ON LIFTING OF EMBEDDINGS BETWEEN TRANSITIVE MODELS OF SET THEORY

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ABSTRACT. Suppose M and N are transitive models of set theory, $\mathbb{P}$ is a forcing notion in M and G is $\mathbb{P}$-generic over M. We say that an elementary embedding $j : (M, \in) \to (N, \in)$ lifts to $M[G]$ if there is $j^+ : (M[G], G, \in) \to (N[j^+(G)], j^+(G), \in)$ such that j^+ restricted to M is equal to j. We survey some basic applications of the lifting method for both large cardinals and small cardinals (such as ω_2, or successor cardinals in general). We focus on results and techniques which appeared after Cummings's handbook article [5]: we for instance discuss a generalization of the surgery argument, liftings based on fusion, and compactness principles such as the tree property and stationary reflection at successor cardinals.

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1. INTRODUCTION

Various results in set theory are derived by means of elementary embeddings between transitive models of set theory (or its fragments). An important part of these argument is the *lifting of elementary embeddings*. By this

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notion we mean the following: Suppose M and N are transitive models (sets or proper classes) of a sufficient fragment of ZFC and

$$j : (M, \in, \dots) \to (N, \in, \dots) \tag{1.1}$$

is an elementary embedding with critical point $\kappa \in M$. Assume further that $\mathbb{P} \in M$ is a forcing notion and G is a $\mathbb{P}$-generic filter over M (i.e. G meets every maximal antichain of $\mathbb{P}$ which is an element of M). We say that j *lifts to* $\mathbb{P}$ *(or G)* if there exists a $j(\mathbb{P})$-generic filter H over N such that j extends to an elementary embedding

$$j^+ : (M[G], G, \in, \dots) \to (N[H], H, \in, \dots). \tag{1.2}$$

Notice that we include G as an additional predicate in $M[G]$. See Theorem 2.1 for a sufficient and necessary condition for a lifting of j to exist.

If $j : M \to N$ is an elementary embedding, we call N the *target model* of j. Let us use V to denote the current ambient universe.

Two main methods are used for lifting, in particular for finding the required generic filter H:

(A) We find H in $V[G]$ to retain the definability of j^+ in $V[G]$ (provided j itself was definable in V and G is $\mathbb{P}$-generic over V). This is used for showing that κ is preserved as a large cardinal.

 In the simplest configuration, it is enough to construct H by a counting argument which ensures that we meet all maximal antichains in $j(\mathbb{P})$ which are elements of the target model, while making sure that H satisfies the necessary criterion for lifting, i.e. $j''G \subseteq H$ (see Theorem 2.1). The latter task is much easier if we can show that there exists $q \in j(\mathbb{P})$ which is below all elements in $j''G$. Such a q is called a *master condition*: if q is a master condition, then $q \in H$ implies $j''G \subseteq H$.

 In other situations an ad hoc argument, or an argument specific for a given class of forcings, is often required for lifting: see Section 3 for a method based on modifying an existing generic filter, and Section 4 for the situation in which $j''G$ generates the required H.

(B) We force H to exist in some further generic extension of $V[G]$. κ may cease to be a large cardinal (depending on the nature of the generic extension), but it can still retain some desirable combinatorial properties (the tree property, stationary reflection, etc.).

 There are two challenges in forcing H to exist: First, we need to argue that $j(\mathbb{P})$ has reasonable properties over $V[G]$ over which we wish to force with it, in particular that it does not collapse cardinals we wish to preserve. This is not automatic even for very simple forcings $\mathbb{P}$ – while $j(\mathbb{P})$ may have nice properties in the target model (by elementarity), its properties over $V[G]$ may be ill-behaved, or difficult to compute. Second, we need to argue that we can choose a $j(\mathbb{P})$-generic filter H over $V[G]$ which contains $j''G$.

In case (B), if $j^+ : M[G] \to N[H]$ exists in some generic extension $V[G^*]$ which contains $V[G]$, we say that j^+ is a *generic elementary embedding*, meaning that it is added by G^*. The critical point of a generic elementary embedding is typically a small cardinal in $V[G]$, and may not be a cardinal in $V[G^*]$.

We will attempt to review the most important examples for both these methods, with focus on those which appeared only after the comprehensive and clearly written [5] was published (but we will often refer to [5] for context and definitions). The selection of the examples is subjective and is limited both by the length of the article and our preferences and knowledge. Here is a brief summary of the topics:

- Silver first showed how to obtain a measurable cardinal κ with $2^\kappa = \kappa^{++}$ starting with a κ^{++}-supercompact cardinal κ (see [5, Section 12] for details). The argument uses a master condition for the lifting, making an essential use of the fact that if $G \subseteq \mathrm{Add}(\kappa, \kappa^{++})$ is a generic filter, then $j''G \subseteq j(\mathrm{Add}(\kappa, \kappa^{++}))$ is an element of the target model and therefore $\bigcup j''G$ is a legitimate condition in $j(\mathrm{Add}(\kappa, \kappa^{++}))$, which is used as a master condition. Magidor (see [5, Section 13]) modified the argument by approximating the master condition by a diagonal construction, starting with just a κ^+-supercompact κ.

  Woodin showed that a much smaller large cardinal is sufficient (and is actually optimal) for obtaining a measurable cardinal which violates GCH: it suffices if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \kappa^{++}$ and M is closed under κ-sequences in V. Such a cardinal is called κ^{++}-*tall*. Tallness is an important weakening of a $\kappa + 2$-*strong* cardinal. Using the so called *surgery argument for the Cohen forcing*, such a j can be lifted using a more complicated argument which is described in [5, Section 25] and also slightly differently in [3]. Woodin's argument follows case (B): first it is shown that a certain forcing of the form $i(\mathbb{P})$ behaves well over the current universe, an $i(\mathbb{P})$-generic filter h is forced over the universe (where i is a normal ultrapower embedding derived from the extender embedding j), a generic filter H for $j(\mathbb{P})$ is constructed from h, and then H is modified to H^* which fits the criterion $j''G \subseteq H^*$.

  In Section 3 we briefly review Woodin's argument and follow up with a description of the technique from [2] which extends Woodin's argument to a more general setting of an Easton-like result for a cardinal κ which is both λ-supercompact and μ-tall for some regular λ, μ with $\kappa \leq \lambda < \lambda^{++} \leq \mu$.

  We also mention that the original Woodin's method can be used to obtain indestructibility of a degree of tallness or strongness under the Cohen forcing or the Mitchell forcing ([16] and [18]).
- The surgery method – powerful as it is – seems to be ill-suited for dealing with general iterations because it requires a manual modification of a generic filter to ensure $j''G \subseteq H$. It is harder to do this if conditions are composed of names.

  As it turns out, a λ-tall embedding with critical point κ can be lifted more easily, provided the forcing notion we are lifting has certain "fusion-like" properties (for instance the generalized κ-Sacks forcing has them, but the κ-Cohen forcing does not).[1] This method

[1] To indicate on which cardinal κ the current forcing lives, we often say κ-Sacks forcing, κ-Cohen forcing, etc.

originated in [13] and has been used since then to deal with more complex iterations. Unlike the surgery method, it does not use a manual modification of a filter; instead, it uses an observation that with a suitable j, for a generic filter $G \subseteq \mathbb{P}$, $j''G$ generates a generic filter for $j(\mathbb{P})$ (this is false for the κ-Cohen forcing but true for a version of the κ-Sacks forcing provided j has certain properties). We briefly review this method in Section 4.

- While Sections 3 and 4 deal with cases (A) + (B) which preserve κ as a large cardinal, Section 5 deals with case (B) in which the critical point is turned into a small successor cardinal. We will review how lifting is used to argue for the consistency of various compactness principles, such as the *tree property* and *stationary reflection*, at small successor cardinals (for instance ω_2).

2. Preliminaries

We will follow the notation from [5], where the reader finds all definitions which we are going to use here.

In this section we briefly summarize some background information which we will use frequently.

2.1. Silver's lifting lemma

An observation due to Silver gives an if and only if condition for the existence of a lifting of an elementary embedding to a generic extension. We include this condition for completeness.

Theorem 2.1 (Silver). *Let $j : M \to N$ be an elementary embedding between transitive models of* ZFC.[2] *Let $\mathbb{P} \in M$ be a forcing notion, let G be $\mathbb{P}$-generic over M and let H be $j(\mathbb{P})$-generic over N. Then the following are equivalent:*

(i) $j''G \subseteq H$,

(ii) There exists an elementary embedding $j^+ : M[G] \to N[H]$, such that $j^+(G) = H$ and $j^+ \restriction M = j$.

It is easy to see that the lifted embedding j^+ has similar properties as j (e.g. if j is an extender embedding, so is j^+, and the supports are the same; see [5, Section 9] for details).

2.2. Regular embeddings from elementary embeddings

Recall that if $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions, then $i : \mathbb{P} \to \mathbb{Q}$ is called *a regular embedding* if for all $p, q \in \mathbb{P}$, (i) $p \le q \to i(p) \le i(q)$, (ii) $p \perp q \leftrightarrow i(p) \perp i(q)$, and for every maximal antichain $A \subseteq \mathbb{P}$, $i''A$ is a maximal antichain in $\mathbb{Q}$.

The following is standard (see for instance [5]).

Fact 2.2. *Assume $\mathbb{P}, \mathbb{Q}$ are forcing notions, G is a $\mathbb{P}$-generic filter, and $i : \mathbb{P} \to \mathbb{Q}$ is a regular embedding. Then $\mathbb{Q}$ is equivalent to $\mathbb{P} * \mathbb{Q}/\dot{G}$, where $\mathbb{Q}/\dot{G}$ is a $\mathbb{P}$-name for a forcing notion with conditions*

$$\{q \in \mathbb{Q} \mid q \text{ is compatible with } i''G\}, \tag{2.3}$$

[2]We assume everything happens in some ambient universe V which contains M, N, j as elements (if they are sets), or M, N, j are definable in V (if they are proper classes).

with the ordering inherited of $\mathbb{Q}$*. We write* $\mathbb{Q}/G$ *for the interpretation of* $\mathbb{Q}/\dot{G}$ *in* $V[G]$ *and call* $\mathbb{Q}/G$ the quotient of $\mathbb{Q}$ over G*. Sometimes, we also write* $\mathbb{Q}/\mathbb{P}$ *if a specific* G *is not important.*

Notice that the quotient $\mathbb{Q}/\dot{G}$ is defined in V and strictly speaking depends on i (which will usually be given by the context). We find it useful to relativize this definition to transitive models of set theory other than V. Let M be a transitive model of set theory and $\mathbb{P} \in M$ a forcing notion; we define $\mathrm{MaxAntichain}(\mathbb{P})^M$ to be the set of all maximal antichains of $\mathbb{P}$ which are elements of M.

Definition 2.3. Let M and N be two transitive models of set theory and $\mathbb{P} \in M$ and $\mathbb{Q} \in N$ partial orders. We say that $i : \mathbb{P} \to \mathbb{Q}$ is an (M,N)*-regular embedding* if i satisfies (i) and (ii) from the definition of the regular embedding and moreover for every $A \in \mathrm{MaxAntichain}(\mathbb{P})^M$, $i''A \in \mathrm{MaxAntichain}(\mathbb{Q})^N$.

It is clear from the definition that if i is an (M,N)-regular embedding, then whenever H is $\mathbb{Q}$-generic over N, then $G = i^{-1''}H$ is $\mathbb{P}$-generic over M.

We will make use of the following fact:

Fact 2.4. *Assume* $j : M \to N$ *is an elementary embedding with critical point* λ *between a pair of transitive models of set theory and let* $\mathbb{P} \in M$ *be a partial order such that* $M \models$ *"*$\mathbb{P}$ *is* λ*-cc". Then the following hold:*

(i) The restriction $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ *is an* (M,N)*-regular embedding. In particular, if* H *is* $j(\mathbb{P})$*-generic over* N *and* $G = j^{-1''}H$*, then* j *lifts to*

$$j : M[G] \to N[H]. \tag{2.4}$$

(ii) Moreover, if

$$j \restriction \mathbb{P} \in N \text{ and } \mathrm{MaxAntichain}(\mathbb{P})^N \subseteq \mathrm{MaxAntichain}(\mathbb{P})^M, \tag{2.5}$$

then

$$N \models j \restriction \mathbb{P} \text{ is a regular embedding from } \mathbb{P} \text{ into } j(\mathbb{P}) \text{ and } j(\mathbb{P}) \text{ is equivalent to } \mathbb{P} * j(\mathbb{P})/\dot{G}. \tag{2.6}$$

Proof. (i) By elementarity, j preserves the ordering relation and compatibility between $\mathbb{P}$ and $j(\mathbb{P})$. To argue for regularity, it suffices to show that if $A \in M$ is a maximal antichain in M, then $j''A \in N$ is a maximal antichain in $j(\mathbb{P})$. This follows immediately by elementarity and the fact $j''A = j(A)$, which holds since $M \models$ "$|A| < \lambda$", and j is the identity below λ.

(ii) First notice that $j \restriction \mathbb{P} \in N$ implies that $\mathbb{P} = \mathrm{dom}(j \restriction \mathbb{P}) \in N$. To be able to carry out the quotient analysis from Fact 2.2 in N, it suffices to assume that $j \restriction \mathbb{P}$ is a regular embedding in N which follows from the fact that it is an (M,N)-regular embedding and (2.5) holds. □

When G is $\mathbb{P}$-generic over N and item (ii) of Fact 2.4 applies, the definition of the quotient $j(\mathbb{P})/G$ is expressible in $N[G]$ and we can write:

$$j(\mathbb{P})/G = \{p^* \in j(\mathbb{P}) \mid N[G] \models p^* \text{ is compatible with } j''G\}. \tag{2.7}$$

One could try to weaken the assumption (2.5) and ask just for $\mathbb{P}$ being an element of N which is easier to ensure in general. With the assumption that $\mathbb{P} \in N$, we could write $N[G]$; however it is not clear under which circumstances the quotient forcing $j(\mathbb{P})/G$ is an element of $N[G]$.

3. Surgery-type arguments

Recall the central part of Woodin's argument for lifting a κ^{++}-tall embedding $j : V \to M$ to the forcing iteration $\mathbb{P} = \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^{++})$,[3] where $\mathbb{P}_\kappa$ is the Easton-support iteration of $\mathrm{Add}(\alpha, \alpha^{++})$ for inaccessible cardinals $\alpha < \kappa$.[4] Suppose $G * g$ is $\mathbb{P}$-generic over V, and there exists a generic filter h_0 over $V[G][g]$ for a certain κ^{++}-cc and κ^+-distributive forcing[5] $\mathbb{R}_0$ so that in $V[G][g][h_0]$:

- j lifts to $j : V[G] \to M[G][g][\tilde{h}]$, for some filter $\tilde{h}$ for the tail of the iteration $j(\mathbb{P})$ defined on the interval$(\kappa, j(\kappa))$.
- $G * g * \tilde{h} * h_1$ is $j(\mathbb{P})$-generic over M, for some filter h_1 for the forcing $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G]}$.

It can be shown that if this configuration arises using the methods described in [3] or [5, Section 25], then $j''g \not\subseteq h_1$, and hence j cannot be lifted. However, an additional argument – *the surgery* – is invoked which uses properties of the Cohen forcing to argue that in $V[G][g][h_0]$, there exists h_2 with the following properties:

- $G * g * \tilde{h} * h_2$ is $j(\mathbb{P})$-generic over M.
- $j''g \subseteq h_2$.

It follows that j can be lifted to $j : V[G][g] \to M[G][g][\tilde{h}][h_2]$, and since h_0 was added by a κ^+-distributive forcing notion,[6] it is possible to lift j further to

$$j : V[G][g][h_0] \to M[G][g][\tilde{h}][h_2][h_0^*], \text{ for some } h_0^*,$$

concluding that κ is still measurable in $V[G][g][h_0]$.

The surgery argument itself uses some specific combinatorial properties of the Cohen forcing (see [3, Subsection 6, Fact 2] for more details) and proceeds as follows: one can manually modify each $p \in h_1$ on the set $\mathrm{dom}(p) \cap j''(\kappa^{++} \times \kappa)$ to match $j''g$ (for any p, this set has size at most κ). Let us call this modified condition p^*. h_2 is the collection $\{p^* \mid p \in h_1\}$. Once it is shown that h_2 is still $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G]})$-generic over $M[G][g][\tilde{h}]$, we are done because $j''g \subseteq h_2$ is now true by the construction.

Cody and Magidor [2] generalized the surgery method to a λ-supercompact cardinal κ which is also μ-tall for some regular μ with $\kappa \le \lambda < \lambda^{++} \le \mu$, performing surgery also on the "ghost coordinates". More precisely, they controlled by means of the Cohen forcing the continuum function on the interval

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[3] We identify conditions in $\mathrm{Add}(\kappa, \kappa^{++})$ with partial functions of size $< \kappa$ from $\kappa^{++} \times \kappa$ to 2.

[4] We assume for simplicity that $\kappa^{++} = (\kappa^{++})^M$. If not, it is possible to define the iteration using a function $f : \kappa \to \kappa$ which satisfies $j(f)(\kappa) = \kappa^{++}$; see [14] or [16] for more details.

[5] In this particular case, $\mathbb{R}_0$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{++})$ defined over a certain submodel of $V[G][g]$. See [18, Section 3.1] for more details.

[6] Strictly speaking, this requires an extender representation of j (see [5, Proposition 15.1].

$[\kappa, \lambda]$ while preserving the initial large-cardinal strength of κ. Woodin's argument does not apply directly because while $j(p) = j''p$ in Woodin's case, in the context of [2], if p is a condition in $\mathrm{Add}(\delta, \alpha)$ for a regular δ in $(\kappa, \lambda]$, then in general $j(p)$ is no longer equal to its pointwise image $j''p$. The elements in $j(p) \setminus j''p$ are called the "ghost coordinates" (of the condition). This generalization is spelled out in [2, Lemma 4]. Note that their method is also limited to the Cohen forcing.

Incidentally, there are two presentations of Woodin's original construction which differ in the sequence of steps for obtaining h_0. The forcing $\mathbb{R}_0$ can be used either over $V[G][g]$ as described above (as is done in [5, Section 25]), or forced beforehand as described in [3] ([3, Subsection 5, Fact 2] makes it possible). In the latter approach, the extra forcing can be tucked-in into a preliminary stage, allowing an indestructibility result for tall cardinals with respect to Cohen forcing of a fixed length (see [16]) or strong cardinals of a given degree with respect to Cohen and Mitchell forcing up to a fixed length (see [18]). However, one should bear in mind that in either approach, the size of 2^{κ^+} is increased non-trivially (proportionally to the length of the Cohen forcing which should preserve the largeness of κ), unlike the analogous Laver's indestructibility result for supercompact cardinals which retains GCH above κ if it holds in V.

It is open whether a similar surgery argument is available for iterations. As we will see in the next section, lifting of iterations can be done using a different method which uses a fusion argument. However, the fusion argument yields only the least possible failure of GCH at a measurable κ: $2^\kappa = \kappa^{++}$. The reason is that the iteration has support of size $\le \kappa$. A surgery argument applied with a κ^+-cc iteration with $< \kappa$-support could possibly achieve $2^\kappa > \kappa^{++}$ in arguments such as [9].

We will review the fusion-based approach in the next section.

4. Fusion-type arguments

While the κ-Cohen forcing for a regular $\kappa \ge \omega$ is usually the easiest test example for many applications, it may not be the case for the lifting of elementary embeddings. In hindsight, Woodin's surgery argument overcomes obstacles which are specific for forcings with supports of size $< \kappa$ (and in particular for the κ-Cohen forcing). There are other forcings which add fresh subsets[7] of κ and can be lifted without the need to provide extra generic filters which need to be modified later.

This was first observed by Friedman and Thompson in [13] for the κ-Sacks forcing. We will briefly review the method, but we will focus on the κ-Grigorieff forcing for more variety. Our exposition follows [21].

4.1. Grigorieff forcing at an inaccessible cardinal

Let κ be an inaccessible cardinal. Unless we say otherwise, I denotes a κ-complete proper ideal on κ.

[7] $x \subseteq \kappa$ is fresh in $V[G]$ if $x \cap \alpha \in V$ for all $\alpha < \kappa$ but $x \notin V$.

Definition 4.1. Let κ be inaccessible[8] and let I be a subset of $\mathscr{P}(\kappa)$. Let us define

$$P_I = \{f \vdots \kappa \to 2 \,|\, \mathrm{dom}(f) \in I\},$$

where $f \vdots \kappa \to 2$ is a partial function from κ to 2. Ordering is by the reverse inclusion: for p, q in P_I, $p \leq q \leftrightarrow p \supseteq q$.

Notice that if we let I be the ideal of bounded subsets of κ, we obtain the usual Cohen forcing.

Definition 4.2. For $\alpha < \kappa$ write

$$p \leq_\alpha q \leftrightarrow p \leq q \,\&\, \mathrm{dom}(p) \cap (\alpha+1) = \mathrm{dom}(q) \cap (\alpha+1).$$

We say that $\langle p_\alpha \,|\, \alpha < \kappa\rangle$ is a *fusion sequence* if for every α, $p_{\alpha+1} \leq_\alpha p_\alpha$ and for limit γ, $p_\gamma = \bigcup_{\alpha<\gamma} p_\alpha$.

The following theorem is standard (see [21, Theorem 2.6]).

Theorem 4.3. *Assume* GCH *and let I be a κ-complete ideal extending the nonstationary ideal on κ (κ inaccessible). Then P_I preserves cofinalities if and only if I is a normal ideal.*

We will consider the following generalization of the definition of $\leq_\alpha$ and of the fusion construction. Let I be a normal ideal on κ and $S \in I^*$, where I^* is the filter dual to I, i.e. $I^* = \{X \subseteq \kappa \,|\, \kappa \setminus X \in I\}$. We will assume that S is composed of limit ordinals; this is without loss of generality because we can always shrink S by intersecting it with the limit ordinals, and still stay in I^*. Let P_I be the forcing defined above.

Definition 4.4. Define the relation $\leq^S_\alpha$ as follows.

(i) if α is in S:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \,\&\, \mathrm{dom}(p) \cap (\alpha+1) = \mathrm{dom}(q) \cap (\alpha+1)$$

(ii) if α is in $\kappa \setminus S$:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \,\&\, \mathrm{dom}(p) \cap \alpha = \mathrm{dom}(q) \cap \alpha.$$

We say that $\langle p_\alpha \,|\, \alpha < \kappa\rangle$ is an S-fusion sequence if $p_{\alpha+1} \leq^S_\alpha p_\alpha$ for every α and $p_\gamma = \bigcup_{\alpha<\gamma} p_\alpha$ for limit γ.

Notice that $S = \kappa$ gives the original definition of $\leq_\alpha$ and fusion.
The following lemma is easy to check.

Lemma 4.5. *Assume I is a normal ideal on κ, and S is a set in I^* which contains only limit ordinals. Then P_I is closed under limits of S-fusion sequences.*[9]

To prevent a possible misunderstanding, notice that to be a fusion sequence or an S-fusion sequence for $S \in I^*$ in P_I are properties of certain sequences of conditions in the same underlying forcing notion $(P_I, \leq)$.

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[8] Much of what follows also holds for a successor $\kappa = \mu^+$ provided $2^\mu = \mu^+$; we focus here on an inaccessible κ for simplicity.

[9] That is, for every S-fusion sequence $\langle p_\alpha \,|\, \alpha < \kappa\rangle$, $p = \bigcup_{\alpha<\kappa} p_\alpha$ is a condition, with $p \leq^S_\alpha p_\alpha$ for every $\alpha < \kappa$.

4.2. Lifting of the Grigorieff forcing

Let us fix some notation first.

Definition 4.6. Assume κ is regular and $\mathrm{Club}(\kappa)$ is the closed unbounded filter on κ. Let S be stationary. Define:

$$\mathrm{Club}(\kappa)[S] = \{X \subseteq \kappa \,|\, \exists C \text{ closed unbounded in } \kappa \text{ and } X \supseteq S \cap C\}.$$

The following is routine.

Lemma 4.7. *For every stationary S,* $\mathrm{Club}(\kappa)[S]$ *is a normal proper filter which contains S and extends* $\mathrm{Club}(\kappa)$.

We will study the forcing P_I with I being the dual ideal of a normal proper filter of the form $\mathrm{Club}(\kappa)[S]$.

Definition 4.8. Let $j : V \to M$ be an elementary embedding with critical point κ from the universe into a transitive class M. We say that a normal ideal I on κ *lifts for* (j, S) if

$$S \in I^* \text{ and } \kappa \in j(\kappa \setminus S).$$

Example 4.9. The nonstationary ideal on κ does not lift for any (j, S) because κ is an element of $j(C)$ for every closed unbounded subset C of κ. For any regular $\mu < \kappa$, let E^μ_κ denote the set of all limit ordinals with cofinality μ. If I is dual to $\mathrm{Club}(\kappa)[E^\mu_\kappa]$, then I lifts for (j, E^μ_κ) for any j.

Definition 4.10. Let P be a forcing notion and let κ be a regular cardinal. Assume that every decreasing sequence of conditions in P of length $\leq \kappa$ has an infimum in P and let $X \subseteq P$ be given. Then

$$\begin{aligned}(4.8)\quad \mathrm{Cl}_{\leq\kappa}X = \{p \in P \mid\ & \text{for some decreasing sequence } \langle p_\alpha \,|\, \alpha < \kappa\rangle \text{ with} \\ & p_\alpha \in X \text{ for all } \alpha < \kappa, \text{ the infimum of } \langle p_\alpha \,|\, \alpha < \kappa\rangle \text{ is less or equal to } p\}\end{aligned}$$

is called the *κ-closure of X*.

It is easy to see that that if X is a directed family (for every x, y in X there exists z in X such that $z \leq x$ & $z \leq y$) closed under limits of sequences of length less than κ, then $\mathrm{Cl}_{\leq\kappa}X$ is a filter in P.

The idea behind the lifting of the Grigorieff forcing is to argue that the κ-closure $\mathrm{Cl}_{\leq\kappa}(j''g)$, where g is a generic for $\mathbb{P}_I$ and I is a normal ideal on κ which lifts for (j, S) for some S, is already a generic filter for $j(P_I)$. This is in stark contrast with the κ-Cohen forcing $\mathrm{Add}(\kappa, 1)$: If g is $\mathrm{Add}(\kappa, 1)$-generic, then the κ-closure of $j''g$ is equal just to $g \cup \{\bigcup g\}$ which yields a function with domain κ while every $j(\mathrm{Add}(\kappa, 1))$-generic must yield a function with domain $j(\kappa)$. The reason is that for every $p \in \mathrm{Add}(\kappa, 1)$, $j(p) = p$ because $|\mathrm{dom}(p)| < \kappa$. Allowing conditions with $|\mathrm{dom}(p)| = \kappa$ as in P_I overcomes this limitation.

Let us show how the argument works for the simple case of a normal measure ultrapower. Assume GCH and let $j : V \to M$ be an ultrapower embedding with critical point κ. In particular $M = \{j(f)(\kappa) \,|\, f : \kappa \to V\}$. Consider a forcing of the form $\mathbb{P} * \dot{P}_I$, where $\mathbb{P}$ is a reverse Easton iteration with $\mathbb{P} \subseteq V_\kappa$ and $\dot{P}_I$ is a $\mathbb{P}$-name for the Grigorieff forcing, where I is a normal ideal which lifts for (j, S) for some S. Think of $\mathbb{P}$ as the standard preparation for P_I. Let $G * g$ be $\mathbb{P} * \dot{P}_I$-generic over V and assume we can lift j partially to $j : V[G] \to M^* = M[G][g][H]$ for some $H \in V[G][g]$. It will hold that

$$M^* = \{j(f)(\kappa) \mid f \in V[G] : \kappa \to V[G]\}. \tag{4.9}$$

Lemma 4.11. *$\mathrm{Cl}_{\leq\kappa}(j''g)$ is a $j(P_I)$-generic filter over M^*.*

Proof. Let us denote $h = \mathrm{Cl}_{\leq\kappa}(j''g)$. It is clear that h is a filter and is well-defined because by standard arguments, M^* is closed under κ-sequences in $V[G * g]$, and $j(P_I)$ is κ^+-closed in M^*.

By (4.9), every dense open set in $j(P_I)$ is of the form $j(f)(\kappa)$ for some $f : \kappa \to H(\kappa^+)^{V[G]}$ in $V[G]$. Moreover, we can assume that $\langle f(\alpha) \mid \alpha < \kappa\rangle$ is in $V[G]$ a sequence of dense open sets in P_I for every such f. Let us fix a dense open set $D = j(f)(\kappa)$. It suffices to show that for any p, there is a condition $p^* \leq p$ which satisfies the following items:

(i) p^* is a limit of an S-fusion sequence $\langle p_\alpha \mid \alpha < \kappa\rangle$ such that $\alpha \in \mathrm{dom}(p_\alpha)$ for every $\alpha \in \kappa \setminus S$.

(ii) For every $\alpha < \kappa$, whenever d is a condition with $\mathrm{dom}(d) = \alpha + 1$ which extends the (partial) condition $p^* \restriction (\alpha + 1)$, then $d \cup (p^* \restriction [\alpha + 1, \kappa))$ is in the dense open set $f(\alpha)$.

It is easy to construct such a sequence using the fusion properties of P_I.

We argue as follows to show that (i) and (ii) are sufficient: by density, there is some such p^* in g. By (i) and (ii), elementarity and by I lifting for (j, S), $p^{**} = \bigcup g \cup j(p^*)$ is a condition whose domain includes $\kappa + 1$ and is therefore an element of $j(f)(\kappa) = D$. It is also easy to see that p^{**} is in h, and we are done. □

Remark 4.12. The lifting of j described in the previous paragraph is not very interesting because j is just a normal measure embedding and $2^\kappa = \kappa^+$. However, the argument naturally generalizes to a (κ, λ)-extender embeddings j, with $\kappa^+ < \lambda$ regular. Let $h : \kappa \to \kappa$ be chosen so that $j(h)(\kappa) \geq \lambda$. We can assume that every dense open set in $j(P_I)$ is of the form $j(f)(\delta)$ for some $f : \kappa \to H(\kappa^+)^{V[G]}$ and $\delta < \lambda$. Let us fix an arbitrary dense open set $D = j(f)(\delta)$. It suffices to modify the properties of the S-fusion sequence $\langle p_\alpha \mid \alpha < \kappa\rangle$ mentioned above so that the condition (ii) above requires that $d \cup (p^* \restriction [\alpha + 1, \kappa))$ should be in $\bigcap_{\beta < h(\alpha)} f(\beta)$. Then $\bigcup g \cup j(p^*)$ meets every dense open set indexed below $j(h)(\kappa)$, which means it meets D. With lifting available for extender embeddings, one can use a $\leq \kappa$-supported product[10] of the forcings P_I to reprove – without a surgery argument – Woodin's original result (see [21] for more details).

4.3. Generalizations and applications

Let us discuss some background information and applications of the method discussed in Section 4.2.

[10]Notice that in dealing with the product of the κ-Grigorieff forcing, we need to deal with "ghost coordinates", similarly as we discussed in Section 3. However, there is a difference: the ghost coordinates for the Cohen forcing appear only if we force over a regular cardinal larger than κ, while with the κ-Grigorieff forcing this phenomenon appears already at stage κ – the reason is of course that the conditions in the κ-Grigorieff forcing have size $\leq \kappa$, and the support of the product has also size $\leq \kappa$.

(1) The lifting via fusion was introduced in [13] using the κ-Sacks forcing (see [24] for more details about the κ-Sacks forcing): the forcing is composed of κ-perfect trees viewed as subsets of $2^{<\kappa}$ which have continuous splitting: whenever $\langle x_\alpha \,|\, \alpha < \delta\rangle$ is a strictly extending sequence of nodes in a tree p with $\delta < \kappa$ a limit ordinal, then if the splitting nodes are unbounded in $x = \bigcup\{x_\alpha \,|\, \alpha < \delta\}$, then x is a splitting node in p. This definition has the effect that $j''g$ (in the notation of the previous section) does not generate a generic filter because every tree $j(p)$, with $p \in g$, splits at level κ (this feature was dubbed the "tuning fork argument" in [13]). While $j''g$ does not generate a generic filter, it "almost" generates it: once we choose for every $p \in g$ whether we go to the left or to the right on the level κ in $j(p)$ (consistently for all p), then we do get a generic filter. On the other hand, we may slightly modify the definition of the forcing to ensure that $j''g$ generates a generic filter: it suffices to modify the definition of the forcing to require the continuous splitting only for δ's of a certain cofinality (such as δ in E^μ_κ in Example 4.9).

The control of cofinality of δ is also used in our treatment of the κ-Grigorieff forcing: the stationary set S in Definition 4.4 controls which ordinals can be added to the domains of conditions in a fusion sequence and which may not be added and consequently controls whether $j''g$ generates a generic filter.

This flexibility of controlling the number of possible generic filters, and consequently the number of liftings, – exactly one for the κ-Grigorieff forcing (e.g. with an ideal containing the complement of E^μ_κ) and exactly two for the κ-Sacks forcing in [13] – was exploited in a paper by Friedman and Magidor [12]. They generalized the definition of the κ-perfect tree and controlled the number of normal measures at κ in the final model by prescribing the size of the set of continuations of a splitting node.

(2) An important advantage of the lifting with fusion is the ability to handle iterations. In [9], a model is constructed in which $2^{\aleph_\omega} = \aleph_{\omega+2}$, $\aleph_\omega$ is strong limit, and there is a well-ordering of the subsets of $\aleph_\omega$ lightface definable in $H(\aleph_{\omega+1})$. The argument starts with a $(\kappa+2)$-strong cardinal κ in an extender model $L[\vec{E}]$. Over this model, a cofinality-preserving Easton-supported iteration $\mathbb{P} = \lim\langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \,|\, \alpha \le \kappa\rangle$ is defined where for each inaccessible $\alpha \le \kappa$, $\dot{\mathbb{Q}}_\alpha$ adds (among other things) α^{++}-many new subsets to α using a version of the α-Sacks forcing. The future well-ordering of the subsets of α is coded by means of selective kills of certain stationary subsets of α^+; this information is in turn localized by a α^+-distributive forcing which ensures the lightface definability in $H(\alpha^+)$. In the context of this survey it is important that the iteration $\dot{\mathbb{Q}}_\kappa$ may be lifted using a fusion-type argument along the lines of Section 4.2, without any surgery. Since κ remains measurable after forcing with $\mathbb{P}$, the proof concludes by using a version of the Prikry forcing with collapses to turn κ to $\aleph_\omega$ while preserving the definability of the well-order.

At the moment, results like this seem to be out of reach for any method based on surgery.[11] For instance, it is an open problem how to achieve a definability result such as [9] with a gap larger than 2, for instance to have $2^\kappa > \kappa^{++}$, κ measurable, and a well-ordering of subsets of κ (lightface) definable in $H(\kappa^+)$.

(3) We may attempt to characterize the class of forcings which add fresh subsets to a measurable cardinal κ and which can be lifted using an argument based on fusion. Interestingly, this characterization (or rather the resulting class of forcings) is very similar to a class of forcings with conditions of size κ for which a reasonable notion of κ-properness may be formulated. See for instance [11], [19], and [28] for more details.

(4) In [8], the κ-Sacks forcing was used to prove a version of Easton's theorem for the continuum function while preserving certain large cardinals.

5. Generic embeddings

In this section, we discuss case (B) from Section 1, with focus on the tree property and stationary reflection. Recall the following definitions:

Definition 5.1. Let λ be a regular cardinal. We say that the *tree property* holds at λ, and we write $\mathsf{TP}(\lambda)$, if every λ-tree has a cofinal branch.

Definition 5.2. Let λ be a cardinal of the form $\lambda = \nu^+$ for some regular cardinal ν. We say that the *stationary reflection* holds at λ, and write $\mathsf{SR}(\lambda)$, if every stationary subset $S \subseteq \lambda \cap \mathrm{cof}(< \nu)$ reflects at a point of cofinality ν; i.e. there is $\alpha < \lambda$ of cofinality ν such that $\alpha \cap S$ is stationary in α.

More information about these properties can be found in [4] and [22].

5.1. The tree property

Let us start with a quick review of a typical argument which uses lifting of an embedding to obtain a large cardinal property at a small cardinal. We sketch the argument that if there is a weakly compact cardinal λ, then there is a generic extension where the tree property holds at ω_2.

Recall the following definition which is implicit in [27] and the present form is taken from [1].

Definition 5.3. Suppose $\omega \le \kappa \le \lambda$ are regular cardinals and λ is inaccessible. Conditions in the Mitchell forcing, $\mathbb{M}(\kappa, \lambda)$, are pairs (p^0, p^1) such that $p^0 \in \mathrm{Add}(\kappa, \lambda)$ and p^1 is a function with domain $\mathrm{dom}(p^1) \subseteq \lambda$ of size at most κ. For α in the domain of p^1, $p^1(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name and

$$1_{\mathrm{Add}(\kappa,\alpha)} \Vdash p^1(\alpha) \in \mathrm{Add}(\kappa^+, 1)^{V[\mathrm{Add}(\kappa,\alpha)]}. \tag{5.10}$$

The ordering is defined as follows: $(p^0, p^1) \le (q^0, q^1)$ iff $p^0 \le q^0$ in $\mathrm{Add}(\kappa, \lambda)$ and the domain of p^1 extends the domain of q^1, and for every $\alpha \in \mathrm{dom}(q^1)$,

$$p^0 \restriction \alpha \Vdash_{\mathrm{Add}(\kappa,\alpha)} p^1(\alpha) \le q^1(\alpha), \tag{5.11}$$

where $p^0 \restriction \alpha$ is the restriction of p^0 to $\mathrm{Add}(\kappa, \alpha)$.

[11] Note that supercompact cardinals – which lift more easily – will not help here since there are no canonical inner models for supercompact cardinals such as $L[\vec{E}]$ for the coding to work properly in the non-GCH context.

If κ, λ are understood from the context, we write just $\mathbb{M}$. For $\alpha < \lambda$, let us denote by $\mathbb{M}_\alpha$ the natural truncation of $\mathbb{M}$ to α (we write $(p^0, p^1) \restriction \alpha$ for the restriction of (p^0, p^1) to $\mathbb{M}_\alpha$).

Using the Abraham's analysis (see [1]), there is a projection onto $\mathbb{M}$ from the product $R^0 \times R^1$ where $R^0 = \mathrm{Add}(\kappa, \lambda)$ is κ^+-Knaster (under the assumption $\kappa^{<\kappa} = \kappa$) and R^1 is κ^+-closed (the "term" forcing). This analysis also holds for the natural quotients of $\mathbb{M}$ and R^0 and R^1: in particular,

(5.12) if $\alpha < \lambda$ is inaccessible, then there is a projection onto $\mathbb{M}/\mathbb{M}_\alpha$ from $R^0_\alpha \times R^1_\alpha$,

where, under the appropriate assumptions, the forcing R^0_α is κ^+-Knaster (in fact, it is equivalent to $\mathrm{Add}(\kappa, \lambda)$) and R^1_α is κ^+-closed in $V[\mathbb{M}/\mathbb{M}_\alpha]$.

Finally recall that if λ is weakly compact, then this fact is witnessed by the existence of elementary embeddings with critical point λ between transitive models of ZFC^- of size λ which are closed under $< \lambda$-sequences, and equivalently, λ satisfies the Π^1_1-reflection (for more details, see [5] or [25]).

Theorem 5.4. *Suppose λ is weakly compact. Then $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ forces $\lambda = \aleph_2$ and $\mathsf{TP}(\omega_2)$.*

Proof. Easton's lemma shows that $R^0 \times R^1$, and hence $\mathbb{M}$, preserves ω_1 and by design $\mathbb{M}$ turns λ to ω_2. Let us now argue for the tree property. Suppose for contradiction there is an ω_2-Aronszajn tree T in $V[G]$, where G is $\mathbb{M}$-generic. It is illustrative to look at $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ as a mixed support iteration which at many inaccessible stages $\alpha < \lambda$ deals with the restriction of $T \restriction \alpha$: for many such α, $T \restriction \alpha$ is an element of $V[G_\alpha]$, it is an $\alpha = \omega_2^{V[G_\alpha]}$-tree, and morevover by the Π^1_1-reflection of λ in the ground model, $T \restriction \alpha$ is an α-Aronszajn tree in $V[G_\alpha]$. However, there must some node of height α in the whole tree T which means that the forcing $\mathbb{M}/G_\alpha$ must add a cofinal branch to $T \restriction \alpha$. This is a contradiction since the product $R^0_\alpha \times R^1_\alpha$ cannot add such a branch on account of the so called "branch lemmas",[12] and hence neither can $\mathbb{M}/G_\alpha$.

The elementary submodel argument (which is behind the argument in the previous paragraph) is more often formulated in the language of elementary expansions and embeddings so it fits into our survey of lifting methods. Since λ is weakly compact, we can choose a transitive model M of size λ closed under $< \lambda$-sequences which contains all necessary parameters, in particular $\mathbb{M}$ and an $\mathbb{M}$-name $\dot{T}$ for an ω_2-Aronszajn tree, and for which there is an elementary embedding $j : M \to N$ with critical point λ into a transitive model N of size λ which is closed under $< \lambda$-sequences. Let H be $j(\mathbb{M})$-generic over V. $N[H]$ is a generic extension by $j(\mathbb{M})$, and since $\mathbb{M}$ is λ-cc, by Fact 2.4 we know that $j^{-1}{}''H = G$ is $\mathbb{M}$-generic over M and j lifts to

$$j : M[G] \to N[H].$$

[12]Variants of the the following two: (1) If $2^\omega = \mu$ for some regular μ (or singular with uncountable cofinality), then no σ-closed forcing can add a cofinal branch to a μ-tree. (2) If $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is ccc, then $\mathbb{P}$ does not add cofinal branches to trees whose height has cofinality ω_1. Useful generalizations appeared for instance in [7, 30, 23].

Now the argument finishes as in the first paragraph when we apply it in $N[H]$ and consider the restriction $j(T) \restriction \lambda = T$. □

Notice that in the previous proof, j restricted to $\mathbb{M}$ is the identity, so we in fact have $H = G * H^*$ where H^* is a generic filter over $N[G]$ for the tail iteration $j(\mathbb{M})$ from λ to $j(\lambda)$. But this simple analysis of H does not suffice in more complex constructions. Let us consider the following example:

Example. This example is a simplified version of [10] and shows how to get the tree property at the double successor of a singular strong limit cardinal with cofinality ω. Suppose $\mathbb{M}(\kappa, \lambda)$ forces that κ is measurable and let $\mathrm{Prk}(\dot{U})$ be the vanilla Prikry forcing which uses a normal measure $\dot{U}$ in $V[\mathbb{M}(\kappa, \lambda)]$ to add a cofinal sequence of type ω to κ without collapsing any cardinals. Then in analogy with Theorem 5.4, we consider $j : M \to N$ and forcing notions $\mathbb{P} = \mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U}) \in M$ and $j(\mathbb{P}) = j(\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})) \in N$. Since the quotient $j(\mathbb{P})/\mathbb{P}$ is no longer a "naturally" defined tail iteration of $j(\mathbb{P})$ and $j \restriction \mathbb{P}$ is not the identity, the lifting of $j : M \to N$ now proceeds as follows: Start by having H which is $j(\mathbb{P})$-generic over V; then $G = j^{-1\prime\prime} H$ is $\mathbb{P}$-generic over M and j lifts to $j : M[G] \to N[H]$. With some additional assumptions, as in Fact 2.4(ii), the quotient forcing $j(\mathbb{P})/G$ is an element of $N[G]$ and the argument finishes by showing that $j(\mathbb{P})/G$ does not add cofinal branches to λ-trees over $N[G]$. The problem now is that the quotient is not a natural forcing, but a very complex one, so the easy branch lemmas do not apply here. This obstacle can be overcome by a "hands-on" argument as in [10], where a Prikry forcing with collapses is considered, or by an appeal to indestructibility of the tree property if we in addition assume that the normal measure $\dot{U}$ for the definition of $\mathrm{Prk}(\dot{U})$ lives already in $V[\mathrm{Add}(\kappa, \lambda)]$ (see Section 5.3 for more details).

Remark 5.5. For the tree property at ω_2, there is a more robust way of "sealing-off" an α-Aronszajn tree $T \restriction \alpha$ mentioned above. Using the ideas from the argument that PFA implies $\mathsf{TP}(\omega_2)$, one can define a countable support iteration which at many inaccessible $\alpha < \lambda$ with $2^\omega = \alpha = (\omega_2)^{V[\mathbb{P}_\alpha]}$ first collapses α to ω_1 and then specialize $T \restriction \alpha$ by a ccc forcing. After specialization, the tail iteration of $\mathbb{P}$ after stage α cannot add a cofinal branch to $T \restriction \alpha$ unless ω_1 is collapsed, so there is no need to use any "branch lemmas". This makes it possible to consider complex countable support iterations $\mathbb{P}$ which preserve ω_1 so that both $\mathsf{TP}(\omega_2)$ and some other properties hold in $V[\mathbb{P}]$ (such as MA). It is not known whether such a robust method of "sealing-off" an α-Aronszajn tree works also for regular cardinals greater than ω_2.

5.2. Stationary reflection

The lifting argument for stationary reflection follows the same pattern as we discussed in Theorem 5.4:

Theorem 5.6. *Suppose λ is weakly compact. Then $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ forces $\lambda = \aleph_2$ and $\mathsf{SR}(\omega_2)$.*

Proof. In analogy with the proof of Theorem 5.4, suppose for contradiction there is a non-reflecting stationary set S in $V[G]$ which concentrates on ordinals with cofinality ω. At many inaccessible $\alpha < \lambda$, $S \cap \alpha$ is in $V[G_\alpha]$ a stationary set in $\alpha = (\omega_2)^{V[G_\alpha]}$ concentrating on ordinals with cofinality ω (by Π^1_1-reflection of λ in the ground model). However, since S is supposed to be non-reflecting, $S \cap \alpha$ cannot be stationary in $V[G]$: the contradiction is achieved by showing that the tail iteration $\mathbb{M}/G_\alpha$ does not destroy the stationarity of $S \cap \alpha$. Instead of "branch lemmas" we use "stationarity preserving lemmas",[13] which we apply to $R^0_\alpha \times R^1_\alpha$. □

Let us note that $\mathsf{SR}(\omega_2)$ does not imply $2^\omega > \omega_1$ so there is a greater variety of forcing notions to obtain stationary reflection. Also, it is known that a Mahlo cardinal is enough to get $\mathsf{SR}(\omega_2)$, see [17]. But a weakly compact cardinal is necessary for stronger forms of stationary reflection (see [26]). Finally note that TP and SR do not imply one another, see [6].

Remark 5.7. In analogy with the tree property at ω_2 – but modified to deal with S-proper forcings which unlike proper forcings (see [29] for more details) may destroy stationary sets – it is possible to "seal-off" $S \cap \alpha$ by shooting a club through $S \cap \alpha$ by means of an ω_1-distributive forcing so that it remains stationary in any extension which preserves ω_1.

5.3. Indestructibility

Instead of "branch lemmas" and "stationarity preserving lemmas" with the – often technically difficult – analysis of quotients, for instance the quotient $j(\mathbb{M}(\kappa,\lambda) * \mathrm{Prk}(\dot{U}))/\mathbb{M}(\kappa,\lambda) * \mathrm{Prk}(\dot{U})$ mentioned in the example in Section 5.1, one can attempt to formulate a more general preservation theorem. With such preservation, or indestructibility, theorems one can argue more easily for instance that $\mathbb{M}(\kappa,\lambda) * \mathrm{Prk}(\dot{U})$ forces $\mathsf{TP}(\lambda)$ and $\mathsf{SR}(\lambda)$ because $\mathbb{M}(\kappa,\lambda)$ does, and the relevant properties are preserved by $\mathrm{Prk}(\dot{U})$. The lifting argument is thus limited to $\mathbb{M}(\kappa,\lambda)$.

Stationary reflection is easier to handle because stationary sets are subsets of ordinals, while trees are binary relations on ordinals. In [20], the following is showed:

Theorem 5.8. *Suppose λ is a regular cardinal, $\mathsf{SR}(\lambda^+)$ holds and $\mathbb{Q}$ is λ-cc. Then $\mathsf{SR}(\lambda^+)$ holds in $V[\mathbb{Q}]$.*

Proof. Suppose for contradiction there are $p_0 \in \mathbb{Q}$ and $\dot{S}$ such that p_0 forces that $\dot{S}$ is a non-reflecting stationary subset of $\lambda^+ \cap \mathrm{cof}(<\lambda)$. Set

$$U_{p_0} = \{\gamma \in \lambda^+ \cap \mathrm{cof}(<\lambda) \mid \exists p \le p_0\ p \Vdash \gamma \in \dot{S}\}. \tag{5.13}$$

U_{p_0} is a stationary set: for every club $C \subseteq \lambda^+$, p_0 forces $C \cap \dot{S} \neq \emptyset$, and because p_0 also forces $\dot{S} \subseteq U_{p_0}$, it forces $C \cap U_{p_0} \neq \emptyset$, which is equivalent to $C \cap U_{p_0}$ being non-empty in V. By $\mathsf{SR}(\lambda^+)$ there is some $\alpha < \lambda^+$ of cofinality λ such that

$$U_{p_0} \cap \alpha \text{ is stationary.} \tag{5.14}$$

By our assumption

$$p_0 \Vdash \dot{S} \cap \alpha \text{ is non-stationary.} \tag{5.15}$$

[13] Most importantly, if κ is regular, than no κ-cc or κ-closed forcing can destroy the stationarity of a subset of κ. In our case we need a variant which says that a countably closed forcing cannot destroy stationarity of sets concentrating on ordinals with countable cofinality.

We will argue that (5.14) and (5.15) are contradictory, which will finish the proof.

First recall that by the λ-cc of $\mathbb{Q}$, every club subset of an ordinal α of cofinality λ in $V[\mathbb{Q}]$ contains a club in the ground model. It follows by (5.15) that there is a maximal antichain A below p_0 such that for every $p \in A$ there is some club D in α in the ground model with $p \Vdash \dot{S} \cap D = \emptyset$. Let us fix for each $p \in A$ some D_p such that $p \Vdash \dot{S} \cap D_p = \emptyset$.

Set

$$C = \bigcap \{D_p \,|\, p \in A\}. \tag{5.16}$$

C is a club subset of α because A has size $< \lambda$ and α has cofinality λ. It holds

$$p_0 \Vdash \dot{S} \cap C = \emptyset \tag{5.17}$$

because conditions forcing $\dot{S} \cap C = \emptyset$ are dense below p_0: for every $q \leq p_0$ there is some $p \in A$ which is compatible with q, and any $r \leq p, q$ forces $\dot{S} \cap D_p = \emptyset$. Since $C \subseteq D_p$, this implies $r \leq q$ forces $\dot{S} \cap C = \emptyset$.

However, by (5.14) there must be $\gamma \in C \cap U_{p_0} \cap \alpha$, and therefore some $p \leq p_0$ such that $p \Vdash \gamma \in \dot{S} \cap C$. This contradicts (5.17). $\square$

In particular $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$ forces $\mathsf{SR}(\lambda)$. By a more technical argument, one can show (see [19]):

Theorem 5.9. *Assume $\omega \leq \kappa < \lambda$ are cardinals, $\kappa^{<\kappa} = \kappa$ and λ is weakly compact. Let $\mathbb{M}$ be the standard Mitchell forcing $\mathbb{M}(\kappa, \lambda)$.*

Suppose $\mathbb{Q} \in V[\mathrm{Add}(\kappa, \lambda)]$ is κ^+-cc in $V[\mathrm{Add}(\kappa, \lambda)]$ (equivalently κ^+-cc in $V[\mathbb{M}]$), then

$$V[\mathbb{M} * \dot{\mathbb{Q}}] \models \mathsf{TP}(\kappa^{++}).$$

In other words, the tree property at κ^{++} is indestructible under any κ^+-cc forcing which lives in $V[\mathrm{Add}(\kappa, \lambda)]$.

This theorem suffices to argue that $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$, with $\dot{U}$ being a measure in $V[\mathrm{Add}(\kappa, \lambda)]$, forces $\mathsf{TP}(\lambda)$. In fact, the same theorem suffices also for the Magidor forcing in place of Prk to obtain a singular κ with uncountable cofinality. It is open, however, whether it applies to Prikry forcing with collapses.

For completeness, let us mention that [15] gives an indestructibility argument for another compactness principle, the failure of the approachability property (see [4] for a definition): the failure of the approachability property at κ^{++} is preserved by κ-centered forcings.

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