GENERALIZED CARDINAL INVARIANTS FOR AN INACCESSIBLE κ WITH COMPACTNESS AT κ^{++}

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ABSTRACT. We study the relationship between non-trivial values of generalized cardinal invariants at an inaccessible cardinal κ and compactness principles at κ^{++}. Let $\mathsf{TP}(\lambda)$, $\neg\mathsf{wKH}(\lambda)$, $\mathsf{SR}(\lambda)$ and $\mathsf{DSS}(\lambda)$ denote the tree property, the negation of the weak Kurepa Hypothesis, stationary reflection and the disjoint stationary sequence property, respectively, at a regular cardinal λ.

We show that if the existence of a supercompact cardinal κ with a weakly compact cardinal λ above κ is consistent, then the following are consistent as well (where $\mathfrak{t}(\kappa)$ and $\mathfrak{u}(\kappa)$ are the tower number and the ultrafilter number, respectively):

(i) There is an inaccessible cardinal κ such that $\kappa^+ < \mathfrak{t}(\kappa) = \mathfrak{u}(\kappa) < 2^\kappa$ and $\mathsf{SR}(\kappa^{++})$ and $\mathsf{DSS}(\kappa^{++})$ hold, and

(ii) There is an inaccessible cardinal κ such that $\kappa^+ = \mathfrak{t}(\kappa) < \mathfrak{u}(\kappa) < 2^\kappa$ and $\mathsf{SR}(\kappa^{++}), \mathsf{DSS}(\kappa^{++}), \mathsf{TP}(\kappa^{++})$ and $\neg\mathsf{wKH}(\kappa^+)$ hold.

The cardinals $\mathfrak{u}(\kappa)$ and 2^κ can have any reasonable values in these models. We obtain these results by combining the forcing construction from [4] due to Brooke-Taylor, Fischer, Friedman and Montoya with the Mitchell forcing and with (new and old) indestructibility results related to $\mathsf{SR}(\lambda)$, $\mathsf{DSS}(\lambda), \mathsf{TP}(\lambda)$ and $\neg\mathsf{wKH}(\lambda)$. Apart from $\mathfrak{u}(\kappa)$ and $\mathfrak{t}(\kappa)$ we also compute the values of $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$, $\mathfrak{s}(\kappa)$, $\mathfrak{r}(\kappa)$, $\mathfrak{a}(\kappa)$, $\mathrm{cov}(\mathcal{M}_\kappa)$, $\mathrm{add}(\mathcal{M}_\kappa)$, $\mathrm{non}(\mathcal{M}_\kappa)$, $\mathrm{cof}(\mathcal{M}_\kappa)$ which will all be equal to $\mathfrak{u}(\kappa)$.

In (ii), we compute $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \kappa^+$ by observing that the κ^+-distributive quotient of the Mitchell forcing adds a tower of size κ^+.

Finally, as a corollary of the construction, we obtain that (i) and (ii) are also true for $\kappa = \omega$ (starting with a weakly compact cardinal in the ground model).

1. INTRODUCTION

There has been an extensive research recently in the area of compactness principles at successor cardinals, and one of the questions is to what extent, if at all, these principles restrict the continuum function in the proximity of these cardinals. See for instance [5], [16], or [20] for some examples. Extending

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this question, we can ask whether there are some restrictions for other cardinal invariants besides the continuum function. This has been done for instance in [15] and [13], where the focus is on the cardinal invariants on singular strong limit cardinals.

In this paper we investigate cardinal invariants on an inaccessible κ with the focus on the ultrafilter number $\mathfrak{u}(\kappa)$ and the tower number $\mathfrak{t}(\kappa)$. We show that for any reasonable choice of cardinals $\mathfrak{u}(\kappa)$ and 2^κ, $\kappa^+ \le \mathfrak{t}(\kappa) = \mathfrak{u}(\kappa) < 2^\kappa$ and $\kappa^+ = \mathfrak{t}(\kappa) < \mathfrak{u}(\kappa) < 2^\kappa$ are both consistent with stationary reflection and the disjoint stationary sequence property at κ^{++}, and $\kappa^+ = \mathfrak{t}(\kappa) < \mathfrak{u}(\kappa) < 2^\kappa$ is consistent with the tree property at κ^{++} and the negation of the weak Kurepa Hypothesis at κ^+ as well. The consistency of $\kappa^+ < \mathfrak{t}(\kappa) = \mathfrak{u}(\kappa) < 2^\kappa$ with the tree property at κ^{++} and/or the negation of the weak Kurepa Hypothesis at κ^+ seems to be open at the moment (see Section 5 with open questions). In addition to $\mathfrak{u}(\kappa)$ and $\mathfrak{t}(\kappa)$ we also compute the values of cardinal invariants $\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{s}(\kappa), \mathfrak{r}(\kappa)$ and $\mathfrak{a}(\kappa)$, and also the invariants of the meager ideal $\mathcal{M}_\kappa$. See Section 2 for definitions related to compactness principles and [3] for details regarding cardinal invariants at a regular uncountable $\kappa > \omega$.

We use indestructibility results available for the above-mentioned compactness principles to argue that they hold in the final models: If κ is regular with $\kappa^{<\kappa} = \kappa$ and $\lambda > \kappa$ is a large cardinal, it is known that the Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ forces many compactness principles at λ with $\lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa, \lambda)]$. Many of these compactness principles can be preserved in a further forcing extension via some $\mathbb{P} \in V[\mathbb{M}(\kappa, \lambda)]$, provided $\mathbb{P}$ has certain nice properties, such as being κ^+-cc. If all is set up correctly, the model $V[\mathbb{M}(\kappa, \lambda) * \dot{\mathbb{P}}]$ can satisfy both the compactness principles at κ^{++} and some additional properties ensured by $\dot{\mathbb{P}}$.

In the present paper, we show how to apply this strategy with the κ^+-Knaster and κ-directed closed forcing notion denoted $\mathbb{P}_\delta \downarrow p_{\dot{U}}$, introduced in Brooke-Taylor, Fischer, Friedman and Montoya [4], which is a simplified version of the original argument of Džamonja and Shelah in [6], and which yields a model where κ is inaccessible and $\mathfrak{u}(\kappa) < 2^\kappa$ (among other things). In both papers, κ is assumed to be a supercompact cardinal and can retain its supercompactness in the final model. The ordinal δ in $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ is the length of the iteration and has the prescribed cofinality which is equal to $\mathfrak{u}(\kappa)$ in the generic extension.

We show that it is possible to use $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ in two different ways, obtaining two different forcing notions $\mathbb{P}_1$ and $\mathbb{P}_2$, and in effect two different patterns of cardinal invariants:

$$\mathbb{P}_1 := \mathbb{M}(\kappa, \lambda) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}, \tag{1.1}$$

where $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is defined in $V[\mathbb{M}(\kappa, \lambda)]$, and

$$\mathbb{P}_2 := \mathrm{Add}(\kappa, \lambda) * (\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}} \times \dot{R}), \tag{1.2}$$

where we use that $\mathbb{M}(\kappa, \lambda)$ is equivalent to $\mathrm{Add}(\kappa, \lambda) * \dot{R}$ for some κ^+-distributive quotient, and define $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ in the smaller model $V[\mathrm{Add}(\kappa, \lambda)]$.

It is straightforward to observe that $\mathbb{P}_1$ forces stationary reflection and the disjoint stationary sequence property (which implies the negation of the approachability property) at κ^{++} by invoking indestructibility results reviewed in Fact 2.7 and Theorem 2.14, together with the pattern of cardinal invariants computed in [4] (see Theorem 4.1).

The forcing $\mathbb{P}_2$ forces in addition the tree property at κ^{++} by Fact 2.8 and the negation of the weak Kurepa Hypothesis at κ^+ by Theorem 2.12, and in contrast to $\mathbb{P}_1$, it forces $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \kappa^+$ due to the fact that $\dot{R}$ introduces a tower of size κ^+. See Theorem 4.2 for details.

The paper is structured as follows: in Section 2 we review basic notions and facts related to compactness principles and prove an indestructibility Theorem 2.12 for the negation of the weak Kurepa Hypothesis. In Section 3 we briefly review the forcing construction from [4] in order to make the paper (relatively) self-contained, and also to clarify some unclear points from [4]. In Section 4 we prove our main Theorems 4.1 and 4.2. In Section 5 we state some open questions and formulate a corollary of our construction which gives the consistency of an analogous configuration at ω using a forcing from [2]: $\omega_1 = \mathfrak{t} < \mathfrak{u} < 2^\omega$ plus $\mathsf{SR}(\omega_2), \mathsf{DSS}(\omega_2), \mathsf{TP}(\omega_2), \neg\mathsf{wKH}(\omega_1)$ (see Theorem 5.1).

2. Preliminaries

Let us define the compactness properties we are going to study in this paper. In this section, let λ^- be an uncountable regular cardinal and $\lambda = (\lambda^-)^+$ (this case is sufficient for our purposes).

Definition 2.1. We say that the *tree property* holds at λ, and we write $\mathsf{TP}(\lambda)$, if every λ-tree has a cofinal branch.

Definition 2.2. We say that the *negation of the weak Kurepa Hypothesis hold* at λ, and we write $\neg\mathsf{wKH}(\lambda)$, if there are no trees of height and width λ which have at least λ^+-many cofinal branches.

Definition 2.3. We say that *stationary reflection* holds at λ, and write $\mathsf{SR}(\lambda)$, if every stationary subset $S \subseteq \lambda \cap \mathrm{cof}(< \lambda^-)$ reflects at a point of cofinality λ^-; i.e. there is $\alpha < \lambda$ of cofinality λ^- such that $\alpha \cap S$ is stationary in α.

For a cardinal λ and sequence $\bar{a} = \langle a_\alpha \,|\, \alpha < \lambda \rangle$ of bounded subsets of λ, we say that an ordinal $\gamma < \lambda$ is approachable with respect to $\bar{a}$ if there is an unbounded subset $A \subseteq \gamma$ of order type $\mathrm{cf}(\gamma)$ and for all $\beta < \gamma$ there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$.

Let us define the *ideal $I[\lambda]$ of approachable subsets of λ*:

Definition 2.4. $S \in I[\lambda]$ if and only if there are a sequence $\bar{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ of bounded subsets of λ and a club $C \subseteq \lambda$ such that every $\gamma \in S \cap C$ is approachable with respect to $\bar{a}$.

Definition 2.5. We say that *the approachability property holds* at λ if $\lambda \in I[\lambda]$ (or equivalently, there is a club subset of λ in $I[\lambda]$), and we write $\mathsf{AP}(\lambda)$.

For cardinals $\kappa \leq \lambda$, we denote by $\mathcal{P}_\kappa(\lambda)$ the set of all subsets of λ of size $< \kappa$. The following property $\mathsf{DSS}(\lambda)$ was introduced in [14]:

Definition 2.6. We say that λ has the *disjoint stationary sequence property*, $\mathsf{DSS}(\lambda)$, if there are a stationary set $S \subseteq \lambda \cap \mathrm{cof}(\lambda^-)$ and a sequence $\langle s_\alpha \mid \alpha \in S \rangle$ such that:

(i) For all $\alpha \in S$, s_α is a stationary subset of $\mathcal{P}_{\lambda^-}(\alpha)$;
(ii) For all $\alpha < \beta$ in S, $s_\alpha \cap s_\beta = \emptyset$.

By [14, Corollary 3.7.], $\mathsf{DSS}(\lambda)$ implies $\neg\mathsf{AP}(\lambda)$.

2.1. Indestructibility of some compactness principles

The strongest form of indestructibility is known for stationary reflection. It works over any model and requires just an appropriate chain condition (we formulate it to fit our present purposes):

Fact 2.7 (Indestructibility of stationary reflection, [13])**.** *Suppose* $\mathsf{SR}(\kappa^{++})$ *holds and* $\mathbb{Q}$ *is* κ^+*-cc. Then* $\mathbb{Q}$ *forces* $\mathsf{SR}(\kappa^{++})$*.*

The indestructibility of the tree property is known only for a specific model, i.e. the Mitchel extension $V[\mathbb{M}(\kappa, \lambda)$. Let us restate it here for reference, starting with a brief review of the Mitchell forcing $\mathbb{M}(\kappa, \lambda)$. Suppose $\kappa = \kappa^{<\kappa}$ and $\lambda > \kappa$ is inaccessible; the Mitchell forcing (or the Mitchell collapse) $\mathbb{M}(\kappa, \lambda)$ collapses cardinals in the interval (κ^+, λ), forces $2^\kappa = \lambda$, and also forces some compactness principles depending on the largeness of λ. It can be decomposed as the Cohen forcing $\mathrm{Add}(\kappa, \lambda)$ followed by a κ^+-distributive quotient forcing $\dot{R}$ so that $\mathbb{M}(\kappa, \lambda)$ is equivalent to $\mathrm{Add}(\kappa, \lambda) * \dot{R}$. Also, there is a κ^+-closed forcing $\mathbb{T}$ (the term forcing) such that $\mathrm{Add}(\kappa, \lambda) \times \mathbb{T}$ projects onto $\mathbb{M}(\kappa, \lambda)$. See [18] for the original definition, [1] for a modern presentation, and Footnote 10 in Lemma 4.10 for specific details relevant for us.

The following is the strongest known indestructibility of the tree property related to the chain condition:

Fact 2.8 (Indestructibility of the tree property, [12])**.** *Suppose* $\kappa = \kappa^{<\kappa}$ *and* $\lambda > \kappa$ *is weakly compact. Suppose* $\mathrm{Add}(\kappa, \lambda)$ *forces that* $\dot{\mathbb{Q}}$ *is a forcing notion which is* κ^+*-cc. Then* $\mathbb{M}(\kappa, \lambda) * \dot{\mathbb{Q}}$ *forces* $\mathsf{TP}(\lambda)$*.*

Let us now extend Fact 2.8 to the negation of the weak Kurepa Hypothesis.

Definition 2.9. Suppose κ is a regular cardinal. We say that a forcing $\mathbb{P}$ is *productively* κ*-cc* iff $\mathbb{P} \times \mathbb{P}$ is κ-cc.

Notice that every κ-Knaster forcing is productively κ-cc, and every productively κ-cc forcing is κ-cc.

The following is well known and will be useful for us:

Fact 2.10. *Suppose κ is a regular cardinal and T is a tree of height κ. If $\mathbb{Q}$ is productively κ-cc, then $\mathbb{Q}$ does not add new cofinal branches to T.*

The productive chain condition is not so well-behaved with regard to preservation under iteration as the regular chain condition or the Knaster condition, but there is a weaker characterization. To formulate it, let us introduce the following notation: Suppose $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name. We can view it artificially as a $\mathbb{P}\times\mathbb{P}$-name by modifying it to depend only on the first coordinate or the second coordinate of $\mathbb{P}\times\mathbb{P}$, obtaining $\dot{\mathbb{Q}}^1$ and $\dot{\mathbb{Q}}^2$, respectively.[1]

Recall the following characterization which holds for the regular chain condition:

$$\mathbb{P} * \dot{\mathbb{Q}} \text{ is } \kappa\text{-cc} \Leftrightarrow \mathbb{P} \text{ is } \kappa\text{-cc and } \mathbb{P} \Vdash \dot{\mathbb{Q}} \text{ is } \kappa\text{-cc.} \tag{2.3}$$

Lemma 2.11. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a forcing notion and κ a regular cardinal. Then the following hold:*

(i) If $\mathbb{P} * \dot{\mathbb{Q}}$ is productively κ-cc, then $\mathbb{P}$ is productively κ-cc and forces that $\dot{\mathbb{Q}}$ is productively κ-cc.

(ii) If $\mathbb{P}$ is productively κ-cc and $\mathbb{P}\times\mathbb{P}$ forces that $\dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^2$ is κ-cc, then $\mathbb{P}*\dot{\mathbb{Q}}$ is productively κ-cc.

Proof. (i). $\mathbb{P}$ is productively κ-cc because there is a natural regular embedding from $\mathbb{P}\times\mathbb{P}$ into $(\mathbb{P}*\dot{\mathbb{Q}})\times(\mathbb{P}*\dot{\mathbb{Q}})$. We use (2.3) repeatedly for the second claim: If $\mathbb{P} * \dot{\mathbb{Q}}$ is productively κ-cc, then $\mathbb{P} * \dot{\mathbb{Q}} \Vdash \mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc. This is equivalent to $\mathbb{P} \Vdash \dot{\mathbb{Q}} \Vdash \mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc, which is in turn equivalent to $\mathbb{P} \Vdash \dot{\mathbb{Q}} * \mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc, which readily implies $\mathbb{P} \Vdash \dot{\mathbb{Q}} \times \dot{\mathbb{Q}}$ is κ-cc.[2]

(ii). Assume for contradiction that $A = \{[(p_\alpha, \dot{q}_\alpha), (p'_\alpha, \dot{q}'_\alpha)] \mid \alpha < \kappa\}$ is an antichain in $(\mathbb{P} * \dot{\mathbb{Q}}) \times (\mathbb{P} * \dot{\mathbb{Q}})$. We shall need the following observation: if $(p_\alpha, \dot{q}_\alpha), (p'_\alpha, \dot{q}'_\alpha)$ and $(p_\beta, \dot{q}_\beta), (p'_\beta, \dot{q}'_\beta)$, $\alpha < \beta$, are in A, and there are conditions p, p' in $\mathbb{P}$ such that $p \le p_\alpha, p_\beta$ and $p' \le p'_\alpha, p'_\beta$, then either $p \Vdash \dot{q}_\alpha \perp \dot{q}_\beta$ or $p' \Vdash \dot{q}'_\alpha \perp \dot{q}'_\beta$, which can be reformulated for the forcing $(\mathbb{P}\times\mathbb{P}) * (\dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^2)$ as:

$$(p, p') \Vdash (\dot{q}_\alpha, \dot{q}'_\alpha) \perp (\dot{q}_\beta, \dot{q}'_\beta), \tag{2.4}$$

by replacing the names $\dot{q}_\alpha, \dot{q}'_\alpha$, etc., as explained in Footnote 1.

[1] Define recursively a function * from $\mathbb{P}$-names to $\mathbb{P}\times\mathbb{P}$-names so that $\sigma^* = \{[(p,1),\tau^*] \mid (p,\tau)\in\sigma\}$ for $\dot{\mathbb{Q}}^1$ and $\{[(1,p),\tau^*] \mid (p,\tau)\in\sigma\}$ for $\dot{\mathbb{Q}}^2$.

[2] We can continue to obtain a partial converse to the implication in (ii): $\mathbb{P} \Vdash \dot{\mathbb{Q}} * \mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc implies $\mathbb{P} \Vdash \mathbb{P} \Vdash \dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^1$ is κ-cc, equivalently $\mathbb{P}\times\mathbb{P} \Vdash \dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^1$ is κ-cc, and by mutual genericity of generic filters for $\mathbb{P}\times\mathbb{P}$, $\mathbb{P}\times\mathbb{P} \Vdash \dot{\mathbb{Q}}^2 \times \dot{\mathbb{Q}}^2$ is κ-cc. But this is still weaker than the condition which implies $\mathbb{P} * \dot{\mathbb{Q}}$ is productively κ-cc in (ii).

Define $A^* = \{[(p_\alpha, p'_\alpha), \mathrm{pair}(\dot{q}_\alpha, \dot{q}'_\alpha)] \,|\, \alpha < \kappa\}$, where $\mathrm{pair}(\dot{q}_\alpha, \dot{q}'_\alpha)$ is the canonical name for the ordered pair $(\dot{q}_\alpha, \dot{q}'_\alpha)$. A^* is a $\mathbb{P} \times \mathbb{P}$-name for a subset of $\dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^2$. Since $\mathbb{P}$ is productively κ-cc, there is a generic filter G such that for some $I \subseteq \kappa$ of size κ, $\{(p_\alpha, p'_\alpha) \,|\, \alpha \in I\} \subseteq G$. Let us write q^1_α and ${q'_\alpha}^2$ for $\dot{q}_\alpha^{G^1}$ and ${\dot{q}'_\alpha}{}^{G^2}$, $\alpha < \kappa$, respectively (where G^1 and G^2 are projections of G to the coordinates). We claim that $\{(q^1_\alpha, {q'_\alpha}^2) \,|\, \alpha \in I\}$ is an antichain, which gives a contradiction: For any $\alpha < \beta \in I$, (p_α, p'_α) and (p_β, p'_β) are compatible because they are in G, and their lower bound $(p, p') \in G$ forces by (2.4) $(\dot{q}_\alpha, \dot{q}'_\alpha) \perp (\dot{q}_\beta, \dot{q}'_\beta)$. □

Theorem 2.12 is slightly weaker than Fact 2.8 because it requires the productive κ^+-cc condition and not just κ^+-cc. The reasons are technical: the tree property deals with κ^{++}-trees, and we used in [12] the fact that a κ^+-cc forcing cannot add a new cofinal branch to a κ^{++}-tree; this is not true in general for trees of height κ^+ which appear in $\neg\mathsf{wKH}(\kappa^+)$. So we use Fact 2.10 instead.

Theorem 2.12 (Indestructibility of the negation of the weak Kurepa Hypothesis)**.** *Assume $\omega \le \kappa < \lambda$ are cardinals, $\kappa^{<\kappa} = \kappa$ and λ is weakly compact. Suppose $\mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{Q}}$ is productively κ^+-cc. Then*

$$V[\mathbb{M}(\kappa, \lambda) * \dot{\mathbb{Q}}] \models \neg\mathsf{wKH}(\kappa^+).$$

Proof. Notice that we require that $\mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{Q}}$ is productively κ^+-cc, and not the potentially weaker condition that $\mathrm{Add}(\kappa, \lambda)$ forces that $\dot{\mathbb{Q}}$ is productively κ^+-cc (see Lemma 2.11(ii)). However, in many situations this is easy to ensure: for instance if $\dot{\mathbb{Q}}$ is forced to be κ^+-Knaster, then $\mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{Q}}$ is κ^+-Knaster, and in particular productively κ^+-cc.

The proof closely follows the proof of [12, Theorem 3.2]. The heart of the argument is [12, Claim 3.5] which must be modified as follows (see [12, Claim 3.5] for notation):

Claim 2.13. *(i) R^1_λ is κ^+-closed in $N[G]$.*
(ii) $j(R^0 * \dot{\mathbb{Q}})/G^0 * h$ is productively κ^+-cc over $N[G][h]$.
(iii) $\dot{\mathbb{Q}}^{G^0} * j(R^0 * \dot{\mathbb{Q}})/G^0 * \dot{h}$ is κ^+-cc over $N[G]$, where $j(R^0 * \dot{\mathbb{Q}})/G^0 * \dot{h}$ denotes a $\dot{\mathbb{Q}}^{G^0}$-name for the quotient.

In (i), the forcing R^1_λ is the κ^+-closed term forcing related to the Mitchell forcing, and this stays true in our case.

For (ii), follow the proof of [12, Claim 3.5]: (3.23) in that proof now reads "...is productively κ^+-cc over N", (3.24) reads "...is productively κ^+-cc over $N[G^1]$ (by Easton's lemma), and the argument in the paragraph below (3.25) follows by Lemma 2.11(i).

With (ii) modified as described, Fact 2.10 is applied to $j(R^0 * \dot{\mathbb{Q}})/G^0 * h$ over $N[G][h]$ to argue that no new cofinal branches are added by the productively κ^+-cc forcing $j(R^0 * \dot{\mathbb{Q}})/G^0 * h$ to a tree T of height and size κ^+ (view T is a possible counterexample to $\neg\mathsf{wKH}(\kappa^+)$ in $V[G][h]$).

Item (iii) still works with the κ^+-cc condition. It uses the fact that a κ^+-closed forcing in a model with $2^\kappa = \kappa^{++}$ cannot add a new cofinal branch to a tree of height and size κ^+ which is added by a κ^+-cc forcing notion (see [12, Fact 2.11] and [21, Lemma 6]).[3] Thus (iii) is applied with the κ^+-cc forcing $\dot{\mathbb{Q}}^{G^0} * j(R^0 * \dot{\mathbb{Q}})/G^0 * h$ and the κ^+-closed forcing R^1_λ to conclude that $R^1_\lambda \times j(R^0 * \dot{\mathbb{Q}})/G^0 * h$ cannot add new cofinal branches to T. The rest of the argument is as in [12, Theorem 3.2]. □

For the disjoint stationary sequence property, the following can be proved:

Theorem 2.14 (Indestructibility of the disjoint stationary sequence property, essentially [14])**.** *Let λ be as fixed above and let $\langle s_\alpha \,|\, \alpha \in S\rangle$ be a disjoint stationary sequence on λ, with $S \subseteq \lambda \cap \mathrm{cof}(\lambda^-)$ stationary. Suppose $\mathbb{P}$ preserves stationary subsets of both λ^- and λ. Then $\mathbb{P}$ forces that $\langle s_\alpha \,|\, \alpha \in S\rangle$ is a disjoint stationary sequence on λ with S stationary.*

Proof. Since $\mathbb{P}$ preserves stationary subsets of λ, S is still stationary. It suffices to check that if $\mathbb{P}$ preserves stationary subsets of λ^-, it preserves stationary subsets of $\mathcal{P}_{\lambda^-}(\alpha)$ for $\alpha \in S$. Let $\langle x_i \,|\, i < \nu\rangle$ in $\mathcal{P}_{\lambda^-}(\alpha)$ be an increasing and continuous sequence of subsets of α of size $< \lambda^-$ whose union is α; then it is easy to verify that s is stationary in $\mathcal{P}_{\lambda^-}(\alpha)$ iff $\{i < \lambda^- \,|\, x_i \in s\}$ is stationary in λ^-. It follows that if $\mathbb{P}$ preserves stationary subsets of λ^-, it also preserves stationary subsets of $\mathcal{P}_{\lambda^-}(\alpha)$. □

Note that [10, Corollary 2.2] gives an indestructibility result for $\neg\mathsf{AP}(\kappa^{++})$, κ regular, but it holds only for κ-centered forcings,[4] and thus is not sufficient for our purposes.

3. A review of the argument which makes $\mathfrak{u}(\kappa)$ small

3.1. The original forcing

Let us briefly review the definition of the forcing $\mathbb{P}$ introduced in [4] by Brooke-Taylor, Fischer, Friedman and Montoya. Let κ be a Laver-indestructible supercompact cardinal, and $\mu \geq \kappa^{++}$ a cardinal of cofinality $> \kappa$ with $\mu^\kappa = \mu$. Let us start by assuming that the ordinals below μ^+ are divided into three disjoint cofinal subsets which are reserved for three different tasks: the first subset I_0 is reserved for the Mathias forcing which controls the ultrafilter number $\mathfrak{u}(\kappa)$, the second subset I_1 is used to control the minimal size of a κ-mad family $\mathfrak{a}(\kappa)$, and the third subset I_2 is used to control the pseudo-intersection number $\mathfrak{p}(\kappa)$.

[3] The referenced [21, Lemma 6] essentially deals with κ^{++}-trees and proceeds by a recursive construction of size κ in a version of Silver's argument to show that if a new cofinal branch is added, then there must be a level of the tree of size $2^\kappa = \kappa^{++}$: this gives a contradiction for our tree of size and height κ^+ as well.

[4] A forcing is κ-centered if it can be written at the union of κ-many filters, in particular it has the κ^+-cc.

For concreteness we may take I_0 to be the set of limit ordinals below μ^+ and I_1 and I_2 to be some sets of odd ordinals such that both I_1 and I_2 are cofinal in every limit ordinal.

The iteration on the segments I_1 and I_2 is defined in the usual way with $< \kappa$ support, but I_0 is more complicated because it uses a lottery among forcings and two types of support to control $\mathfrak{u}(\kappa)$. In what follows we will suppress the details for iterands on $I_1 \cup I_2$ and focus on I_0.

Definition 3.1. The forcing $\mathbb{P}$ has the following structure:

- $\mathbb{P} = \langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \,|\, \alpha < \mu^+ \rangle$ is an iteration of length μ^+ with $< \kappa$-support on $I_1 \cup I_2$ such that for each $\alpha \in I_0$, $\dot{\mathbb{Q}}_\alpha$ is the lottery over Mathias forcings Mathias$(\dot{U})$ defined with respect to normal ultrafilters $\dot{U}$ on κ existing in $V[\mathbb{P}_\alpha]$. Since we start with a Laver-indestructible supercompact κ, there will always be some normal ultrafilters available.
- For a condition $p \in \mathbb{P}$, the support of the lottery (on I_0) is called Lottery(p) and it is an initial segment of I_0; for ease of notation we will identify it with an ordinal below μ^+, i.e. we write Lottery$(p) = \gamma$ to mean that Lottery$(p) = \gamma \cap I_0$: if $\alpha \in$ Lottery(p), then the lottery has chosen at stage $\alpha \in I_0$ a normal ultrafilter (denoted $\dot{U}^p_\alpha$).
- Only at most $< \kappa$-many coordinates in Lottery(p) are allowed to be non-trivial in the sense that they are not equal to the weakest condition in Mathias$(\dot{U}^p_\alpha)$: we call this set the Mathias support and denote it Mathias(p).
- If Mathias$(p) = \emptyset$ and Lottery$(p) = \gamma$, we write $p^{\to\gamma}$ to indicate that p has made its choice regarding the normal ultrafilters below γ, but has not chosen any (non-trivial) conditions in the respective Mathias forcings.

We will work with restrictions of the form $\mathbb{P}_\delta \!\downarrow\! p^{\to\delta}$ (the conditions in $\mathbb{P}_\delta$ which extend $p^{\to\delta}$) which always have a dense subset of size at most μ and are κ^+-Knaster (unlike $\mathbb{P}_\delta$ which has antichains of size 2^{2^κ} due to the lottery over all normal ultrafilters on κ).[5]

The key idea in [4] is to identify a suitable name $\dot{U}$ for a normal ultrafilter on κ in $V[\mathbb{P}]$, and truncate $\mathbb{P}$ at some ordinal $\delta \in (\mu, \mu^+)$ of the required cofinality such that for a certain condition $p_{\dot{U}}$ (which is of the form $p^{\to\delta}$), the forcing $\mathbb{P}_\delta \!\downarrow\! p_{\dot{U}}$ forces $\kappa^+ < \mathfrak{u}(\kappa) < 2^\kappa = \mu$. The ultrafilter number $\mathfrak{u}(\kappa)$ will be equal to the cofinality of δ.

[5]The paper [4] only states that $\mathbb{P}_\delta \!\downarrow\! p^{\to\delta}$ is κ^+-cc, but it is easy to see that it is κ^+-Knaster: This forcing has $< \kappa$-support, and all iterands are κ-directed closed and are κ-centered: their compatibility is determined by stems $s \in \kappa^{<\kappa}$ (the Mathias forcing on $I_0 \cup I_2$ and an almost-disjointness forcing on I_1). To argue that $\mathbb{P}_\delta \!\downarrow\! p^{\to\delta}$ is κ^+-Knaster, first use a Δ-system lemma on the supports and then use the κ-centeredness of the iterands on the root of the system (the stems s can be without the loss of generality represented by checked names).

The main tool in the proof is to argue that any normal ultrafilter on κ in $V[\mathbb{P}]$ reflects sufficiently often below μ^+. Lemma 3.2 below is implicit in [4, Lemma 10]:

We say that $S \subseteq \mu^+$ is a $\mathrm{cof}(>\kappa)$-club if S is unbounded in μ^+ and closed at limit points of cofinality $>\kappa$.

Lemma 3.2. *Assume κ, μ and $\mathbb{P}$ are is in Definition 3.1. Assume that*

$$1_{\mathbb{P}} \Vdash \dot{U} \text{ is a normal ultrafilter on } \kappa. \tag{3.5}$$

Then there is a $\mathrm{cof}(>\kappa)$-club $S_{\dot{U}} \subseteq \mu^+$ where $\dot{U}$ reflects. More precisely, there is a decreasing sequence $\langle p^{\to\alpha} \mid \alpha \in S_{\dot{U}}\rangle$ continuous at points of cofinality $>\kappa$ which chooses in the lottery the restrictions of the ultrafilter $\dot{U}$ at the relevant stages:

(i) For every $\alpha \in S_{\dot{U}}$,

$$p^{\to\alpha} \Vdash \dot{U} \cap V[\mathbb{P}_\alpha] \in V[\mathbb{P}_\alpha]. \tag{3.6}$$

(ii) For all $\alpha < \alpha^* \in S_{\dot{U}}$

$$p^{\to\alpha^*} \Vdash \dot{U}_\alpha = \dot{U} \cap V[\mathbb{P}_\alpha] \in V[\mathbb{P}_\alpha], \tag{3.7}$$

where $\dot{U}_\alpha$ is a name of the ultrafilter selected by the lottery at stage α by $p^{\to\alpha^*}$.

(iii) The sequence is continuous at ordinals of cofinality $>\kappa$: for any limit δ in $S_{\dot{U}}$ of cofinality $>\kappa$, $p^{\to\delta}$ is the infimum of $\langle p^{\to\beta} \mid \beta < \delta\rangle$ such that (3.7) holds for all $\alpha < \alpha^* < \delta$.

3.2. Obtaining the right normal ultrafilter for our proof

The existence of normal ultrafilters in $V[\mathbb{P}]$ follows by assuming that κ is Laver-indestructibly supercompact. In order to find a suitable name $\dot{U}$ for which the iteration on I_0 – below some well-chosen condition $p_{\dot{U}}$ – generates a base of a uniform ultrafilter and ensures the desired value of $\mathfrak{u}(\kappa)$, we need to go into the details of the Laver preparation. We follow the structure of the argument in [4] while including the forcings $\mathbb{M}(\kappa,\lambda)$ and $\mathrm{Add}(\kappa,\lambda)$ in the preparation (we also provide a clarification of certain points in the proof; see Footnote 6).

In preparation for Theorems 4.1 and 4.2, we need to consider not just the forcing $\mathbb{P}$, but forcings of the form

$$\mathbb{M}(\kappa,\lambda) * \dot{\mathbb{P}} \tag{3.8}$$

and

$$\mathrm{Add}(\kappa,\lambda) * \dot{\mathbb{P}}. \tag{3.9}$$

The argument is the same in both cases, so let us write this forcing as

$$P * \dot{\mathbb{P}}, \tag{3.10}$$

with the understanding that P is either $\mathrm{Add}(\kappa,\lambda)$ or $\mathbb{M}(\kappa,\lambda)$.

Suppose κ is supercompact in V', and L is the Laver preparation for κ which makes κ in $V[L]$ indestructible under all κ-directed closed forcings. Let $H * F * G$ be an $L * P * \mathbb{P}$-generic over V'.

Fix a suitable supercompact embedding

$$j : V' \to M \tag{3.11}$$

with critical point κ such that the iteration $j(L)$ chooses at stage κ in M the forcing $P * \dot{\mathbb{P}}$.

Using a standard master condition argument we can lift in $V'[H][F][G]$ further to j'

$$j' : V'[H][F] \to M[H][F][G][H^*][F^*], \tag{3.12}$$

where H^* is any generic over $M[H][F][G]$ for the tail of $j(L)$ and F^* is any generic over $M[H][F][G][H^*]$ containing a master condition for F.

For the lifting of j' to j^*, we need a specific master condition p^* extending the pointwise image of $j'[G]$, the properties of which are detailed in Remark 3.3 below. Since p^* is a master condition, j' lifts to j^*:

$$j^* : V'[H][F][G] \to M[H][F][G][H^*][F^*][G^*], \tag{3.13}$$

where G^* contains p^*.

In $V'[H][F]$, let $\dot{U}$ be a $\mathbb{P}$-name forced by $1_{\mathbb{P}}$ to be a normal ultrafilter on κ generated by j^*. This name $\dot{U}$ is the one to which Lemma 3.2 is applied, and which determines the ordinal $\delta \in S_{\dot{U}}$, the condition $p_{\dot{U}} = p^{\to\delta}$, and finally the forcing $\mathbb{P}_\delta \downarrow p_{\dot{U}}$.[6]

Remark 3.3. Let us briefly review the definition of p^*. We work in the generic extension $V'[H][F][G]$, but all we say can be translated into $\mathbb{P}$-forcing statements dealing with names for j' and G over $V'[H][F]$. Let p^*_0 be some master condition for $j'[G]$; we extend p^*_0 by some $p^* \leq p^*_0$ which in addition has the following property (*):

- (*) Whenever $\alpha < \mu^+$ has the property that for every $A \in (\dot{U}_\alpha)^G$ (where $\dot{U}_\alpha$ is a name for the normal ultrafilter selected by the lottery at stage α by a condition in G) there is some name $\dot{A}$ for A and a condition $p_A \in G_\alpha$ with $j'(p_A) \Vdash \kappa \in j'(\dot{A})$, then $p^*(j(\alpha))$ is obtained from

[6] The argument in [4, Theorem 12] seems to suggest that one can start with an arbitrary name $\dot{U}$ and apply [4, Lemma 10] with it, and control the interpretation of $\dot{U}$ by choosing the right master condition to lift j' to j^* in (3.13). But $\dot{U}$ is a $\mathbb{P}$-name and the lifting from j' to j^* does not effect its interpretation which is fixed by the $\mathbb{P}$-generic filter G (and so $\dot{U}^G$ may not contain the Mathias generic subsets of κ which ensure a base of size κ^*). Instead, we should argue that $1_{\mathbb{P}}$ forces over $V'[H][F]$ that there is a normal ultrafilter $\dot{U}$ with the required properties (*) reviewed in Remark 3.3, apply Lemma 3.2 with this name $\dot{U}$ to secure a condition $p_{\dot{U}}$ of the form $p^{\to\delta}$, and only then choose a generic filter G for $\mathbb{P}_\delta$ which contains the condition $p_{\dot{U}}$.

$p_0^*(j(\alpha))$ by adding $\{\kappa\}$ into the stem of the Mathias forcing at stage $j(\alpha)$.

Let j^* be the lifting of j' with any G^* which contains p^*, and let $\dot{U}$ be a name for the normal ultrafilter generated by j^*. Let us apply Lemma 3.2 with this name $\dot{U}$, obtaining the appropriate $S_{\dot{U}}$, δ and $p^{\to\delta} = p_{\dot{U}}$. Let G be any $\mathbb{P}_\delta \downarrow p_{\dot{U}}$-generic filter. The Mathias-generic subsets of κ, denoted x_α, at all stages $\alpha \in S_{\dot{U}}$ are in $\dot{U}^G$ because $p_{\dot{U}} \in G$ and $p^*(j(\alpha))$ was defined to contain the stem $x_\alpha \cup \{\kappa\}$ (and $\kappa \in j^*(x_\alpha)$ is equivalent to x_α being in $\dot{U}^G$).

4. Compactness and generalized cardinal invariants

Using the indestructibility results for $\mathsf{SR}(\lambda)$ and $\mathsf{DSS}(\lambda)$ reviewed in Section 2 and the properties of the forcing $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ reviewed earlier, one can immediately observe the following:

Theorem 4.1. *Suppose κ is a supercompact cardinal, $\lambda > \kappa$ is a weakly compact cardinal, $\mu \geq \lambda$ is a cardinal with cofinality $> \kappa$ with $\mu^\kappa = \mu$, and κ^* is a regular cardinal with $\kappa < \kappa^* < \mu$. Then there is a generic extension $V[G]$ which satisfies the following:*

(i) Exactly the cardinals in the open interval (κ, λ) are collapsed, with $\lambda = (\kappa^{++})^{V[G]}$,
(ii) $2^\kappa = \mu$,
(iii) $\mathsf{SR}(\kappa^{++})$, $\mathsf{DSS}(\kappa^{++})$ and $\neg\mathsf{AP}(\kappa^{++})$,

And the following identities hold:

$$\kappa^* = \mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{u}(\kappa) = \\ = \mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa). \tag{4.14}$$

Proof. Let us define

$$\mathbb{P}_1 := \mathbb{M}(\kappa, \lambda) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}},$$

where $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is defined in $V[\mathbb{M}(\kappa,\lambda)]$ following the review in Section 3.1 with the condition $p_{\dot{U}}$ determined by Lemma 3.2 with respect to the name $\dot{U}$ obtained through the Laver preparation (3.8) and the construction described in Remark 3.3. In particular $\delta \in (\mu, \mu^+)$ is such that $S_{\dot{U}} \cap \delta$ has cofinality κ^* and $p_{\dot{U}}$ is equal to $p^{\to\delta}$ in the construction in Lemma 3.2 applied with $\dot{U}$.

Let $G = F * G_\delta$ be an $\mathbb{M}(\kappa, \lambda) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$-generic filter.

Using standard arguments, $\mathsf{SR}(\kappa^{++})$ and $\mathsf{DSS}(\kappa^{++})$ hold in $V[F]$, and by Fact 2.7 and Theorem 2.14 continue to hold in $V[G]$ because $(\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}})^F$ is κ-directed closed and κ^+-cc in $V[F]$.

The desired pattern of the cardinal invariants follows exactly as in [4]. $\square$

We may obtain $\mathsf{SR}(\kappa^{++})$ and $\mathsf{DSS}(\kappa^{++})$ in a different way, and in addition have also $\mathsf{TP}(\kappa^{++})$ and $\neg\mathsf{wKH}(\kappa^+)$, if we modify the forcing $\mathbb{P}_1$. This modification results in a different pattern of cardinal invariants:

Theorem 4.2. *Suppose κ is a supercompact cardinal, $\lambda > \kappa$ is a weakly compact cardinal, $\mu \geq \lambda$ is a cardinal with cofinality $> \kappa$ with $\mu^\kappa = \mu$, and κ^* is a regular cardinal with $\kappa < \kappa^* < \mu$. Then there is a generic extension $V[G]$ which satisfies the following:*

(i) Exactly the cardinals in the open interval (κ, λ) are collapsed, with $\lambda = (\kappa^{++})^{V[G]}$,
(ii) $2^\kappa = \mu$,
(iii) $\mathsf{SR}(\kappa^{++})$, $\mathsf{DSS}(\kappa^{++})$, $\neg\mathsf{AP}(\kappa^{++})$, $\mathsf{TP}(\kappa^{++})$ and $\neg\mathsf{wKH}(\kappa^+)$.

And the following hold:

$$\begin{aligned}(4.15)\quad \kappa^+ = \mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) \leq \\ \leq \kappa^* = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{u}(\kappa) = \mathfrak{r}(\kappa) = \\ = \mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa).\end{aligned}$$

Proof. Let us define

$$\mathbb{P}_2 := \mathrm{Add}(\kappa, \lambda) * (\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}} \times \dot{R}), \tag{4.16}$$

where $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is defined in $V[\mathrm{Add}(\kappa, \lambda)]$ following the review in Section 3.1 with the condition $p_{\dot{U}}$ determined by Lemma 3.2 with respect to the name $\dot{U}$ obtained through the Laver preparation (3.9) and the construction described in Remark 3.3. In particular $\delta \in (\mu, \mu^+)$ is such that $S_{\dot{U}} \cap \delta$ has cofinality κ^* and $p_{\dot{U}}$ is equal to $p^{\to\delta}$ in the construction in Lemma 3.2 applied with the name $\dot{U}$. The forcing $\dot{R}$ is forced by $\mathrm{Add}(\kappa, \lambda)$ to be κ^+-distributive, with $\mathrm{Add}(\kappa, \lambda) * \dot{R}$ being forcing equivalent to $\mathbb{M}(\kappa, \lambda)$.

Let $F = F_0 * F_1$ be $\mathrm{Add}(\kappa, \lambda) * \dot{R}$-generic and let G_δ be $\mathbb{P}_\delta \downarrow p_{\dot{U}} := (\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}})^{F_0}$-generic over $V[F_0]$. By Lemma 4.3(ii), $F_0 * (G_\delta \times F_1)$ is $\mathbb{P}_2$-generic and satisfies the following:

Lemma 4.3. *The following hold:*

(i) $\mathbb{M}(\kappa, \lambda)$ forces $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is productively κ^+-cc.
(ii) Suppose F_1 is $R = \dot{R}^{F_0}$-generic over $V[F_0]$ and G_δ is $\mathbb{P}_\delta \downarrow p_{\dot{U}} = (\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}})^{F_0}$-generic over $V[F_0]$. Then F_1 and G_δ are mutually generic over $V[F_0]$.
(iii) F_1 does not add new κ-sequences over $V[F_0][G_\delta]$.

Proof. (i). It suffices to show that $\mathrm{Add}(\kappa, \lambda) \times \mathbb{T}$ forces $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is productively κ^+-cc, because there is a projection from $\mathrm{Add}(\kappa, \lambda) \times \mathbb{T}$ onto $\mathbb{M}(\kappa, \lambda)$ (see the paragraph before Fact 2.8). $\mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is productively κ^+-cc because it is κ^+-Knaster, and by the Easton's lemma the κ^+-closed $\mathbb{T}$ forces $\mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is still productively κ^+-cc, and hence $\mathbb{T} \times \mathrm{Add}(\kappa, \lambda)$ forces $\dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is productively κ^+-cc (compare with Lemma 2.11(i)).

(ii). Suppose F_1 is $V[F_0]$-generic for R, and G_δ is $\mathbb{P}_\delta \downarrow p_{\dot{U}}$-generic over $V[F_0]$. It suffices to show that G_δ is generic over the model $V[F_0][F_1]$ because then $F_1 \times G_\delta$ is a generic filter over $V[F_0]$ for the product $\mathbb{P}_\delta \downarrow p_{\dot{U}} \times R$, and hence

$V[F_0][F_1][G_\delta] = V[F_0][G_\delta][F_1]$. By (i), $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ is still κ^+-cc over $V[F_0][F_1]$, and since R does not add new κ-sequences of elements over the model $V[F_0]$, it follows that $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ has the same maximal antichains in $V[F_0]$ as it has in $V[F_0][F_1]$, and so G_δ is generic over the larger model $V[F_0][F_1]$ as well.

(iii). Suppose for contradiction that x is a κ-sequence in $V[F_0][G_\delta][F_1]$ which is not in $V[F_0][G_\delta]$. This means that there is a $\mathbb{P}_\delta \downarrow p_{\dot{U}}$-name $\dot{x}$ in $V[F_0][F_1]$ such that $\dot{x}^{G_\delta} = x$ in $V[F_0][G_\delta][F_1] = V[F_0][F_1][G_\delta]$ and $\dot{x}$ is not in $V[F_0]$. However, this name is itself a κ-sequence of elements in $V[F_0]$ because $\mathbb{P}_\delta \downarrow p_{\dot{U}}$ is κ^+-cc. This is a contradiction because F_1 does not add new κ-sequences over $V[F_0]$. □

Remark 4.4. The argument in Lemma 4.3(iii) actually shows the following more general claim: Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions with $\mathbb{P}$ being κ^+-cc and $\mathbb{Q}$ being κ^+-distributive. Then if $\mathbb{Q}$ forces that $\mathbb{P}$ is κ^+-cc, then $\mathbb{P}$ forces that $\mathbb{Q}$ is κ^+-distributive.

Let us denote the filter $F_0 * (G_\delta \times F_1)$ by G. Lemma 4.3(i) together with Facts 2.7, Theorem 2.14, Fact 2.8 and Theorem 2.12 immediately imply that $\mathsf{SR}(\kappa^{++})$, $\mathsf{DSS}(\kappa^{++})$, $\mathsf{TP}(\kappa^{++})$ and $\neg\mathsf{wKH}(\kappa^+)$, respectively, hold in $V[G]$.

It remains to check that the required pattern of cardinal invariants holds in $V[G]$. By the same observation as in the proof of Theorem 4.1, the equalities (4.14) hold in $V[F_0][G_\delta]$. By Lemma 4.3(iii), the quotient forcing R does not add new sequences of length κ, so the space ${}^\kappa\kappa$ is the same in $V[F_0][G_\delta]$ and $V[G]$. However, R may add new subsets of ${}^\kappa\kappa$ and consequently change the values of some cardinal invariants. We will argue that this happens only for $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ which will be equal to κ^+ in $V[G]$ disregarding the value of κ^* (see Lemma 4.10).

Let us start by showing that all cardinal invariants in our list except for $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ (if $\kappa^* > \kappa^+$) continue to have value κ^* in $V[G]$:

Lemma 4.5. *The following identities hold in $V[G]$:*

$$\kappa^* = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{u}(\kappa) = \operatorname{cov}(\mathcal{M}_\kappa) = \operatorname{add}(\mathcal{M}_\kappa) = \operatorname{non}(\mathcal{M}_\kappa) = \operatorname{cof}(\mathcal{M}_\kappa). \tag{4.17}$$

Proof. Let us first focus on the invariants $\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{s}(\kappa), \mathfrak{r}(\kappa), \mathfrak{a}(\kappa), \mathfrak{u}(\kappa)$. We know that they are all equal to κ^* in $V[F_0][G_\delta]$, and this fact is witnessed for each invariant by an appropriate subset of $({}^\kappa\kappa)^{V[F_0][G_\delta]}$ of size κ^*. Since $({}^\kappa\kappa)^{V[F_0][G_\delta]} = ({}^\kappa\kappa)^{V[G]}$, these witnesses are still relevant and imply

$$\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{s}(\kappa), \mathfrak{r}(\kappa), \mathfrak{a}(\kappa), \mathfrak{u}(\kappa) \le \kappa^* \text{ in } V[G].$$

It thus suffices to show $\kappa^* \le \mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{s}(\kappa), \mathfrak{r}(\kappa), \mathfrak{a}(\kappa), \mathfrak{u}(\kappa)$.

As the Mathias generic subsets of κ are added cofinally often below δ, we have

$$\kappa^* \le \mathfrak{b}(\kappa) \text{ and } \kappa^* \le \mathfrak{s}(\kappa)$$

in $V[G]$: If an unbounded or a splitting family B of size $< \kappa^*$ were added by R, then because δ has cofinality κ^* in $V[G]$, it would follow

$$V[G] \models B \subseteq ({}^{\kappa}\kappa)^{V[F_0][G_\alpha]} \tag{4.18}$$

for some $\alpha < \delta$ – but this is impossible because a Mathias generic subset of κ added at any stage after α dominates all functions in $V[F_0][G_\alpha]$ and is unsplit by any subset of κ in $V[F_0][G_\alpha]$.

Since $\mathfrak{b}(\kappa) = \kappa^*$, the remaining inequalities $\kappa^* \le \mathfrak{d}(\kappa), \mathfrak{r}(\kappa), \mathfrak{u}(\kappa), \mathfrak{a}(\kappa)$ follow by ZFC inequalities (4.19)–(4.21) which were proved in [4]:

$$\kappa^+ \le \mathfrak{b}(\kappa) \le \mathfrak{a}(\kappa), \tag{4.19}$$

$$\mathfrak{b}(\kappa) \le \mathfrak{r}(\kappa) \le \mathfrak{u}(\kappa), \tag{4.20}$$

$$\mathfrak{b}(\kappa) \le \mathfrak{d}(\kappa). \tag{4.21}$$

Let us now turn to the invariants related to the meager ideal $\mathcal{M}_\kappa$. We first observe that $\mathbb{P}_\delta \!\downarrow\! p_{\dot{U}}$ adds a sequence of Cohen generic subsets of κ of order-type κ^*, $\langle c_\beta \,|\, \beta < \kappa^* \rangle$, which are cofinal in δ. By standard arguments, for every nowhere dense set A in κ^κ in $V[F_0][G_\delta]$, there is some $\alpha < \kappa^*$ such that for every $\beta > \alpha$, $c_\beta \notin A$. This implies $\mathrm{cov}(\mathcal{M}_\kappa) \ge \kappa^*$ and also $\mathrm{non}(\mathcal{M}_\kappa) \le \kappa^*$ (because $\{c_\beta \,|\, \beta < \kappa^*\}$ is seen to be non-meager) in $V[F_0][G_\delta]$. For the proof, see for instance [3, Proposition 47] which works in our case with obvious modifications. We need to argue that both inequalities still hold in $V[G]$. By [3, Proposition 47], one can work with (closed) nowhere dense sets A_f determined by certain functions $f : 2^{<\kappa} \to 2^{<\kappa}$ (because for every nowhere dense set D there is f such that $D \subseteq A_f$).[7] Since R does not add new κ-sequences, these functions f and the associated nowhere dense sets A_f are the same in $V[F_0][G_\delta]$ and $V[G]$. Now we use a similar argument which we used to show that κ^* is less or equal to $\mathfrak{b}(\kappa)$ in (4.18): If B is a collection of $< \kappa^*$-many nowhere dense sets A_f in $V[G]$, then B is contained as a subset in $V[F_0][G_\alpha]$ for some $\alpha < \delta$, and consequently there is some $\alpha' < \kappa^*$ such that for every $\beta > \alpha'$, $c_\beta \notin \bigcup B$. This implies that $\bigcup B$ does not cover the whole space and that $\{c_\beta \,|\, \beta < \kappa^*\}$ is still non-meager, and so $\mathrm{cov}(\mathcal{M}_\kappa) \ge \kappa^*$ in $V[G]$ and $\mathrm{non}(\mathcal{M}_\kappa) \le \kappa^*$ in $V[G]$.

To finish the argument, we use the following inequalities which are provable in ZFC (see [4] and [3] for details):

$$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}, \mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}(\kappa), \mathrm{non}(\mathcal{M}_\kappa)\}, \tag{4.22}$$

[7] To avoid a possible confusion: In [3, Proposition 47], $\{c_\beta \,|\, \beta < \kappa^+\}$ is shown to be non-meager. This is because in that paper only the Cohen forcing at κ is used, and any κ-many A_f's have a name which uses only $< \kappa^+$-many Cohen coordinates, which leaves some c_β, $\beta < \kappa^+$, outside of these A_f's. For our iteration, this product-type argument is not possible (all we can say is that every A_f appears in the iteration by some stage $\alpha < \delta$), so only $\{c_\beta \,|\, \beta < \kappa^*\}$ is seen to be non-meager.

$$\mathfrak{b}(\kappa) \leq \mathrm{non}(\mathcal{M}_\kappa), \mathrm{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}(\kappa). \tag{4.23}$$

Then $\kappa^* = \mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa)$ holds in $V[G]$ by (4.22), (4.23), and $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \kappa^*$. □

Let us now discuss the values of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$. Since $\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa)$ follows easily, we will show $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \kappa^+$ by arguing that R adds a tower of size κ^+ (if $\kappa^* > \kappa^+$).

Let us review the definition of the tower number $\mathfrak{t}(\kappa)$. Recall that we write $A \subseteq^* B$ for $A, B \in [\kappa]^\kappa$ if $|A \setminus B| < \kappa$.

Definition 4.6. We say that $\mathcal{T} \subseteq [\kappa]^\kappa$ is a *tower* if $\mathcal{T}$ is reversely well-ordered by $\subseteq^*$,[8] for every $X \subseteq \mathcal{T}$ of size $< \kappa$, $|\bigcap X| = \kappa$ (we say that $\mathcal{T}$ satisfies the *strong intersection property, SIP*), and $\mathcal{T}$ has no pseudo-intersection, i.e. there is no $A \in [\kappa]^\kappa$ such that $A \subseteq^* T$ for all $T \in \mathcal{T}$.

We will use the following special case of [19, Main Lemma 2.1]:

Fact 4.7. *Suppose $\kappa = \kappa^{<\kappa}$, $\kappa > \beth_\omega$, and $\mathfrak{t}(\kappa) > \kappa^+$. Then there is an injective map $\varphi : 2^{<\kappa^+} \to [\kappa]^\kappa$ such that for each $\alpha < \kappa^+$ and $f \in 2^\alpha$, $\{\varphi(f \restriction \beta) \mid \beta < \alpha\}$ is reversely well-ordered by $\subseteq^*$, satisfies SIP, and*

$$\varphi(f^\frown 0) \cap \varphi(f^\frown 1) = \emptyset. \tag{4.24}$$

Fact 4.7 gives a sufficient condition for a κ^+-distributive forcing to add a tower of size κ^+.

Lemma 4.8. *Suppose $\kappa^{<\kappa} = \kappa$, $\kappa > \beth_\omega$ and $\mathfrak{t}(\kappa) > \kappa^+$. Suppose $\mathbb{P}$ is a κ^+-distributive forcing which adds a new cofinal branch to the tree $(2^{<\kappa^+}, \subseteq)$.*[9] *Then $\mathbb{P}$ adds a tower of size κ^+ and thus forces $\mathfrak{t}(\kappa) = \kappa^+$.*

Proof. In V, let φ be the mapping from Fact 4.7. Suppose $b \in 2^{\kappa^+}$ is a new cofinal branch through $(2^{<\kappa^+}, \subseteq)^V$ in $V[\mathbb{P}]$. Since $\mathbb{P}$ is κ^+-distributive, $([\kappa]^\kappa)^V = ([\kappa]^\kappa)^{V[\mathbb{P}]}$. Let $\mathcal{T}_b = \{\varphi(b \restriction \alpha) \mid \alpha < \kappa^+\}$. By the properties of φ, $\mathcal{T}_b$ is reversely well-ordered by $\subseteq^*$ and satisfies SIP. Moreover, $\mathcal{T}_b$ is a tower because it cannot have a pseudo-intersection in $V[\mathbb{P}]$: If A is a pseudo-intersection of $\mathcal{T}_b$, then $\{\varphi^{-1}(B) \mid A \subseteq^* B, B \in \mathrm{rng}(\varphi)\}$ defines the cofinal branch b in the ground model due to (4.24). This is a contradiction, and so such an A cannot exist. □

Note that the assumption $\kappa > \beth_\omega$ for an uncountable κ in Fact 4.7 can be replaced by weaker conditions (see [19, Main Lemma 2.1]), and that the argument in Fact 4.7 and Lemma 4.8 also works for $\omega = \kappa$ without additional assumptions.

[8] I.e. $\mathcal{T}$ can be enumerated as a $\subseteq^*$-decreasing sequence $\langle T_\alpha \mid \alpha < \gamma\rangle$ for some ordinal γ.

[9] This is the same as saying that $\mathbb{P}$ adds a fresh subset of κ^+, or that it fails to have the κ^+-approximation property.

Remark 4.9. Lemma 4.8 provides a way of showing $\mathfrak{t}(\kappa) = \kappa^+$ in $V[P]$ for a variety of forcing notions related to trees of height κ^+: the tree $(2^{<\kappa^+}, \subseteq)$ embeds many trees of height κ^+, and adding a new cofinal branch to any of them by a κ^+-distributive forcing results in $\mathfrak{t}(\kappa) = \kappa^+$. An example of this argument is in Lemma 4.10 below.

We use Lemma 4.8 to argue that R adds a tower of size κ^+ over $V[F_0][G_\delta]$ (if $\kappa^* > \kappa^+$):

Lemma 4.10. *R adds a tower of size κ^+ over $V[F_0][G_\delta]$ if $\kappa^* > \kappa^+$. It follows $V[G]$ satisfies:*

$$\kappa^+ = \mathfrak{p}(\kappa) = \mathfrak{t}(\kappa). \tag{4.25}$$

Proof. Since $\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa)$ is always true, it suffices to prove $\mathfrak{t}(\kappa) = \kappa^+$. For $\alpha < \lambda$, let $F_0|\alpha$ denote the restriction of F_0 to $\mathrm{Add}(\kappa, \alpha)$. By the definition of $\mathbb{M}(\kappa, \lambda)$,[10] there is a complete embedding i_α:

$$i_\alpha : \mathrm{Add}(\kappa, \alpha) * \mathrm{Add}(\kappa^+, 1) \to \mathbb{M}(\kappa, \lambda)$$

for every successor cardinal $\alpha < \lambda$. Fix any such α. Let T denote the tree $(2^{<\kappa^+}, \subseteq)$ in $V[F_0|\alpha]$ of height κ^+. Since the tail iteration $\mathrm{Add}(\kappa, \lambda \backslash \alpha) * \dot{\mathbb{P}}_\delta \downarrow p_{\dot{U}}$ is forced to be κ^+-Knaster, T has the same cofinal branches in $V[F_0|\alpha]$ as it has in $V[F_0][G_\delta]$.

Let us now work in $V[F_0][G_\delta]$. If $\kappa^* = \kappa^+$, then $\mathfrak{t}(\kappa) = \kappa^+$, and there is nothing to prove because $V[G]$ will still satisfy $\mathfrak{t}(\kappa) = \kappa^+$. So assume $\mathfrak{t}(\kappa) = \kappa^* > \kappa^+$. Let φ be the mapping from Fact 4.7 from $\tilde{T} = (2^{<\kappa^+})^{V[F_0][G_\delta]}$ to $([\kappa]^\kappa)^{V[F_0][G_\delta]}$. Note that T is a subtree of $\tilde{T}$.

Due to the existence of i_α, the generic filter F_1 for the κ^+-distributive quotient forcing R adds over $V[F_0][G_\delta]$ a new cofinal branch through T determined by the generic filter for the forcing $\mathrm{Add}(\kappa^+, 1)^{V[F_0|\alpha]}$ (which is induced by F_1 and which adds a new cofinal branch to T). Now the result follows by Lemma 4.8 applied in $V[F_0][G_\delta]$ with $R = P$, using the fact that a new cofinal branch through T is a new cofinal branch through $\tilde{T}$ as well. □

This ends the proof of Theorem 4.2. □

[10] See [1] for the details regarding the properties of $\mathbb{M}(\kappa, \lambda)$. Briefly, conditions in $\mathbb{M}(\kappa, \lambda)$ have the form (p, q) where p is a condition in $\mathrm{Add}(\kappa, \lambda)$ and q is a function on λ with domain of size $\leq \kappa$ containing only successor cardinals such that for every $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is a condition in $\mathrm{Add}(\kappa^+, 1)^{V[\mathrm{Add}(\kappa,\alpha)]}$. The ordering is defined by $(p', q') \leq (p, q) \Leftrightarrow p' \leq p, \mathrm{dom}(q) \subseteq \mathrm{dom}(q')$ and for all $\alpha \in \mathrm{dom}(q)$, $p'|\mathrm{Add}(\kappa, \alpha) \Vdash q'(\alpha) \leq q(\alpha)$. It is easy to check that the mapping which sends $(p, q) \in \mathrm{Add}(\kappa, \alpha) * \mathrm{Add}(\kappa^+, 1)$ to (p, q') where q' is a function with domain $\{\alpha\}$ and $q'(\alpha) = q$ is a complete embedding.

5. Open questions and further results

We end the paper with some open questions and further results.

Question 1. The indestructibility of the tree property at κ^{++} in Fact 2.8 and of the negation of the weak Kurepa Hypothesis at κ^+ in Theorem 2.12 only work for the iteration $\mathbb{P}_2$ in Theorem 4.2. At the moment we do not know whether it is consistent for an inaccessible κ to have:

$$\kappa^+ < \mathfrak{t}(\kappa) \le \mathfrak{u}(\kappa) < 2^\kappa \text{ with } \mathsf{TP}(\kappa^{++}) \text{ and/or } \neg\mathsf{wKH}(\kappa^+). \tag{5.26}$$

A similar limitation of the indestructibility method appears in [13] where it is left open whether it is consistent to have a strong limit $\aleph_\omega$ with

$$2^{\aleph_\omega} > \aleph_{\omega+1}, \mathfrak{u}(\aleph_\omega) = \aleph_{\omega+1} \text{ and } \mathsf{TP}(\aleph_{\omega+2}). \tag{5.27}$$

Sometimes an ad hoc argument can be found in these contexts (as in [7] where (5.27) is obtained without $\mathfrak{u}(\kappa) = \aleph_{\omega+1}$), but technical difficulties are usually substantial and increase with the complexity of the forcing (in our case $\mathbb{P}_\delta \downarrow p_{\dot{U}}$).

An underlying open question – and arguably more interesting – is therefore whether Fact 2.8 and/or Theorem 2.12 can be extended to include all κ^+-cc or at least κ^+-Knaster forcings in $V[\mathbb{M}(\kappa,\lambda)]$.

Question 2. (The situation at ω.) Let us briefly sketch an argument that our results also apply to ω: By [2], one can obtain the consistency of $\omega_1 \le \mathfrak{u} = \nu < \mathfrak{d} = \delta = 2^\omega$ for any regular uncountable cardinals $\nu < \delta$ in the ground model with GCH. The forcing is $\mathrm{Add}(\omega,\delta)$ followed by an iteration of the Mathias forcing of length ν, with carefully chosen ultrafilters. Let us write this forcing as $\mathbb{C} * \dot{\mathbb{Q}}$. This forcing is ω_1-Knaster, so mimicking the argument in Theorem 4.2, using Lemma 4.8 with $\kappa = \omega$, we obtain:

Theorem 5.1. *Suppose* GCH *holds,* λ *is a weakly compact cardinal and* $\mu \ge \lambda$ *and* $\omega_1 \le \kappa^* < \mu$ *are regular cardinals. Then* $P_3 := \mathrm{Add}(\kappa,\lambda) * [(\mathbb{C} * \dot{\mathbb{Q}}) \times \dot{R}]$ *forces*

$$\omega_1 = \mathfrak{t} \le \mathfrak{u} = \kappa^* < 2^\omega = \mu$$

and the compactness principles

$$\mathsf{SR}(\omega_2), \mathsf{DSS}(\omega_2), \mathsf{TP}(\omega_2), \neg\mathsf{wKH}(\omega_1).$$

It is possible to compute the values of other cardinal invariants in this model as well. As in Question 1, we do not know how to construct a model in which all these compactness principles hold and yet $\omega_1 < \mathfrak{t} = \mathfrak{u} < 2^\omega$ holds as well.

Question 3. We have obtained $\neg\mathsf{AP}(\kappa^{++})$ in our model as a consequence of a stronger principle $\mathsf{DSS}(\kappa^{++})$, for which we have an indestructibility result available (to our knowledge it is open whether $\neg\mathsf{AP}(\kappa^{++})$ implies $\mathsf{DSS}(\kappa^{++})$).

We do not know whether Theorems 4.1 or 4.2 can be proved with $\neg\mathsf{AP}(\kappa^{++})+\neg\mathsf{DSS}(\kappa^{++})$.

Question 4. There are other compactness principles which may hold at κ^{++} with suitably modified Theorems 4.1 and 4.2.

(a) The club stationary reflection at ν^+ for a regular ν, $\mathsf{CSR}(\nu^+)$, states that for every stationary set S in ν^+ which contains only ordinals of cofinality $< \nu$, there is a club C in ν^+ such that C intersected with ordinals of cofinality ν is contained in the set of reflection points of S. $\mathsf{CSR}(\nu^+)$ can be obtained by an iteration which shoots clubs through the sets of reflection points of all stationary sets with ordinals of cofinality $< \nu$, see [17] for more details. Honzik and Stejskalova showed in [13] a limited indestructibility result for CSR: if κ is regular, then Cohen forcing at κ of arbitrary length preserves $\mathsf{CSR}(\kappa^{++})$, and so does the simple Prikry forcing at κ if κ is measurable. Recently, this has been extended to κ^+-linked forcings in [9] by Gilton and Stejskalova, but this is still not good enough for the application in this paper.

Remark. $\mathsf{CSR}(\lambda)$ does not hold in $V[\mathbb{M}(\kappa,\lambda)]$, so in order to have $\mathsf{CSR}(\lambda)$ a (variant) of the iteration from [17] should be considered. Also note that $\mathsf{CSR}(\lambda)$ is compatible with $2^\kappa = \kappa^+$, and this is the setup of [17]. A generalization of the method in [17] for the Mitchell forcing and $2^\kappa = \kappa^{++} = \lambda$ appeared in [8].

(b) The Guessing model principle at κ^{++}, $\mathsf{GMP}(\kappa^{++})$ is a strong principle which implies $\neg\mathsf{wKH}(\kappa^+)$ and $\mathsf{TP}(\kappa^{++})$. $\mathsf{GMP}(\omega_2)$ is a consequence of PFA, and $\mathsf{GMP}(\kappa^{++})$ holds in the Mitchell collapse $V[\mathbb{M}(\kappa,\lambda)]$ if λ is supercompact in the ground model. Honzik, Lambie-Hanson, and Stejskalova proved in [11] that $\mathsf{GMP}(\kappa^{++})$ is preserved over any model of $\mathsf{GMP}(\kappa^{++})$ by adding any number of Cohen subsets of κ, and its consequence $\neg\mathsf{wKH}(\kappa^+)$ is preserved over any model of $\mathsf{GMP}(\kappa^{++})$ by any κ-centered forcing. Preservation uder the κ-centered forcing notions is still not good enough for the present applications, but may be the indestructibility result from [11] can be strengthened accordingly to have $\mathsf{GMP}(\kappa^{++})$ in Theorems 4.1 or 4.2 (with λ now being supercompact in the ground model).

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