INDESTRUCTIBILITY OF SOME COMPACTNESS PRINCIPLES OVER MODELS OF PFA AND GMP

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Abstract. We show that over any model of PFA, the tree property at ω_2 is preserved by the single Cohen forcing at ω, and the negation of the weak Kurepa Hypothesis is preserved by any σ-centered forcing. We do not need the full strength of PFA – the Guessing Model Principle, GMP, is enough. GMP can be formulated also for larger cardinals, and we give some applications of our results to larger cardinals, such as the negation of the weak Kurepa Hypothesis at $\aleph_{\omega+1}$.

1. Introduction

Suppose κ is a regular cardinal. We say that the *tree property* holds at κ if every κ-tree has a cofinal branch, equivalently, there are no κ-Aronszajn trees. We say that the *Kurepa Hypothesis* at κ holds if there is a κ-tree with at least κ^+-many cofinal branches (we call such a tree a κ-Kurepa tree); we say that the *weak Kurepa Hypothesis* holds if there is a tree of height and size at most κ with at least κ^+-many cofinal branches (we call such a tree a weak κ-Kurepa tree). A κ-Aronszajn tree is an incompact object because it has chains of every size $< \kappa$, but no chains of size κ; similarly a (weak) κ-Kurepa tree is an incompact object because every level of the tree has size $< \kappa$ (or $\leq \kappa$), yet there are κ^+-many cofinal branches.

It is easy to check that both the tree property at κ^{++} and the negation of the weak Kurepa Hypothesis at κ^+ imply $2^\kappa > \kappa^+$. In many natural models, for instance in the Michell model, both the tree property at κ^{++} and the negation of the weak Kurepa hypothesis at κ^+ hold simultaneously. But it is also known that these principles are mutually independent, see Section 6 below Question 1 for details.

It has been open whether the tree property or the negation of the weak Kurepa Hypothesis can be shown to be indestructible by a class of forcing notions over all transitive models V which satisfy some theory T which extends ZFC. In this short paper we show that over all transitive models of $T = \mathsf{ZFC} + \mathsf{PFA}$, the tree property at ω_2 is preserved by the single Cohen forcing at ω, $\mathrm{Add}(\omega, 1)$, and the negation of the weak Kurepa Hypothesis is preserved by any σ-centered forcing. We do not need the full strength of PFA, the Guessing model principle, GMP, is enough (see Section 2.4 for definitions). Unlike PFA, GMP can be formulated for larger cardinals. This

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makes it possible to apply our result for the negation of the weak Kurepa hypothesis to the Prikry collapse at a measurable κ, which gives an easy proof of the negation of the weak Kurepa hypothesis at $\aleph_{\omega+1}$.

Remark 1.1. There are results in the literature which mention indestructibility of the tree property, but they are formulated for specific forcing extensions, such as the Mitchell model, not theories. See [7] for more details. A similar result is available for the negation of the Kurepa Hypothesis over the Levy collapse, see [8].

Remark 1.2. Indestructibility results for theories are available for other compactness principles: In [4], Gitik and Krueger showed that the negation of the approachability property at κ^{++} (κ regular) is indestructible under all κ^+-centered forcings[1] $\mathbb{P}$ over any transitive model of ZFC. In [5], we showed that *stationary reflection* at κ^+ (κ regular) is indestructible under all κ-cc forcing notions over any transitive model of ZFC.

2. Background

2.1. Centered forcings

Definition 2.1. Let $\mathbb{P}$ be a forcing and suppose κ is a cardinal. We say that $\mathbb{P}$ is κ^+-*centered* if $\mathbb{P}$ can be written as the union of a family $\{\mathbb{P}_\alpha \subseteq \mathbb{P} \,|\, \alpha < \kappa\}$ such that for every $\alpha < \kappa$:

$$\text{for every } p, q \in \mathbb{P}_\alpha \text{ there exists } r \in \mathbb{P}_\alpha \text{ with } r \le p, q. \tag{2.1}$$

If $\kappa = \omega$, we say that $\mathbb{P}$ is σ-*centered*.

It follows that $\mathbb{P}$ can be written as a union of κ-many filters if we close each $\mathbb{P}_\alpha$ upwards. We require (2.1) to ensure nice properties of the system $S(\dot{T})$ defined in Section 2.3 (in particular the transitivity of $<_i$).

Some definitions of κ^+-centeredness require just the compatibility of the conditions, with a witness not necessarily in $\mathbb{P}_\alpha$. The condition (2.1) in this case reads:

$$\begin{gathered}\text{for every } n < \omega \text{ and every sequence } p_0, p_1, \ldots, p_{n-1} \\ \text{of conditions in } \mathbb{P}_\alpha \text{ there exists } r \in \mathbb{P} \text{ with } r \le p_i \text{ for every } 0 \le i < n.\end{gathered} \tag{2.2}$$

The conditions (2.1) and (2.2) are not in general equivalent (see Kunen [9], before Exercise III.3.27), but the distinction is not so important for us because the forcings we will discuss – the Cohen forcing and the Prikry forcings – are all centered in the stronger sense of (2.1). Also note that the conditions are equivalent for Boolean algebras: the definition (2.2) means that each $\mathbb{P}_\alpha$ is a system with FIP (finite intersection property), and as such can be extended into a filter.

2.2. Systems

Suppose $\mathbb{P}$ is a forcing notion. In order to show that certain objects cannot exist in a generic extension $V[\mathbb{P}]$ (such as a weak Kurepa tree), we will work in the ground model and work with a system derived from a $\mathbb{P}$-name for the object in question. We give a general definition of a system here in Section

[1] Defined as in our Definition 2.1

2.2, and discuss systems derived from names in Section 2.3. The idea to use derived systems goes back to [11].

Definition 2.2. Let $\kappa \leq \lambda$ be cardinals and let $D \subseteq \lambda$ be unbounded in λ. For each $\alpha \in D$, let $S_\alpha \subseteq \{\alpha\} \times \kappa$ and let $S = \bigcup_{\alpha \in D} S_\alpha$.[2] Moreover, let I be an index set of cardinality $< \kappa$ and $\mathcal{R} = \{<_i \mid i \in I\}$ a collection of binary relations on S. We say that $\langle S, \mathcal{R} \rangle$ is a (κ, λ)-*system* if the following hold for some D:

(i) For each $i \in I$, $\alpha, \beta \in D$ and $\gamma, \delta < \kappa$; if $(\alpha, \gamma) <_i (\beta, \delta)$ then $\alpha < \beta$.
(ii) For each $i \in I$, $<_i$ is irreflexive and transitive.
(iii) For each $i \in I$, and $\alpha < \beta < \gamma$, $x \in S_\alpha$, $y \in S_\beta$ and $z \in S_\gamma$, if $x <_i z$ and $y <_i z$, then $x <_i y$.
(iv) For all $\alpha < \beta$ there are $y \in S_\beta$ and $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

We call a (κ, λ)-system $\langle S, \mathcal{R} \rangle$ a *strong* (κ, λ)-*system* if the following strengthening of item (iv) holds:

(iv') For all $\alpha < \beta$ and for every $y \in S_\beta$ there are $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

If $\langle S, \mathcal{R} \rangle$ is a (κ, λ)-system, we say that the system has height λ and width κ. We call S_α the α-th level of S.

For the purposes of this paper we introduce the following definition:

Definition 2.3. Suppose $\kappa \leq \lambda$ are cardinals and let $\langle S, \mathcal{R} \rangle$ be a (κ, λ)-system. We call $\langle S, \mathcal{R} \rangle$ *well-behaved* if $|\mathcal{R}| < \kappa$, i.e. the number of relations is strictly smaller than the width of the system.

A *branch* of the system is a subset B of S such that for some $i \in I$, and for all $a \neq b \in B$, $a <_i b$ or $b <_i a$. A branch B is *cofinal* if for each $\alpha < \lambda$ there are $\beta \geq \alpha$ and $b \in B$ on level β.

2.3. Systems derived from forcing notions

Systems appear naturally when we wish to analyse in the ground model a $\mathbb{P}$-name $\dot{T}$ for a tree which is added by a forcing notion $\mathbb{P}$. We give the definition for the context we will use it (more general definitions are possible).

Definition 2.4. Assume κ is a regular cardinal. Assume $\mathbb{P}$ is a κ^+-centered forcing notion; let $\mathbb{P} = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$ where each $\mathbb{P}_\alpha$ is a filter. Assume further that $\mathbb{P}$ forces $\dot{T}$ is a tree of height and size λ, where $\lambda \geq \kappa^+$ is regular. We assume that the domain of T is $\lambda \times \lambda$, where the β-th level of $\dot{T}$ consistes of pairs $\{\beta\} \times \lambda$. We say that $S(\dot{T}) = \langle \lambda \times \lambda, \mathcal{R} \rangle$ is a *derived system* (with respect to $\mathbb{P}$ and $\dot{T}$) if it is a system with domain $\lambda \times \lambda$ which is equipped with binary relations $\mathcal{R} = \{<_\alpha \mid \alpha < \kappa\}$, where

$$x <_\alpha y \leftrightarrow (\exists p \in \mathbb{P}_\alpha)\, p \Vdash x <_{\dot{T}} y.$$

Following the terminology of Definitions 2.2 and 2.3, $S(\dot{T})$ is a strong well-behaved (λ, λ)-system. We will give some examples of derived systems.

Example 1. Assume $\mathbb{P}$ is a countable forcing notion and $\mathbb{P}$ forces that $\dot{T}$ is an ω_2-tree. For each $p \in \mathbb{P}$, let $\mathbb{P}_p = \{q \in \mathbb{P} \mid p \leq q\}$. Then $\mathbb{P}$ is (trivially)

[2] The elements of S are therefore ordered pairs of ordinals; if the ordinals are not important, we denote the pairs of ordinals by letters $x, y, \ldots$, etc.

the union of the filters $\mathbb{P}_p$, and hence is σ-centered. Since $\dot{T}$ is an ω_2-tree, i.e. has levels of size $< \omega_2$, the domain of the system can be thinned out to $\omega_2 \times \omega_1$. The derived system $S(\dot{T})$ is equipped with binary relations $<_p$, $p \in \mathbb{P}$, where

$$x <_p y \Leftrightarrow p \Vdash x \dot{<}_T y. \tag{2.3}$$

If $\mathbb{P}$ is non-trivial, it is be equivalent to the Cohen forcing $\mathrm{Add}(\omega, 1)$. This system will be useful for our proof that the tree property at ω_2 is indestructible over transitive models of PFA for the Cohen forcing $\mathrm{Add}(\omega, 1)$.

Example 2. Let $\mathbb{P}$ be a κ^+-centered forcing which forces that $\dot{T}$ is a tree of height and size κ^+. Let us write $\mathbb{P} = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$. The derived system has domain $\kappa^+ \times \kappa^+$, and is equipped with binary relations $<_\alpha$ for $\alpha < \kappa$, where

$$x <_\alpha y \Leftrightarrow (\exists p \in \mathbb{P}_\alpha)\, p \Vdash x \dot{<}_T y. \tag{2.4}$$

$S(\dot{T})$ is a strong well-behaved (κ^+, κ^+)-system. With $\kappa = \omega$, this system will be useful for our proof that the negation of the weak Kurepa Hypothesis at ω_1 is indestructible over transitive models of PFA for any σ-centered forcing. We will also use this system with κ being measurable, see Section 5.

2.4. Guessing models

In the proof, we will analyse derived systems $S(\dot{T})$ using the notion of a guessing model, introduced in [13]. We first give the definitions formulated for the case of ω_2 and PFA, and follow up with a more general notion in Definition 2.8.

Definition 2.5. Let θ be a regular cardinal, and let $M \prec H(\theta)$ and $z \in M$.
- (i) A set $d \subseteq z$ is *M-approximated* if $d \cap a \in M$ for all countable $a \in M$.
- (ii) A set $d \subseteq z$ is *M-guessed* if there is an $e \in M$ such that $d \cap M = e \cap M$.
- (iii) M is a *guessing model* if for every $z \in M$, if $d \subseteq z$ is M-approximated, it is M-guessed.
- (iv) Let $\mathcal{G}_{\omega_2} H(\theta)$ denote the set

$$\{M \prec H(\theta) \,|\, |M| < \omega_2 \text{ and } M \text{ is a guessing model}\}.$$

Viale and Weiss in [13] proved following:

Definition 2.6. We say that the *Guessing model principle* holds at ω_2, and write $\mathsf{GMP}(\omega_2)$, if $\mathcal{G}_{\omega_2} H(\theta)$ is stationary in $\mathcal{P}_{\omega_2} H(\theta)$ for every $\theta \geq \omega_2$

Fact 2.7 ([13])**.** *PFA implies $\mathsf{GMP}(\omega_2)$.*

The notion of a guessing model can be generalised to larger cardinals:

Definition 2.8. Suppose κ is a regular cardinal. We say that the *Guessing model principle* holds at κ^{++}, and write $\mathsf{GMP}(\kappa^{++})$, if $\mathcal{G}_{\kappa^{++}} H(\theta)$ is stationary in $\mathcal{P}_{\kappa^{++}} H(\theta)$ for every $\theta \geq \kappa^{++}$, where $\mathcal{G}_{\kappa^{++}} H(\theta)$ is formulated in analogy with Definition 2.5 with $\kappa = \omega$.

Models with $\mathsf{GMP}(\kappa^{++})$ can be obtained starting with sufficiently large cardinals (see [14] and [13] for more details).

Fact 2.9. *Suppose κ is regular and $\lambda > \kappa$ is supercompact. Then in the Mitchell model $V[\mathbb{M}(\kappa, \lambda)]$ which turns λ to κ^{++}, $\mathsf{GMP}(\kappa^{++})$ holds.*

Remark 2.10. For our purposes, we will need that the guessing models are stationary in $\mathcal{P}_{\kappa^{++}} H(\kappa^{+3})$ (for the tree property), or $\mathcal{P}_{\kappa^{++}} H(\kappa^{++})$, for the negation of the weak Kurepa hypothesis. These weakenings are true in the model $V[\mathbb{M}(\kappa, \lambda)]$ if λ is λ^+-supercompact (and $2^\lambda = \lambda^+$ in the ground model, to ensure $H(\kappa^{+3})$ has size κ^{+3} in the extension), or λ is an ineffable cardinal, respectively.

3. The tree property

We prove the following theorem:

Theorem 3.1. $\mathsf{GMP}(\omega_2)$*, and hence* PFA*, imply that the tree property at* ω_2 *is indestructible under the single Cohen forcing at* ω*, i.e. if* V *is a transitive model satisfying* $\mathsf{GMP}(\omega_2)$ *and* G *is* $\mathrm{Add}(\omega, 1)$*-generic over* V*, then* $V[G]$ *satisfies the tree property at* ω_2*.*

Proof. Let us assume that $\mathsf{GMP}(\omega_2)$ holds in V and suppose for contradiction that $\mathbb{P} = \mathrm{Add}(\omega, 1)$ forces that $\dot{T}$ is an ω_2-Aronszajn tree. We view $\dot{T}$ as having domain $\omega_2 \times \omega_1$, where the β-th level of $\dot{T}$ consists of pairs $\{\beta\} \times \omega_1$. Let $\dot{<}_T$ be a name for the tree-order in $\dot{T}$ on $\omega_2 \times \omega_1$.

Let $S(\dot{T})$ be the derived system as in Example 1, Section 2.3.

To prove Theorem 3.1, it is enough to show that $S(\dot{T})$ has a cofinal branch because this implies that $\dot{T}^G$ has a cofinal branch, contradicting the fact that $\mathbb{P}$ forces $\dot{T}$ is Aronszajn.

Lemma 3.2. *Suppose* b *is a cofinal branch in* $S(\dot{T})$ *with respect to* $<_p$ *for some* $p \in \mathbb{P}$*, and let* G *be a* $\mathbb{P}$*-generic containing* p*. Then* b *is a cofinal branch in* $\dot{T}^G$ *in* $V[G]$*.*

Proof. Obvious. □

To finish the proof, we will argue that $S(\dot{T})$ has a cofinal branch. By Fact 2.7 there is $M \in \mathcal{G}_{\omega_2} H(\kappa^{+3})$ such that $S(\dot{T}) \in M$ and $\omega_1 \subseteq M$. Note that $\delta = M \cap \omega_2$ has cofinality ω_1 because M is a guessing model (see [10, Claim 10.4]).

Take an arbitrary $x_\delta \in S(\dot{T})$ on level δ, and for every $p \in \mathbb{P}$ consider

$$b_p = \{y \in S(\dot{T}) \cap M \,|\, y <_p x_\delta\}.$$

Since δ has cofinality ω_1, there is $p \in \mathbb{P}$ such that b_p meets unboundedly many levels below δ. Let us fix some such p. Notice that for all $y \neq y'$ in b_p, either $y <_p y'$ or $y' <_p y$, and so b_p is a cofinal branch through

$$S(\dot{T}) \cap M = \langle \delta \times \omega_1, \{<_p \cap (\delta \times \omega_1) \,|\, p \in \mathbb{P}\}\rangle.$$

Lemma 3.3. b_p *is* M*-approximated.*

Proof. Let $a \in M$ be a countable set; we want to show that $a \cap b_p \in M$. Consider the set $A \subseteq \delta$ of all $\alpha < \delta$ such that there is x on level α in $a \cap b_p$; since a is countable, A is bounded below δ which has cofinality ω_1. On the other hand, b_p is unbounded below x_δ and therefore there is some y on level above A such that $y \in b_p$ and therefore $y \in M$. Then $a \cap b_p$ can be defined in M as the set $\{z \in a \cap S(\dot{T}) \,|\, z <_p y\}$. □

Since b_p is M-approximated, it is also guessed, and therefore there is $b \in M$ such that $b \cap M = b_p \cap M = b_p$. It means that M thinks that b is a cofinal branch in $S(\dot{T})$ in the ordering $<_p$. By elementarity, b is a cofinal branch in $<_p$ in the real $S(\dot{T})$ in $H(\theta)$. By Lemma 3.2 this proves Theorem 3.1. □

The proof in particular implies the known fact that the tree property at ω_2 is a consequence of GMP (with $\mathbb{P}$ being a trivial forcing). This weaker result is proved for instance in [13].

Note that we only need that guessing models are stationary in $\mathcal{P}_{\omega_2} H(\omega_3)$; see Remark 2.10 for more discussion of this weakening.

Remark 3.4. Let us say a few words about obstacles to generalising this argument to σ-centered forcings: Suppose $S(\dot{T})$ is the derived system with respect to some σ-centered forcing $\mathbb{P}$. Arguing as we did, one can show that there is a cofinal branch b in $S(\dot{T})$ with respect to some $<_i$, $i < \omega$. However the analogue of Lemma 3.2 may fail because if G is $\mathbb{P}$ generic, then it may be false that for all (or sufficiently many) x, y in b there is some $p \in G \cap \mathbb{P}_i$ forcing $x <_{\dot{T}} y$. See Section 6 with open questions.

Remark 3.5. Theorem 3.1 can be generalised to any regular κ and the forcing $\mathrm{Add}(\kappa, 1)$, provided we start with $\mathsf{GMP}(\kappa^{++})$.

4. The negation of the weak Kurepa Hypothesis

We have a more general result for the negation of the weak Kurepa hypothesis at ω_1: we show that it is indestructible over any transitive model of GMP under any σ-centered forcing. Note that the class of σ-centered forcings includes the Cohen forcing $\mathrm{Add}(\omega, \omega_2)$, because we have $2^\omega = \omega_2$, or the Mathias forcing at ω.

Theorem 4.1. $\mathsf{GMP}(\omega_2)$*, and hence also* PFA*, imply that the negation of the weak Kurepa Hypothesis is indestructible under any σ-centered forcing, i.e. if V is a transitive model satisfying* $\mathsf{GMP}(\omega_2)$*, $\mathbb{P}$ is σ-centered, and G is $\mathbb{P}$-generic over V, then $V[G]$ satisfies the negation of the weak Kurepa Hypothesis at ω_1.*

Proof. First note that under our assumption, the negation of the weak Kurepa Hypothesis holds in V (this appeared already in [1]): Suppose for contradiction that T is an ω_1-tree with ω_2 cofinal branches and let M be a guessing model of size ω_1 with $\omega_1 \subseteq M$ and $T \in M$. It is obvious that each cofinal branch b through T is M-approximated; it follows $b \in M$ because M is a guessing model. This implies $|M| > \omega_1$, contradicting our initial assumption.

Suppose $\mathbb{P} = \bigcup_{i<\omega} \mathbb{P}_i$ is a σ-centered forcing. Assume for contradiction that $\dot{T}$ is forced by the weakest condition in $\mathbb{P}$ to be a weak ω_1-Kurepa tree, and let $S(\dot{T})$ be the derived system with respect to $\dot{T}$, as in Example 2 in Section 2.3. Let M be a guessing model in $\mathcal{G}_{\omega_2} H(\omega_2)$ of size ω_1 with $\omega_1 \subseteq M$ and $S(\dot{T}) \in M$. Since $S(\dot{T})$ has domain $\omega_1 \times \omega_1$, the system with the relations is a subset of M.

Let us fix a sequence $\langle \dot{b}_\alpha \mid \alpha < \omega_2 \rangle$ of $\mathbb{P}$-names such that

$$1_{\mathbb{P}} \Vdash \langle \dot{b}_\alpha \mid \alpha < \omega_2 \rangle \text{ are pairwise distinct cofinal branches in } \dot{T}. \tag{4.5}$$

Working in V, there must be some $i < \omega$, such that for some $I \subseteq \omega_2$ of size ω_2, and all $\alpha \in I$, there are cofinally many x for which there are $p_x \in \mathbb{P}_i$ with

$$p_x \Vdash x \in \dot{b}_\alpha.$$

Let us fix such an $i < \omega$.

For each $\alpha \in I$, let us define

$$B_\alpha = \{x \in S(\dot{T}) \,|\, (\exists p \in \mathbb{P}_i)\; p \Vdash x \in \dot{b}_\alpha\}.$$

Note that B_α is a cofinal branch in $S(\dot{T})$. We finish the proof by showing:

(i) For each $\alpha \in I$, B_α is an element of M.
(ii) For all $\alpha \neq \beta \in I$, $B_\alpha \neq B_\beta$.

The items (i) and (ii) imply that M has size at least ω_2, which is a contradiction.

With regard to (i), we will show that each B_α is M-approximated, and therefore is an element of M.[3] Let us fix $\alpha \in I$ and a countable $a \in M$. We need to show that $B_\alpha \cap a$ is in M. Since $M \cap \omega_2$ has cofinality ω_1, there is some $y \in B_\alpha \cap M$ which is above $B_\alpha \cap a$ in $<_i$. It follows

$$B_\alpha \cap a = \{x \in a \cap S(\dot{T}) \,|\, (\exists p \in \mathbb{P}_i)\; p \Vdash x \in \dot{b}_\alpha\} = \{x \in a \cap S(\dot{T}) \,|\, x <_i y\}.$$

For the identity between the second and third set, note that if $p \Vdash x \in \dot{b}_\alpha$ and $p' \Vdash y \in \dot{b}_\alpha$, then the existence of a lower bound of p, p' in $\mathbb{P}_i$ implies $x <_i y$; and conversely, if $p \Vdash x <_{\dot{T}} y$ for some $p \in \mathbb{P}_i$ and $p' \Vdash y \in \dot{b}_\alpha$, then the existence of a lower bound in $\mathbb{P}_i$ implies that for some $r \in \mathbb{P}_i$, $r \Vdash x \in \dot{b}_\alpha$. Since the third expression determines a set in M (because all parameters are in M), $B_\alpha \cap a$ is in M.

With regard to (ii): suppose for contradiction $B_\alpha = B_\beta$ for some $\alpha \neq \beta \in I$. Fix for every $x \in B_\alpha = B_\beta$ some conditions p^α_x and p^β_x in $\mathbb{P}_i$ such that

$$p^\alpha_x \Vdash x \in \dot{b}_\alpha \text{ and } p^\beta_x \Vdash x \in \dot{b}_\beta.$$

Let $p_x \in \mathbb{P}_i$ be some lower bound of p^α_x, p^β_x.

Suppose first that $\{p_x \,|\, x \in B_\alpha = B_\beta\}$ is countable. Then there exists some p such that $p = p_x$ for uncountably many x. This p forces $\dot{b}_\alpha = \dot{b}_\beta$, which contradicts (4.5).

Suppose now that $\{p_x \,|\, x \in B_\alpha = B_\beta\}$ is uncountable. Since it is an uncountable collection of conditions, the ccc of $\mathbb{P}$ implies that there is a condition p which forces that $\dot{G}$ has an uncountable intersection with $\{p_x \,|\, x \in B_\alpha = B_\beta\}$. In particular, p forces $\dot{b}_\alpha = \dot{b}_\beta$, which contradicts (4.5). □

Note that we only need that guessing models are stationary in $\mathcal{P}_{\omega_2} H(\omega_2)$; see Remark 2.10 for more discussion of this weakening.

Remark 4.2. Let us say a few words regarding the comparison of the proofs for the tree property and the negation of the weak Kurepa Hypothesis.

If we assume the negation of the tree property at ω_2, we work with a name $\dot{T}$ for a tree which does *not* have a cofinal branch. The proof proceeds by

[3] In general, if some set d is M-approximated, and M is guessing, there is some $e \in M$ such that $d \cap M = e \cap M$, where $d \neq e$ is possible. This was for instance the case for $b_p \neq b$ in the proof of Theorem 3.1. However, in the present case, since $S(\dot{T}) \subseteq M$, if B_α is M-approximated, and M is guessing, then $B_\alpha \in M$.

finding a cofinal branch in the derived system $S(\dot{T})$, and then arguing that it determines a cofinal branch in the tree T. This line of argument seems to work only for the single Cohen forcing.

If we assume the weak Kurepa Hypothesis, we work with a name $\dot{T}$ for a tree which *has* many cofinal branches. We can thus fix names $\langle \dot{b}_\alpha \,|\, \alpha < \omega_2 \rangle$ for the cofinal branches in $\dot{T}$ and get many cofinal branches in the system $S(\dot{T})$. This argument works for all σ-centered forcings.

5. Some applications

We assume the reader is familiar with the Mitchel forcing $\mathbb{M}(\kappa, \lambda)$; see for instance [7] for definitions and details. $\mathbb{M}(\kappa, \lambda)$ can be written as $\mathrm{Add}(\kappa, \lambda) * \dot{Q}$ for some quotient forcing $\dot{Q}$ which is forced to be κ^+-distributive. If G is $\mathbb{M}(\kappa, \lambda)$-generic, we write $G_0 * G_1$ to denote the corresponding $\mathrm{Add}(\kappa, \lambda) * \dot{Q}$-generic.

Theorem 5.1. *Suppose $\kappa < \lambda$ are supercompact cardinals and κ is Laver-indestructibly supercompact. Let G be $\mathbb{M}(\kappa, \lambda)$-generic, where $\mathbb{M}(\kappa, \lambda)$ is the Mitchell forcing. In $V[G]$, κ is supercompact, $2^\kappa = \kappa^{++}$, and $\mathsf{GMP}(\kappa^{++})$ holds. Let $\mathbb{Q}$ be the Prikry forcing with interleaved collapses which turns κ into $\aleph_{\omega+1}$. Suppose F is $\mathbb{Q}$-generic over $V[G]$. Then in $V[G][F]$, the negation of the weak Kurepa Hypothesis holds at $\aleph_{\omega+1}$.*

Proof. For the properties of the model $V[G]$, see Fact 2.9. With regard to the forcing $\mathbb{Q}$: $\mathbb{Q}$ is defined with respect to some guiding generic whose existence follows from the facts that κ is still κ^+-supercompact in $V[G]$, $2^\kappa = \kappa^{++}$ in $V[G]$, and $V[G]$ can be written as $V[G_0][G_1]$, where G_0 is $\mathrm{Add}(\kappa, \lambda)$-generic. See [3, Lemma 4.1] for more details. Since the compatibility of conditions in $\mathbb{Q}$ depends only on the stems, $\mathbb{Q}$ is κ^+-centered. Then the theorem follows as in Theorem 4.1 using $\mathsf{GMP}(\kappa^{++})$ instead of $\mathsf{GMP}(\omega_2)$. □

Remark 5.2. By an argument using a quotient analysis in [3, Lemma 4.6], the tree property holds at $\aleph_{\omega+2}$ in the model $V[G][F]$.

If κ is not collapsed, but only singularised (to an arbitrary cofinality), we can apply an indestructibility result also for the tree property, following our [7].

Theorem 5.3. *Suppose $\kappa < \lambda$ are supercompact cardinals and κ is Laver-indestructibly supercompact. Let G be $\mathbb{M}(\kappa, \lambda)$-generic, where $\mathbb{M}(\kappa, \lambda)$ is the Mitchell forcing. In $V[G] = V[G_0][G_1]$, κ is supercompact, $2^\kappa = \kappa^{++}$, and $\mathsf{GMP}(\kappa^{++})$ holds. Let $\mathbb{Q}$ be the Prikry or Magidor forcing which turns κ into a singular cardinal without collapsing any cardinals; choose $\mathbb{Q}$ to be an element of $V[G_0]$ (this is possible). Suppose F is $\mathbb{Q}$-generic over $V[G]$. Then in $V[G][F]$, the negation of the weak Kurepa Hypothesis holds at κ^+ and the tree property holds at κ^{++}.*

Proof. The part regarding the negation of the weak Kurepa Hypothesis is as in Theorem 5.1 (but it is easier since no guiding generic needs to be constructed), and the tree property holds because the forcing $\mathbb{Q}$ lives in $V[G_0]$, so the indestructibility result from [7] applies. □

6. Open questions

We showed in Theorem 3.1 that the tree property at ω_2 is indestructible under the single Cohen forcing at ω if we assume $\mathsf{GMP}(\omega_2)$. We also showed in Theorem 4.1 that under the same assumption, the negation of the weak Kurepa hypothesis is indestructible under all σ-centered forcings. It is natural to ask whether this holds also for the tree property (see also Remark 3.4):

Question 1. Is the tree property at ω_2 indestructible under all σ-centered forcings over every model which satisfies $\mathsf{GMP}(\omega_2)$ or PFA? Note that because $2^\omega = \omega_2$ in every model satisfying PFA, the Cohen forcing $\mathrm{Add}(\omega, \omega_2)$ is σ-centered.

It is hard to guess whether the answer to Question 1 is positive or negative. On one hand, all the natural models satisfy both the tree property at ω_2 and the negation of the weak Kurepa Hypothesis at ω_1, so the principles seem similar. On the other hand, over the Mitchell model $V[\mathbb{M}(\omega, \lambda)]$, where λ is weakly compact, one can force an ω_1-Kurepa tree without destroying the tree property at ω_2 (see [2]), and also an ω_2-Suslin tree without destroying the negation of the weak Kurepa Hypothesis,[4] so the two principles are independent. Moreover, the difference between a single Cohen and many Cohens may be important: for instance, $\mathrm{Add}(\omega, 1)$ preserves MA for σ-centered forcings, while $\mathrm{Add}(\omega, \omega_1)$ makes $\mathfrak{b} = \omega_1$, and hence destroys MA for σ-centered forcings (see [12] for more details); while this is not directly relevant for our case, it indicates that one should be careful in trying to generalize arguments from the single Cohen to ω_1-many Cohens on ω.

In [1], Cox and Krueger introduce an "indestructible" version of $\mathsf{GMP}(\omega_2)$, and call it IGMP (Indestructible guessing model principle). IGMP says that for every $\theta \geq \omega_2$, there are stationarily many guessing models in $\mathcal{P}_{\omega_2} H(\theta)$ which remain guessing in any forcing extension which preserves ω_1. They further show that PFA implies IGMP.

However, to prevent a misunderstanding, IGMP does not say that GMP is true in $V[\mathbb{P}]$ if $\mathbb{P}$ preserves ω_1 (this is clearly false because one can force CH in this way, which contradicts $\mathsf{GMP}(\omega_2)$). Moreover, if a guessing model $M \prec H(\theta)$ is still guessing in $V[G]$, where G is $\mathbb{P}$-generic, it is typically no longer elementary in $H(\theta)^{V[G]}$, so the correspondence between M and the universe $V[G]$ is lost.[5] We do not see an obvious way to use IGMP to improve the results in our paper, but still we can ask:

Question 2. Is it possible to use IGMP, or its modification, to extend Theorem 3.1 to include more forcing notions?

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[4]This is similar to argument in [2] or [6], using the fact that an ω_2-Suslin tree can be forced with an ω_2-distributive forcing.

[5]Seeing that $M[G]$ is typically elementary in $H(\theta)^{V[G]}$, it might be more useful for our purposes to know that $M[G]$ is a guessing model (for some class of forcing notions), but this is not the subject of [1].

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