter-model of D_n. Hence it is at least twice as big. This shows that the formulas D_n are as desired: their sizes grow only polynomially, whereas the sizes of their smallest counter-models grow exponentially.

Statman's theorem and the construction above confirm what one would intuitively think about classical and intuitionistic logic: the latter is algorithmically (strictly) more complicated. This is an instance of a more general phenomenon, observed in various areas of logic. Stronger theories or axiomatic systems, resulting from weaker ones by adding axioms (the reader should feel free to think about, say, set theory and its extensions) cannot be algorithmically more complicated than the weaker ones: the additional axioms forbid something, and their addition simplifies, never complicates, the situation.

There are logics that extend intuitionistic logic but are weaker than classical logic. These logics are called *intermediate*. A natural question reads: where, on the path from intuitionistic logic to classical logic, the *PSPACE*-complete decision problem turns to the simpler *coNP*-complete decision problem? Other thing to note is that we need

more and more atoms to construct the formula D_n above, and that these formulas contain the connectives $\to$, $\vee$, $\neg$, but no conjunctions. So other natural questions are the following. What happens if the number of atoms is restricted (i.e., fixed)? What happens if the use of logical connectives is restricted?

3 Restricting connectives or the number of atoms

The question whether the decision problem simplifies if the number of atoms is fixed is interesting because in classical logic it does simplify. If, for example, the number of possible atoms is 3, then the number of truth evaluations is 8, and the time needed to check whether a formula A with length n is satisfied by all the 8 evaluations grows only moderately (polynomially) with n. So the decision problem of classical logic with a fixed number of atoms is in *P*. However, the following theorem shows that intuitionistic logic is different in this respect.

Theorem 2 (Rybakov, 2006) *The decision problem of intuitionistic logic remains PSPACE-complete even if the number of atoms is restricted to* 2.

The case where there is only one propositional atom is also interesting. In classical logic and with one atom p only, there exist only 4 non-equivalent formulas: p and its negatin $\neg p$, their conjunction $p \& \neg p$ (i.e. $\bot$), and their disjunction $p \vee \neg p$ (i.e. $\top$). In intuitionistic logic, the situation is more complicated since, e.g., $\neg\neg p$ is not equivalent to p, and formulas $\neg\neg p \to p$, $p \vee \neg p$, $\neg p \vee \neg\neg p$, $(\neg\neg p \to p) \to p \vee \neg p$ are not intuitionistic tautologies and are not mutually equivalent. There are infinitely many non-equivalent formulas built up from the atom p only. However, these infinitely many formulas form an interesting and rather well-organized structure called *Rieger-Nishimura lattice*, invented independently by Rieger, Nishimura, de Jongh, ..., see Rieger (1949). Closer inspection of properties of this structure shows that this one-atom fragment of intuitionistic logic is in *P*. As such it can be neither *coNP*-complete nor *PSPACE*-complete.

It should be noted that the precise computational complexity status of the one atom fragment of intuitionistic logic might be an interesting problem. *LOG* is another complexity class, of problems decidable in logarithmic space. *LOG* is a subclass of *P*. While it is known that the decision problem of the one atom fragment of intuitionistic logic in in *P*, it is not known whether it belongs to *LOG*.

The following theorem speaks about the situation where the use of logical connectives is restricted.

Theorem 3 *The purely implicational fragment of intuitionistic logic, i.e. the set of all intuitionistic tautologies built up from (any number of) atoms using implication $\rightarrow$ as the only connective, is PSPACE--complete.*

The proof of this theorem is based on the fact that the construction of the formulas D_n, given above, can be improved so that the formulas are built up using implication only. For the full proof see Švejdar (2003). The proof is also implicit (in fact, almost explicit) in the earlier Statman's paper (Statman, 1979).

An interesting question is what happens if both restrictions apply, i.e. if implication is the only connective and simultaneously the number of atoms is restricted. The answer is given by the following theorem, that immediately follows from results in (Urquhart, 1974):

Theorem 4 (Urquhart, 1974) *For any natural number n, the implicational fragment of intuitionistic logic with n atoms only is decidable in polynomial time.*

In fact, for any fixed n the number of non-equivalent implicational formulas in n atoms, as well as the number of different Kripke models relevant for these formulas, is finite. For example, with one atom p only, p and $p \rightarrow p$ are the only two non-equivalent formulas. With two atoms p and q, it is not quite trivial to verify that the number of non-equivalent formulas, like p, q, $p \rightarrow (q \rightarrow p)$, $p \rightarrow (p \rightarrow q)$, ... is exactly 14. For more information, see the thesis (Blicha, 2010).

4 Remarks on intermediate logics

A *logic* is sometimes defined as a set of propositional formulas closed under the rules modus ponens and substitution. A logic is *consistent* if it is not the set of all formulas. Consistent logics extending the intuitionistic logic (i.e. containing the set IPL) are called *intermediate*. It can be verified that all formulas in an intermediate logic are classical tautologies. Thus CPL is the strongest intermediate logic. Specific intermediate logic are often defined by adding one or more axiom

schemas to intuitionistic logic. For example *Gödel-Dummett* logic G (also denoted LC) is obtained by adding the schema $(A \rightarrow B) \vee (B \rightarrow A)$. *Testability logic* KC (also called Jankov's logic, or De Morgan logic, or the Logic of weak excluded middle) is obtained by adding the schema $\neg A \vee \neg\neg A$ to intuitionistic logic. Recall that the classical logic is obtained from the intuitionistic logic by adding the schema $A \vee \neg A$ (or equivalently, $\neg\neg A \rightarrow A$). A sound argument in intuitionistic logic is this: if $\neg\neg A \rightarrow \neg A$ then $\neg A$; if $\neg A \rightarrow \neg\neg A$ then $\neg\neg A$. This argument shows that testability logic KC is a sublogic of Gödel-Dummett logic G. Thus from computational complexity point of view, G might be simpler than KC but not vice versa, both logics might be simpler than IPL.

One can easily verify that the logic KC is sound with respect to Kripke models with a greatest element. In fact, a completeness theorem with respect to this class holds. Since $\neg A \vee \neg\neg A$ is not an intuitionistic tautology, KC is stronger than IPL. Since it is easy to construct a counter-model of the formula $(p \rightarrow q) \vee (q \rightarrow p)$ having a greatest element, KC is weaker than G. The logic G can be verified to be sound with respect to linearly ordered Kripke models, and it is known to be complete with respect to this class.

Since linearly ordered Kripke models correspond to linearly ordered truth values, Gödel-Dummett logic G is studied as one of *fuzzy logics*. It was originally considered in connection with the question whether intuitionistic logic can be characterized as a logic with finite number of truth values, see Gödel (1932). M. Dummett (Dummett, 1959) showed that, in G, disjunction $\vee$ is expressible in terms of the remaining connectives; neither IPL nor KC have this property.

Testability logic KC is discussed in (Gabbay, 1981). This logic is important in connection with the question which completeness theorems (various formulations for various logics) can be proved if the given logic is accepted on metamathematical level (accepted as metalogic) instead of the classical logic. For more on this see Carter (2008).

A linearly ordered Kripke counter-model of a formula A has size not (significantly) exceeding the size of A. So in the logic G, one cannot construct an expansive sequence of formulas similar to the sequence $\{ D_n ; n \in \mathrm{N} \}$ above. This fact has a consequence that G is a decision problem in *coNP*; thus from algorithmic point of view, G is simpler than IPL. The following theorem says that KC represents the same level of algorithmic complexity as IPL.

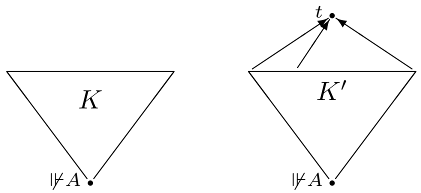

Figure 4: Adding a top node to a Kripke model

Theorem 5 KC *is PSPACE-complete. Its purely implicational fragment is PSPACE-complete as well.*

Proof To prove this theorem, it is sufficient to show that KC is conservative over IPL with respect to purely implicational formulas. So let A be a purely implicational formula such that IPL $\nvdash A$. We have to show that KC $\nvdash A$. Let $K = \langle W, \leq, \Vdash \rangle$ be an intuitionistic counter-model of A. Let $K' = \langle W \cup \{t\}, \leq', \Vdash' \rangle$ be a model as in Fig. 4, constructed from K by adding a new greatest element t, i.e. an alement accessible from everywhere in K. Let all atoms be evaluated positively in t and have the same value in all remaining nodes as they had in K. It is immediate that every purely implicational formula B is satisfied in t. An easy induction shows that for every purely implicational formula B and for each node $a \in W$ we have $a \Vdash B \Leftrightarrow a \Vdash' B$. ∎

The same result is also claimed in (Rybakov, 2006). Out of other popular intermediate logics, see their entry http://en.wikipedia.org/wiki/Intermediate_logic in Wikipedia, Kreisel-Putnam logic and Scott's logic are *PSPACE*-complete as well, because these two logics are sublogics of KC.

It is clear from Theorem 5 and *coNP*-completeness of G that G cannot be conservative over IPL with respect to purely implicational formulas. Indeed,

$$((p \to q) \to r) \to (((q \to p) \to r) \to r)$$

is an example of a purely implicational formula in G which is not an intuitionistic tautology. This example is, of course, derived from the formula $(p \to q) \vee (q \to p)$, and it suggests how it is possible to simulate disjunctions using additional atoms. One can check, using the results

in (Urquhart, 1974), that no such example is possible with two atoms only.

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