On strong forms of reflection in set theory

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The authors acknowledge the generous support of JTF grant Laboratory of the Infinite ID35216.

Abstract. In this paper we review the most common forms of reflection and introduce a new form which we call *sharp-generated reflection*. We argue that sharp-generated reflection is the strongest form of reflection which can be regarded as a natural generalization of the Lévy reflection theorem. As an application we formulate the principle *sharp-maximality* with the corresponding hypothesis IMH$^{\#}$. IMH$^{\#}$ is an analogue of the IMH (Inner Model Hypothesis, introduced in [3]) which is compatible with the existence of large cardinals.

Keywords: Reflection, Inner Model Hypothesis, sharps, indiscernibles.

AMS subject code classification: 03E35,03E55.

1 Introduction

Vertical reflection for the universe V can be intuitively formulated as the following principle, denoted (Refl):

(Refl) Any property which holds in V already holds in some initial segment of V.

(Refl) says that V cannot be described as the unique initial segment of the universe satisfying a given property. The strength of reflection depends on what we consider by *property*; by varying the notion of property we obtain a hierarchy of reflection principles. We say that a given V is *vertically maximal* if it satisfies a formalization of (Refl) which can be viewed, arguably, as being the strongest possible.[1]

The weakest form of reflection, with first-order notion of property, is Lévy's theorem which is provable in ZF.

[1] Indeed, we propose the notion of sharp-generation developed in Section 2.2 as a candidate for an ultimate form of (Refl).

Theorem 1.1 (Lévy) *Let $\varphi(x_1, \dots, x_n)$ be a first-order formula with free variables shown. Then the following is a theorem of* ZF*:*

(1.1)
$$\forall \alpha\, \forall x_1, \dots, x_n \in V_\alpha\, \exists \beta \geq \alpha\, \big(\varphi(x_1, \dots, x_n) \leftrightarrow (V_\beta, \in) \models \varphi(x_1, \dots, x_n)\big).$$

Since the language of ZF is first-order, there is no direct way of generalizing Lévy's theorem to higher-order formulas applied to V. Lévy resolved this problem by studying structures of the form $(V_\kappa, \in, R)$, with R ranging over subsets of V_κ. We say that $\varphi(R)$ true in $(V_\kappa, \in, R)$ reflects if there is some $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha)$ satisfies $\varphi(R \cap V_\alpha)$. For an inaccessible κ, V_κ is thus identified with an approximation of the universe V and higher-order properties attributable to V_κ are expressed as first-order properties in V. It is known that by postulating a range of reflection principles for $(V_\kappa, \in, R)$, one can obtain large cardinals compatible with L (such as weakly compact cardinals).[2]

Reflection principles discussed in the previous paragraph allow φ to be higher-order, but the parameter R itself is always just second-order. Our motivation in this paper is to look for strengthenings of reflection with potential to yield vertical maximality, and which in particular should allow parameters of order higher than 2. For instance, for a third-order parameter $\mathscr{R} \subseteq \mathscr{P}(V_\kappa)$ one is tempted to formulate the following natural-looking principle:

$(*)_3$: If $\varphi(\mathscr{R})$ is true in $(V_\kappa, \in, \mathscr{R})$, then for some $\alpha < \kappa$, $(V_\alpha, \in, \bar{\mathscr{R}})$ satisfies $\varphi(\bar{\mathscr{R}})$, where $\bar{\mathscr{R}} = \{R \cap V_\alpha \mid R \in \mathscr{R}\}$.

However, an easy example shows that $(*)_3$ is inconsistent.[3] In order to retain some sort of reflection with higher-order parameters, we need to tread more carefully. First in Section 2.1, we reformulate $(*)_3$ (and its generalizations) using elementary embeddings internal to V (see Definition 2.1). Seeing that this reformulation has certain drawbacks (in particular it is not compatible with L), we will develop the idea of elementary embeddings in a different way, making the resulting notion compatible with L. This construction – based on indiscernibles and sharp-generation – is described in Section 2.2. An application of a sharp-generated reflection is given in Section 3.

[2]Instead of working with V_κ, one can work directly with V in theories with classes, such as GB. Let R range over classes. We say that $\varphi(R)$ true in V reflects if for some $\alpha < \kappa$, $(V_\alpha, V_{\alpha+1})$ satisfies $\varphi(R \cap V_\alpha)$.

[3]Consider the following example. For any infinite ordinal κ, let $\mathscr{R}$ be the collection of all $\alpha < \kappa$ (viewed as subsets of κ), and consider $\varphi(\mathscr{R})$ which says that every element of $\mathscr{R}$ is bounded in κ (φ is first-order with a third-order parameter $\mathscr{R}$). Clearly, $\varphi(\mathscr{R})$ is true in V_κ. However, $\varphi(\bar{\mathscr{R}})$ is false in V_α for every $\alpha < \kappa$. See [6] for more discussion of reflection with higher-order parameters.

2 Reflection with elementary embeddings

2.1 Embeddings internal to V

To make the following discussion more standard, we will work with structures of the form $H(\kappa)^{+n}$, $0 < n < \omega$. Let $\mathscr{R}$ range over subsets of $H(\kappa^{+n})$; we write

$$(H(\kappa^{+n}), \in, \mathscr{R}) \models \varphi(\mathscr{R}) \tag{2.2}$$

instead of $(H(\kappa), \in, \mathscr{R}) \models \varphi(\mathscr{R})$ to express that $\varphi(\mathscr{R})$ holds in $H(\kappa)$ with appropriately interpreted higher-order quantifiers.[4] The notation in (2.2) has the advantage that it emphasizes that the properties of order $n+1$ over $H(\kappa)$ actually reduce to first-order properties over $H(\kappa^{+n})$, with $\mathscr{R}$ being second-order over $H(\kappa^{+n})$.

The known concept of a *subcompact* cardinal can be used to make sense of reflection for higher-order parameters:

Definition 2.1 *Let κ be an uncountable regular cardinal. We say that κ* satisfies reflection with parameters of order $n+2$, *$0 < n < \omega$, if for every $\mathscr{R} \subseteq H(\kappa^{+n})$ there are a regular uncountable cardinal $\bar{\kappa} < \kappa$, $\bar{\mathscr{R}} \subseteq H(\bar{\kappa}^{+n})$, and an embedding $\pi : H(\bar{\kappa}^{+n}) \to H(\kappa^{+n})$ with critical point $\bar{\kappa}$, $\pi(\bar{\kappa}) = \kappa$, such that*

$$\pi : (H(\bar{\kappa}^{+n}), \in, \bar{\mathscr{R}}) \to (H(\kappa^{+n}), \in, \mathscr{R}) \tag{2.3}$$

is elementary.

Note that demanding $(H(\bar{\kappa}^{+n}), \in, \bar{\mathscr{R}}) \prec (H(\kappa^{+n}), \in, \mathscr{R})$ is contradictory;[5] thus the requirement that π is not the identity is essential.

Remark 2.2 For n, $0 < n < \omega$, κ is κ^{+n}-subcompact iff κ satisfies reflection for parameters of order $n+2$ according to Definition 2.1. Subcompact cardinals were defined by Jensen,[6] and apparently for different reasons than the study of reflection (Jensen isolated the concept of subcompact cardinals for his study of the failure of the square). α-subcompact cardinals can be defined for any cardinal $\alpha > \kappa$, not just the κ^{+n}'s for $n < \omega$, and are therefore suitable for expressing reflection with parameters of transfinite order. For more details about subcompact cardinals, see [2].

[4] For simplicity, we restrict our attention in this section to higher-order properties of finite order.

[5] Set $n = 1$ and choose $\mathscr{R}$ as in the example in Footnote 3. By elementarity, $\bar{\mathscr{R}}$ is equal to $H(\bar{\kappa}^{+}) \cap \mathscr{R}$, which leads to contradiction as in Footnote 3.

[6] Jensen defined κ to be subcompact if it is κ^{+}-subcompact according to our definition.

The definition 2.1 forces no "canonicity" on π; any embedding which satisfies the requirements will do. One might wonder whether more stringent requirements on π, such as demanding constructibility in some sense, might give the definition more structure. However, this cannot be done if by canonicity we mean constructibility in L-like models: by Theorem 2.3, reflection for parameters of order three implies failure of square and for higher orders we get supercompact cardinals (of specific degrees):

Theorem 2.3 *(GCH) The following hold:*

- *(i) For all n, $0 < n < \omega$: κ satisfies reflection with parameters of order $n+4$ iff κ is κ^{n+2}-subcompact iff κ is κ^{+n}-supercompact.*
- *(ii) κ satisfies reflection for parameters of order 4 iff κ is κ^{++}-subcompact iff κ is measurable.*
- *(iii) If κ satisfies reflection for parameters of order 3 (which is the same as being κ^{+}-subcompact), then $\square_\kappa$ fails.*

Proof. For proofs, see for instance [2]. □

There are other versions of strong forms of reflection implying transcendence over L; see for instance [7].

Definition 2.1 seems very natural, but – in our opinion – the postulation of non-canonical elementary embeddings as elements of the universe V turned out to make the resulting principle too strong. Theorem 2.3 contradicts our original intuition regarding (Refl) and its formalization: while we would like to extend the usual form of reflection to higher-order parameters, we wish to retain compatibility with L (see Remark 2.8). A more suitable form of reflection compatible with L is described in next section.

2.2 Sharp-generated reflection

Let us start with V which we view as a transitive set which approximates the real universe. This viewpoint allows us to consider end-extensions $V \subseteq V^*$ of a larger ordinal length. Constructions of this type can be carried out in certain axiomatic theories more complicated than ZF or GB (for example Ackermann's, or theories developed by Reinhardt; see [5], Section 23, for more details). However we think that by treating V as a transitive set model (often countable), we obtain a much stronger (indeed the strongest possible) form of reflection.[7]

Let us extrapolate from the usual reflection and see where it takes us. It is natural to strengthen the reflection of individual first-order properties from

[7]Recall that standard forms of reflection are also formulated with set approximations of the form $(V_\kappa, \in, R)$; however, we do not require V to be a rank-initial segment of the universe which makes it possible to consider countable V's.

V to some V_α to the simultaneous reflection of all first-order properties of V to some V_α, even with parameters from V_α. Thus V_α is an elementary submodel of V. Repeating this process suggests that in fact there should be an increasing, continuous sequence of ordinals $(\kappa_i \mid i < \infty)$ such that the models $(V_{\kappa_i} \mid i < \infty)$ form a continuous chain $V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots$ of elementary submodels of V whose union is all of V (where ∞ denotes the ordinal height of the universe V).

But the fact that for a closed unbounded class of κ's in V, V_κ can be "lengthened" to an elementary extension (namely V) of which it is a rank initial segment suggests via reflection that V itself should also have such a lengthening V^*. But this is clearly not the end of the story, because we can also infer that there should in fact be a continuous increasing sequence of such lengthenings $V = V_{\kappa_\infty} \prec V^*_{\kappa_{\infty+1}} \prec V^*_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals. For ease of notation, let us drop the *'s and write W_{κ_i} instead of $V^*_{\kappa_i}$ for $\infty < i$ and instead of V_{κ_i} for $i \leq \infty$. Thus V equals W_∞.

But which tower $V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of lengthenings of V should we consider? Can we make the choice of this tower "canonical"?

Consider the entire sequence $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$. The intuition is that all of these models resemble each other in the sense that they share the same first-order properties. Indeed by virtue of the fact that they form an elementary chain, these models all satisfy the same first-order sentences. But again in the spirit of "resemblance", it should be the case that any two pairs $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$, $(W_{\kappa_{j_1}}, W_{\kappa_{j_0}})$ (with $i_0 < i_1$ and $j_0 < j_1$) satisfy the same first-order sentences, even allowing parameters which belong to both $W_{\kappa_{i_0}}$ and $W_{\kappa_{j_0}}$. Generalising this to triples, quadruples and n-tuples in general we arrive at the following situation:

$(*)$ Our approximation V to the universe should occur in a continuous elementary chain $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals, where the models W_{κ_i} form a *strongly-indiscernible chain* in the sense that for any n and any two increasing n-tuples $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$, $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$, the structures $W_{\vec{i}} = (W_{\kappa_{i_{n-1}}}, W_{\kappa_{i_{n-2}}}, \cdots, W_{\kappa_{i_0}})$ and $W_{\vec{j}}$ (defined analogously) satisfy the same first-order sentences, allowing parameters from $W_{\kappa_{i_0}} \cap W_{\kappa_{j_0}}$.

But this is again not the whole story, as we would want to impose higher-order indiscernibility on our chain of models. For example, consider the pair of models $W_{\kappa_0} = V_{\kappa_0}, W_{\kappa_1} = V_{\kappa_1}$. Surely we would want that these models satisfy the same second-order sentences; equivalently, we would want $H(\kappa_0^+)^V$ and $H(\kappa_1^+)^V$ to satisfy the same first-order sentences. But as with the pair $H(\kappa_0)^V$, $H(\kappa_1)^V$ we would want $H(\kappa_0^+)^V$, $H(\kappa_1^+)^V$ to satisfy the same first-order sentences *with parameters*. How can we formulate this? For example, consider κ_0, a parameter in $H(\kappa_0^+)^V$ that is second-order with respect to $H(\kappa_0)^V$; we cannot simply require $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_0)$, as κ_0 is the largest cardinal in $H(\kappa_0^+)^V$ but not in $H(\kappa_1^+)^V$. Instead we need to replace the occurrence of κ_0 on the left side with a "corresponding" parameter on the right side, namely κ_1, resulting in the natural requirement $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_1)$. More generally, we should be able to replace each parameter in $H(\kappa_0^+)^V$ by a "corresponding" element of $H(\kappa_1^+)^V$ and conversely, it should be the case that, to the maximum extent possible, all elements of $H(\kappa_1^+)^V$ are the result of such a replacement.This also should be possible for $H(\kappa_0^{++})^V$, $H(\kappa_0^{+++})^V$, ... and with the pair κ_0, κ_1 replaced by any pair κ_i, κ_j with $i < j$.

It is natural to solve this parameter problem using embeddings, as in the last subsection. But the difference here is that there is no assumption that these embeddings are internal to V; they need only exist in the "real universe", outside of V. In this way we will arrive at a principle compatible with $V = L$ in which the choice of embeddings is indeed "canonical".

Thus we are led to the following.

Definition 2.4 *Let W be a transitive set-size model of ZFC of ordinal height ∞. We say that W is* indiscernibly-generated *iff W satisfies the following:*

(i) There is a continuous sequence $\kappa_0 < \kappa_1 < \ldots$ of the length ∞ such that $\kappa_\infty = \infty$ and there are commuting elementary embeddings $\pi_{ij} : W \to W$ where π_{ij} has critical point κ_i and sends κ_i to κ_j.

(ii) For any $i \leq j$, any element of W is first-order definable in W from elements of the range of π_{ij} together with κ_k's for k in the interval $[i, j)$.

The last clause in the above definition formulates the idea that to the maximum extent possible, elements of W are in the range of the embedding π_{ij} for each $i \leq j$; notice that the interval $[\kappa_i, \kappa_j)$ is disjoint from this range, but by allowing the κ_k's in this interval as parameters, we can first-order definably recover everything.

Indiscernible-generation as formulated in the above definition does indeed give us our advertised higher-order indiscernibility: For example, in the notation of the definition, if $\vec{i} = i_0 < i_1 < \ldots < i_{n-1}$ and $\vec{j} = j_0 < j_1 < \ldots < j_{n-1}$ with $i_0 \leq j_0$, and $x_k \in H(\kappa_{i_0}^+)^W$ for $k < n$ then the structure $W_{\vec{i}}^+ = (H(\kappa_{i_{n-1}}^+)^W, H(\kappa_{i_{n-2}}^+)^W, \cdots, H(\kappa_{i_0}^+)^W)$ satisfies a sentence with parameters $(\pi_{i_0,i_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,i_0}(x_0))$ iff $W_{\vec{j}}^+$ satisfies the same sentence with corresponding parameters $(\pi_{i_0,j_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,j_0}(x_0))$. There is a similar statement with W^+ replaced by higher-order structures $W^{+\alpha}$ for arbitrary α.

Indiscernible-generation has a clearer formulation in terms of #*-generation*, which we explain next.

Definition 2.5 *A structure $N = (N, U)$ is called a* sharp with critical point κ*, or just a #, if the following hold:*

(i) N is a model of ZFC^- (ZFC minus powerset, with replacement replaced by the collection principle) in which κ is the largest cardinal and κ is strongly inaccessible.
(ii) (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).
(iii) U is a normal measure on κ in (N, U).
(iv) N is iterable, i.e., all of the successive iterated ultrapowers starting with (N, U) are well-founded, yielding iterates (N_i, U_i) and Σ_1 elementary iteration maps $\pi_{ij} : N_i \to N_j$ where $(N, U) = (N_0, U_0)$.

We will use the convention that κ_i denotes the the largest cardinal of the i-th iterate N_i.

If N is a # and λ is a limit ordinal then $LP(N_\lambda)$ denotes the union of the $(V_{\kappa_i})^{N_i}$'s for $i < \lambda$. (LP stands for "lower part".) $LP(N_\infty)$ is amodel of ZFC.

Definition 2.6 *We say that a transitive model V of ZFC is* #-generated *iff for some sharp $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$, V equals $LP(N_\infty)$ where ∞ denotes the ordinal height of V.*

Fact 2.7 *The following are equivalent for transitive set-size models V of ZFC:*

(i) V is indiscernibly-generated.
(ii) V is #-generated.

Proof. The last clause in the definition of indiscernible-generation ensures that the embeddings π_{ij} in that definition in fact arise from iterated ultrapowers of the embedding π_{01}, itself an ultrapower by the measure U_0 on κ_0 given by $X \in U_0$ iff $\pi_{01}(X)$ contains κ_0 as an element. Conversely, if (N, U) generates V, then the chain of embeddings given by iteration of (N, U) witnesses that V is indiscernibly-generated. □

In our opinion, #-generation fulfils our intuition for being vertical maximal, with powerful consequences for reflection. L is #-generated iff $0^\#$ exists, so this principle is compatible with $V = L$. If V is #-generated via (N, U) then there are embeddings witnessing indiscernible-generation for V which are canonically-definable through iteration of (N, U). Although the choice of # that generates V is not in general unique, it can be taken as a fixed parameter in the canonical definition of these embeddings. Moreover, #-generation evidently provides the maximum amount of vertical reflection: If V is generated by (N, U) as $LP(N_\infty)$ where ∞ is the ordinal height of V, and x is any parameter in a further iterate $V^* = N_{\infty^*}$ of (N, U), then any first-order property $\varphi(V, x)$ that holds in V^* reflects to $\varphi(V_{\kappa_i}, \bar{x})$ in N_j for all sufficiently large $i < j < \infty$, where $\pi_{j,\infty^*}(\bar{x}) = x$. This implies any known form of vertical reflection and summarizes the amount of reflection one has in L under the assumption that $0^\#$ exists, the maximum amount of reflection in L.

Thus #-generation tells us what lengthenings of V to look at, namely the initial segments of V^* where V^* is obtained by further iteration of a # that generates V. And it fully realises the idea that V should look exactly like closed unboundedly many of its rank initial segments as well as its "canonical" lengthenings of arbitrary ordinal height.

Therefore we believe that #-generated models are the strongest formalization of the principle of reflection (Refl) – we call this form of reflection *sharp-generated reflection*, and we shall call these models *vertically maximal*.

Remark 2.8 Notice that a sharp-generated model can satisfy $V = L$, and hence our reflection principle is compatible with L. The reason is that the non-trivial embeddings obtained from the sharp-iteration are external to the model in question. This contrasts with the use of nontrivial embeddings in Section 2.1. Compatibility with L agrees with our intuition that a natural formulation of vertical reflection (Refl) should be determined by the *height* of the universe, and not its *width* (and L has the same height as V).

3 An application

We now apply sharp-generated reflection to formulate an analogue of the IMH principle in [3].

3.1 Vertically maximal models and IMH

The Hyperuniverse is the collection of all countable transitive models of ZFC. We view members of the Hyperuniverse as possible pictures of V which mirror all possible first-order properties of V. The Hyperuniverse Programme, which originated in [3], is concerned with the formulation of natural criteria for the selection of preferred members of the Hyperuniverse. First-order sentences holding in the preferred universes can be taken to be true in the "real V"; in other words, preferred universes may lead to adoption of new axioms. Models satisfying IMH, and IMH$^\#$ introduced below, are examples of such preferred universes.

Definition 3.1 *We say that a #-generated model M is #-maximal if and only if the following hold. Whenever M is a definable inner model of M' and M' is #-generated, then every sentence φ, i.e. without parameters, which holds in a definable inner model of M' already holds in some definable inner model of M.*

We say that a #-generated model M satisfies IMH$^\#$ if it is #-maximal.[8]

Note that IMH$^\#$ differs from IMH by demanding that both M and M', the outer model, are of a specific kind, i.e. should be #-generated (while the outer models considered in IMH are arbitrary). The motivation behind this requirement is that not all outer models count as "maximal"; if our main motivation is formulated in terms of maximality, consideration of non-maximal models as the outer models seems counterintuitive. Indeed, inclusion of such non-maximal models leads to incompatibility of maximal universes satisfying IMH with inaccessible cardinals (see [3]).

The following theorem is a sharp-generated analogue of the argument in [4].

Theorem 3.2 *Assume there is a Woodin cardinal with an inaccessible above. Then there is a model satisfying IMH$^\#$.*

Proof. For each real R let $M^\#(R)$ be $L_\alpha[R]$ where α is least so that $L_\alpha[R]$ is #-generated. Note that $R^\#$ exists for each $R \subseteq \omega$ by our large cardinal assumption. The Woodin cardinal with an inaccessible above implies enough projective determinacy to enable us to use Martin's theorem, see [5] Proposition 28.4, to find $R \subseteq \omega$ such that the theory of $(M^\#(S), \in)$ for $R \leq_T S$ stabilizes. By this we mean that for $R \leq_T S$, where $\leq_T$ denotes the Turing reducibility relation, the theories of $(M^\#(R), \in)$ and $(M^\#(S), \in)$ are the same.[9]

We claim that $M^\#(R)$ satisfies IMH$^\#$: Indeed, let M be a #-generated outer model of $M^\#(R)$ with a definable inner model satisfying some sentence φ. Let α be the ordinal height of $M^\#(R)$ (= the ordinal height of M). By Theorem 9.1 in [1], M has a #-generated outer model W of the form $L_\alpha[S]$

[8] We thus give two names two a single concept; denotation IMH$^\#$ is used to emphasize the family resemblance to the earlier principle IMH.

[9] In more detail, given a sentence σ in the language with $\{\in\}$ consider the set of Turing degrees $X_\sigma = \{S \,|\, (M^\#(S), \in) \models \sigma\}$. X_σ has a projective definition ($\boldsymbol{\Delta}^1_2$). By Martin's theorem, X_σ or $X_{\neg\sigma}$ contains a cone of degrees. Denote Y_{σ^*} the unique set of the two X_σ and $X_{\neg\sigma}$ which contains the cone. Then $\bigcap_\sigma Y_{\sigma^*}$ contains a cone. Take R to be the base of this cone.

for some real S with $R \leq_T S$. Of course α is least so that $L_\alpha[S]$ is #-generated as it is least so that $L_\alpha[R]$ is #-generated.So W equals $M^\#(S)$. By the choice of R, $M^\#(R)$ also has a definable inner model satisfying φ. So $M^\#(R)$ is #-maximal.[10] □

3.2 IMH$^\#$ is compatible with large cardinals

Finally, we show that – unlike IMH – IMH$^\#$ is compatible with large cardinals.

Theorem 3.3 *Assume there is a Woodin cardinal with an inaccessible above. Then for some real R, any #-generated transitive model M containing R also models IMH$^\#$.*

Proof. Let R be as in the proof of Theorem 3.2. Thus $M^\#(R) = L_\alpha[R]$ is a #-generated model of IMH$^\#$. Now suppose that $M^* = L_{\alpha^*}[R]$ is obtained by iterating $L_\alpha[R]$ past α; we claim that M^* is also a model of IMH$^\#$: Indeed, suppose that W is a #-generated outer model of M^* which has a definable inner model satisfying some sentence φ. Again by Jensen's Theorem 9.1 in [1], we can choose W to be of the form $L_{\alpha^*}[S]$ for some real $S \geq_T R$. But then $L_{\alpha^*}[S]$ is an iterate of $M^\#(S)$ (via the iteration given by $S^\#$) and therefore $M^\#(S)$ also has a definable inner model of φ. By the choice of R, $M^\#(R)$, and therefore by iteration also $L_{\alpha^*}[R]$, has a definable inner model of φ. This verifies the IMH$^\#$ for M^*.

Now any #-generated transitive model M containing R is an outer model of such a model of the form $L_{\alpha^*}[R]$ as above and therefore is also a model of IMH$^\#$. □

Corollary 3.4 *Assume the existence of a Woodin cardinal with an inaccessible above and suppose that φ is a sentence that holds in some V_κ with κ measurable. Then there is a transitive model which satisfies both the IMH$^\#$ and the sentence φ.*

Proof. Let R be as in Theorem 3.3 and let U be a normal measure on κ. The structure $N = (H(\kappa^+), U)$ is a #; iterate N through a large enough ordinal ∞ so that $M = LP(N_\infty)$, the lower part of the model generated by

[10] Woodin noticed that this theorem and also Theorem 3.3 can be proved without recourse to Jensen's coding theorem: let R be a real such that every nonempty lightface Σ^1_3 set contains a member recursive in R. Then any M which is #-generated and contains R satisfies IMH$^\#$. However, Jensen's coding theorem does seem necessary for a modification of IMH$^\#$ which is formulated for ω_1-preserving #-generated extensions (this modification is not discussed in this paper).

N, has ordinal height ∞. Then M is $\#$-generated and contains the real R. It follows that M is a model of the IMH$^\#$. Moreover, as M is the union of an elementary chain $V_\kappa = V_\kappa^N \prec V_{\kappa_1}^{N_1} \prec \cdots$ where φ is true in V_κ, it follows that φ is also true in M. □

Note that in Corollary 3.4, if we take φ to be any large cardinal property which holds in some V_κ with κ measurable, then we obtain models of the IMH$^\#$ which also satisfy this large cardinal property. This implies the compatibility of the IMH$^\#$ with arbitrarily strong large cardinal properties.

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