A Laver-like indestructibility for hypermeasurable cardinals

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The author was supported by FWF/GAČR grant I 1921-N25.

Abstract: We show that if κ is $H(\mu)$-hypermeasurable for some cardinal μ with $\kappa < \mathrm{cf}(\mu) \leq \mu$ and GCH holds, then we can extend the universe by a cofinality-preserving forcing to obtain a model V^* in which the $H(\mu)$-hypermeasurability of κ is indestructible by the Cohen forcing at κ of any length up to μ (in particular κ is $H(\mu)$-hypermeasurable in V^*). The preservation of hypermeasurability (in contrast to preservation of mere measurability) is useful for subsequent arguments (such as the definition of Radin forcing). The construction of V^* is based on the ideas of Woodin (unpublished) and Cummings [3] for preservation of measurability, but suitably generalised and simplified to achieve a more general result. Unlike the Laver preparation [18] for a supercompact cardinal, our preparation non-trivially increases the value of 2^{κ^+}, which is equal to μ in V^* (or μ^+ if $\mathrm{cf}(\mu) = \kappa^+$), but $2^\kappa = \kappa^+$ is still true in V^* if we start with GCH.

Keywords: Laver indestructibility, hypermeasurable cardinals, strong cardinals.

AMS subject code classification: 03E35,03E55.

Date: July 25, 2019

1 Introduction

Laver showed in [18] that one can extend any universe where κ is supercompact by a forcing of size κ such that in the resulting generic extension the supercompactness of κ is indestructible by any κ directed-closed forcing notion. It is natural to try to formulate an analogue of Laver's indestructibility for other large cardinals. There is ample literature on the subject, see for instance Gitik and Shelah [12] for strong cardinals (with respect to Prikry-style forcing notions), Hamkins [14] for strongly compact cardinals, Johnston [16] for strongly unfoldable cardinals and Apter [2] for a strong cardinal (the strong cardinal was supercompact in the ground model); these results usually try to achieve the greatest possible level of generality. However, it is often useful to have indestructibility for a smaller family of forcing notions, such as the Cohen forcing at κ of a prescribed length α (we denote this forcing $\mathrm{Add}(\kappa, \alpha)$), or forcings which in some sense contain the Cohen forcing (such as the Mitchell forcing discussed in Section 3.2.1). This is for instance the case of obtaining the tree property at the double successor of a singular strong limit cardinal κ in [5] where the large cardinal assumption of supercompactness is invoked only to ensure that κ is still measurable after forcing with $\mathrm{Add}(\kappa, \lambda)$, where λ is the least weakly compact above κ. With such a limited goal, one can study large cardinals for which a more general version of indestructibility may not be available.

Recall the definition of a hypermeasurable cardinal: If $\kappa < \mu$ are cardinals, we call κ *$H(\mu)$-hypermeasurable* if there is $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$ and $H(\mu) \subseteq M$, where $H(\mu)$ denotes the collection of sets whose transitive closure has size less than μ.[1] Hypermeasurable cardinals, and related notions, feature predominantly in many arguments involving the structure of the continuum function on all cardinals, and in arguments dealing with combinatorial properties at the double successor of a singular strong limit cardinal; see for instance [10], [11], [19], and [7]. In particular, the paper [7] provided the incentive for the results in this paper as the assumption of indestructibility simplified substantially the argument for the tree property (building on the indestructibility, [13] achieves a more general result).

Woodin first discovered an argument for preparing a universe with an $H(\kappa^{++})$-hypermeasurable cardinal so that the further forcing with $\mathrm{Add}(\kappa, \kappa^{++})$ yields the measurability (in fact $H(\kappa^{++})$-hypermeasurability) of κ.[2] There exist two different presentations of this argument: [4] and [3] (we will use the latter); both presentations force over the universe a generic object for the image of $\mathrm{Add}(\kappa, \kappa^{++})$ by a measure ultrapower embedding which is then used to obtain the desired generic filter to be able to lift a certain embedding (the generic object needs to be modified; this step is often called the "surgery"). It seemed crucial in this argument that the image of $\mathrm{Add}(\kappa, \kappa^{++})$ via a measure ultrapower embedding is in fact equivalent to $\mathrm{Add}(\kappa^+, \kappa^{++})$ in the universe.

Building on the method described in [3], we reformulate and simplify the argument to work with an arbitrary $\kappa < \mathrm{cf}(\mu) \leq \mu$ and to preserve the original degree of hypermeasurability. Since there seems to be no uniform representation of the image of $\mathrm{Add}(\kappa, \mu)$ via a measure ultrapower embedding for larger μ (see Question **Q1** in Section 3.1 for more details), we use a lottery of forcing notions to circumvent this step (see Definition 2.2). This change necessitated another modification in the subsequent argument: it was necessary to avoid the use of the measure ultrapower embedding in the argument for Theorem 2.11 (see Remark 2.8 for more details). This modification led to a more elegant argument and allows one to handle more complicated forcing notions (see Question **Q2** and Section 3.2.1).

The paper is structured as follows: In Section 2.2 we prepare the universe by forcing a generic object (denoted $\tilde{g}^*$ in Corollary 2.7) which is then used in Section 2.3 and Theorem 2.11 to lift a hypermeasurable embedding. The preparation in Section 2.2 must be carefully set up so that the hypermeasurability of κ is preserved at this stage. In section 3, we list some open questions.

[1] The same cardinal is also called $\kappa + \alpha$-strong for an appropriate α, i.e. for an α such that $V_{\kappa+\alpha}$ is included in M if and only if $H(\mu)$ is included in M, where M is an inner model.

[2] Woodin's construction may also be formulated for the weaker notion of the *κ^{++}-tall cardinal* (there exists $j : V \to M$ with critical point κ such that ${}^{\kappa}M \subseteq M$ and $\kappa^{++} < j(\kappa)$). A μ-tall cardinal for $\mu > \kappa$ is defined analogously. Our results apply to μ-tall cardinals as well, with a suitable simplification to the preparation P^1 in Section 2.2 (see Remark 2.9).

2 Construction of the model

2.1 Initial assumptions

Let us assume that GCH holds in V. Let us for the rest of the paper fix cardinals κ an μ with

$$\kappa < \mathrm{cf}(\mu) \leq \mu$$

and assume that κ is $H(\mu)$-hypermeasurable.

If $\mu = \kappa^+$, then κ is just measurable and the indestructibility argument is much simpler (see Remark 2.12). Let us therefore further assume

$$\kappa^{++} \leq \mu.$$

Let us first review some basic properties of embeddings associated with large cardinals (a general reference for this review is [4] which contains details regarding the lifting of embeddings).

Let us fix $j : V \to M$ which witnesses the $H(\mu)$-hypermeasurability of κ and has the (κ, μ)-extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \ \&\ \alpha < \mu\}.$$

In particular, $H(\mu)$ is included in M and M is closed under κ-sequences in V (this may be assumed since $\mathrm{cf}(\mu) > \kappa$). Let us also assume there exists $\mathfrak{f} : \kappa \to \kappa$ such that $j(\mathfrak{f})(\kappa) = \mu$ and for each measurable $\alpha < \kappa$, $\mathfrak{f}(\alpha)$ has cofinality at least α^+; let us write μ_α for $\mathfrak{f}(\alpha)$.[3]

Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have width κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, for some $\alpha < \mu$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has width $i(\mathfrak{f})(\kappa) < i(\kappa)$, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain $i(\mathfrak{f})(\kappa)$;[4] in particular $(\kappa^{++})^N \leq i(\mathfrak{f})(\kappa) < i(\kappa) < \kappa^{++}$.

We need the following simple lemma:

Lemma 2.1 *Assume GCH and that κ, μ, i, j are as above. Then*

$$\mu \leq i(\mu) < j(\kappa) < \mu^+ \text{ and } j(\kappa) < j(\mu) < \mu^{++}; \tag{2.1}$$

more precisely:

(i) If μ is the successor of a singular cardinal ν with cofinality κ, then $\mu \leq i(\mu) < j(\kappa)$.
(ii) Otherwise $\mu = i(\mu) < j(\kappa)$.

[3] Such an $\mathfrak{f}$ can be forced if necessary using the fast-function forcing of Woodin; see [14].

[4] Every $j(f)(\alpha)$ can be written as $k(i(f))(\alpha)$; if we set $f^* = i(f) \restriction i(\mathfrak{f})(\kappa)$, the domain of $k(f^*)$ is equal to $k(i(\mathfrak{f})(\kappa)) = \mu$, and $k(f^*)(\alpha) = j(f)(\alpha)$.

Proof. All inequalities in (2.1) – except for $i(\mu) < j(\kappa)$ – follow by simple counting arguments and elementarity using the fact that the cofinality μ is at least κ^+. The claim $i(\mu) < j(\kappa)$ will follow from the proofs of (i) and (ii).

(i). Every $\alpha < i(\nu)$ in the ultrapower can be written as $[f]_U$ for some $f : \kappa \to \nu$ (where $[f]_U$ denotes the collapse of the equivalence class of f). Since M contains U as an element and is closed under κ-sequences in V, it can compute that $|i(\nu)| \le \nu^\kappa = (\nu^+)^M = \mu$ so that:

$$i(\nu) < (\nu^{++})^M.$$

Since ν is less than $j(\kappa)$, $j(\kappa)$ is inaccessible in M, and $i(\mu)$ is the successor of $i(\nu)$ we obtain:

$$i(\mu) = (i(\nu^+))^M < (\nu^{+++})^M < j(\kappa).$$

(ii). Every $\alpha < i(\mu)$ can be written as $[f]_U$ for some $f : \kappa \to \mu$, and this gives $|i(\mu)| \le \mu^\kappa = \mu$. To argue that $i(\mu) = \mu$, it suffices to show that $\langle i(\alpha) \,|\, \alpha < \mu \rangle$ is cofinal in $i(\mu)$ and its limit is μ. For the first claim, note that if $[f]_U < i(\mu)$, then without the loss of generality there is some $\alpha < \mu$ such that the range of f is included in α (since the cofinality of μ is at least κ^+), and therefore $[f]_U \le i(\alpha)$. For the second claim notice that each $i(\alpha)$ has size strictly less than μ and therefore $i(\alpha) < \mu$. (The second claim does not go through for case (i).) □

Note that since U is in M, an argument similar to Lemma 2.1(i), (ii) implies that the forcing $i(\mathrm{Add}(\kappa, \mu)) = \mathrm{Add}(i(\kappa), i(\mu))^N$ computes the same way in V and M, and is an element of $H(j(\kappa))^M$.

2.2 Preparation

Let $C(\mathfrak{f})$ be the closed unbounded set of all closure points of $\mathfrak{f}$: if α is in $C(\mathfrak{f})$ and $\beta < \alpha$, then $\mathfrak{f}(\beta) < \alpha$. Let $\mathcal{M}$ be the unbounded set in κ which contains the measurable cardinals in $C(\mathfrak{f})$, and let $\mathcal{M}^+$ be the set $\mathcal{M} \cup \{\kappa\}$. Recall that we write μ_α instead of $\mathfrak{f}(\alpha)$.

Let us review the notion of the *sum of forcing notions* or the *lottery sum of forcing notions*, in the terminology of [14]. Let I be a non-empty set and $\{P_i \,|\, i \in I\}$ a set of forcing notions. Then the lottery

$$\bigoplus \{P_i \,|\, i \in I\} \tag{2.2}$$

is a forcing notion with the domain $\{(i,p) \,|\, i \in I \ \&\ p \in P_i\} \cup \{1\}$, where 1 is not in $\bigcup \{P_i \,|\, i \in I\}$, and the ordering defined as follows: (i) 1 is the greatest element and (ii) $(i,p) \le (j,q)$ if and only if $i = j$ and $p \le q$ in P_i. The intuition is that the generic filter first chooses a forcing notion, and then forces with it.

We will use the lottery twice: in Definition 2.2 and in Definition 2.10.

Definition 2.2 *Define in V an Easton product*

$$P^1_\kappa = \prod_{\alpha \in \mathcal{M}} Q^1_\alpha \tag{2.3}$$

of length κ such that at every $\alpha \in \mathcal{M}$,

$$Q^1_\alpha = \bigoplus\{R \,|\, R \text{ satisfies } (*)\},$$

where $(*)$ is the following condition:

- $(*)$ *There is a normal measure W on α in V and R is equal to $i_W(\mathrm{Add}(\alpha, \mu_\alpha))$, where i_W is the embedding generated by W.*

Let us further define

$$P^1 = P^1_\kappa \times i(\mathrm{Add}(\kappa, \mu)). \tag{2.4}$$

Let us write $i(\mathrm{Add}(\kappa, \mu))$ as Q^1_κ, and let us use μ_κ to denote μ in the next paragraph and in Lemma 2.3 to simplify notation.

For $\alpha \in \mathcal{M}^+$, let us write P^1_α for the product P^1 below α, and $P^1_{>\alpha}$ for the product indexed above α so that $P^1 = P^1_\alpha \times Q^1_\alpha \times P^1_{>\alpha}$.[5] For $\alpha \in \mathcal{M}^+$, let G^1_α, $G^1_{>\alpha}$ and g^1_α denote generic filters for $P^1_\alpha, P^1_{>\alpha}$, and Q^1_α over V, respectively. For $\alpha \in \mathcal{M}$, let us write α^* for the next element in $\mathcal{M}$ above α.

We state some facts concerning P^1.

Lemma 2.3 *Assume GCH.*

(i) For every $\alpha \in \mathcal{M}$, Q^1_α has size less than α^*.
(ii) For every $\alpha \in \mathcal{M}^+$, Q^1_α preserves cofinalities over V.
(iii) P^1 preserves cofinalities over V.
(iv) For every $\alpha \in \mathcal{M}^+$, G^1_α, g^1_α, and $G^1_{>\alpha}$ are pairwise mutually generic.

Proof. (i). There are at most α^{++}-many normal measures W at stage α and each R in the lottery has size less than α^* (by an analogue of Lemma 2.1, $|i_W(\mu_\alpha)| = \mu_\alpha$, and therefore $|R| = \mu_\alpha$).

(ii). Let R be a variable for the forcings in the lottery Q^1_α for $\alpha \in \mathcal{M}$ and for the forcing Q^1_κ for $\alpha = \kappa$. It suffices to show that R is α^+-closed and α^{++}-cc in V, and therefore is cofinality-preserving. Closure is obvious by the fact that a measure ultrapower via a normal measure at α is closed under α-sequences in V. Regarding the chain condition, notice that every element of R can be identified with the equivalence class of some function $f : \alpha \to \mathrm{Add}(\alpha, \mu_\alpha)$. For $f, f' : \alpha \to \mathrm{Add}(\alpha, \mu_\alpha)$, set $f \le f'$ if for all $i < \alpha$, $f(i) \le f'(i)$; it suffices to check that the ordering $\le$ on these f's is α^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size α^+ and is closed under α-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most α^+.

(iii). For every $\alpha \in \mathcal{M}$, P^1_α is α-cc, Q^1_α is α^+-closed and has size less than α^*, and $P^1_{>\alpha}$ is $(\alpha^*)^+$-closed. For $\alpha = \kappa$, the same facts are true (and the size of Q^1_κ is just μ). By standard

[5] For $\alpha = \kappa$, we consider $P^1_{>\kappa}$ as the trivial forcing.

arguments, it suffices to show that each R (R is a variable as in (ii)) is cofinality preserving over $V[G^1_\alpha]$. Since P^1_α is α-cc, R remains α^+-distributive in $V[G^1_\alpha]$; R is α^{++}-cc in V, and since P^1_α has size just α, it forces that R is still α^{++}-cc (if there were an antichain in $V[G^1_\alpha]$ of size α^{++}, a single condition in P^1_α would determine a cofinal part of it, which would yield an antichain in V of size α^{++}).

(iv). This follows by the combination of the right chain condition and closure using the Easton's lemma (which implies that if P is α-cc and Q is α-closed, then the respective generic filters are mutually generic). □

We will show now that Q^1_κ provides an important generic object (which we denote $\tilde{g}$) which will be later used for the lifting of the Cohen forcing at κ (see Theorem 2.11).

Notational remark. From now on, let us write Q instead of Q^1_κ, and g instead of g^1_κ for easier reading.

Let us fix a $P^1_\kappa \times Q$-generic filter $G^1_\kappa \times g$.

Lemma 2.4 *Assume GCH. The following hold:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.
(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.
(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.
(iv) g is Q-generic over $N[i^1(g)]$.
(v) There is $\tilde{g}$ in $V[g]$ such that $\tilde{g}$ is $k(Q) = j(\mathrm{Add}(\kappa,\mu))$-generic over $M[j^1(g)]$.

Proof. (i) and (ii). These follow by the κ^+-closure of Q in V and the fact that j, i have width κ: the pointwise image of g generates a generic filter for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $i(\mathfrak{f})(\kappa) < i(\kappa) < i(\kappa^+)$, we use the closure of $i(Q)$ and the fact that k has width $i(\mathfrak{f})(\kappa)$ to argue that the pointwise image $k``(i^1(g))$ generates a generic filter which is equal to the generic filter generated by $j``g$ by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. Therefore g and $i^1(g)$ are mutually generic over N by Easton's lemma.

(v). Q is $i(\kappa)$-closed in $N[i^1(g)]$ since the generic $i^1(g)$ does not add new sequences of length $i(\kappa)$; it follows as in (iii) that $k^1``g$ generates a $j(\mathrm{Add}(\kappa,\mu))$-generic filter $\tilde{g}$ over $M[j^1(g)]$. □

After using k to secure the generic object $\tilde{g}$, we will forget i and k, and only lift j^1 further to $V[g][G^1_\kappa]$ (which is the same as $V[G^1_\kappa][g]$ by mutual genericity). For convenience, let us write h^* for $j^1(g)$.

Lemma 2.5 *Assume j^1 is as in Lemma 2.4.*

(i) j^1 lifts in $V[g][G^1_\kappa]$ to

$$j^2 : V[g][G^1_\kappa] \to M[h^*][G^1_\kappa][g][h], \tag{2.5}$$

for some h.

(ii) *There exists in $V[g][G^1_\kappa]$ an object $\tilde{g}^*$ which is $j^2(\mathrm{Add}(\kappa,\mu)^{V[g][G^1_\kappa]})$-generic over the model $M[h^*][G^1_\kappa][g][h]$.*

Proof. In $V[G^1_\kappa][g]$, lift j^1 to

$$j^2 : V[g][G^1_\kappa] \to M[h^*][G^1_\kappa][g][h], \tag{2.6}$$

choosing in the lottery at stage κ of $j(P^1_\kappa)$ the forcing Q which is available here since $H(\mu)$ is included in M (see the paragraph below the proof of Lemma 2.1). The generic object h is constructed using the extender representation of M and the fact that $j(P^1_\kappa)$ in the interval $(\kappa, j(\kappa))$ is κ^+-closed in V, and more than μ^+-closed in the sense of M. In more detail: every dense open set can be written as $j(f)(\alpha)$ for some $f : \kappa \to H(\kappa^+)$ and $\alpha < \mu$. For every such f, the collection of all dense open sets $j(f)(\alpha)$, $\alpha < \mu$, has size at most μ, and its intersection is therefore dense. Since there are only κ^+-many such f, h can be constructed recursively in κ^+-many steps.

By mutual genericity, h constructed in V as generic over M is actually generic over $M[h^*][G^1_\kappa][g]$. Let us denote $M[h^*]$ by M^1, and $M[h^*][G^1_\kappa][g][h]$ by M^2.

Note that $\mathrm{Add}(\kappa,\mu) = \mathrm{Add}(\kappa,\mu)^{V[g]}$, but $\mathrm{Add}(\kappa,\mu)^{V[g][G^1_\kappa]}$ properly contains $\mathrm{Add}(\kappa,\mu)$. We know from Lemma 2.4 that the object $\tilde{g}$ is $j(\mathrm{Add}(\kappa,\mu))$-generic over M^1. We wish to find an object which is generic for $j^2(\mathrm{Add}(\kappa,\mu)^{V[g][G^1_\kappa]})$ over M^2. We use the following Fact, proved in [3, Fact 2], to achieve this aim:

Fact 2.6 *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any γ, the term forcing $Q_S = \mathrm{Add}(\kappa,\gamma)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\gamma)$.*

Proof. See [3]. □

The import of Fact 2.6 is that any generic filter for $\mathrm{Add}(\kappa,\gamma)$ adds a generic filter for $\mathrm{Add}(\kappa,\gamma)$ as defined in $V[S]$, provided that S is κ-cc and has size κ.

We wish to apply this fact to $j(\mathrm{Add}(\kappa,\mu)) = \mathrm{Add}(j(\kappa), j(\mu))^{M^1}$ and $j(P^1_\kappa)$ in the model M^1. $j(\kappa)^{<j(\kappa)} = j(\kappa)$ holds in M^1, $j(P^1_\kappa)$ has size $j(\kappa)$, and is $j(\kappa)$-cc. It follows by Fact 2.6 that $\tilde{g}$ generates an object $\tilde{g}^*$ which is $\mathrm{Add}(j(\kappa), j(\mu))^{M^2} = j^2(\mathrm{Add}(\kappa,\mu)^{V[g][G^1_\kappa]})$-generic over M^2.

Since G^1_κ and g are in M^2, j^2 witnesses that κ is still $H(\mu)$-hypermeasurable in $V[g][G^1_\kappa]$. □

We can summarise the construction above as follows (removing the superscripts for clarity):

Corollary 2.7 *Assume V' is a ground model where GCH holds and κ, μ and j' satisfy the initial assumptions in Section 2.1: in particular $\kappa < \mathrm{cf}(\mu) \le \mu$, $\kappa^{++} \le \mu$, and κ is $H(\mu)$-hypermeasurable and this is witnessed by a (κ,μ)-extender embedding $j' : V' \to M'$. Then there is a cofinality-preserving generic extension V of V' and an embedding $j : V \to M$ which lifts j' such that the following hold:*

(i) GCH holds on an unbounded set of measurable cardinals below κ, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = \mu$.
(ii) j is a (κ, μ)-extender embedding, i.e.

$$M = \{j(f)(\alpha) \,|\, f : \kappa \to V \ \&\ \alpha < \mu\},$$

M is closed under κ-sequences, $\mu < j(\kappa) < \mu^+$, and $H(\mu)$ is included in M.
(iii) There exists in V a function $\mathfrak{f} : \kappa \to \kappa$ such that $\mu = j(\mathfrak{f})(\kappa)$ (we denote $\mathfrak{f}(\alpha)$ by μ_α).
(iv) There exists in V an object $\tilde{g}^*$ which is $j(\mathrm{Add}(\kappa, \mu))$-generic over M.

Remark 2.8 Notice that unlike in [3], Corollary 2.7 does not mention the lifting of k. In fact k may not lift in this context because the forcing $k(P^1)$ cannot choose the right forcing in the lottery at stage κ: the normal measure U generating k is certainly not available in the measure ultrapower via U. The dropping of k from the statement of Corollary 2.7 simplifies the argument in Theorem 2.11 since we deal directly with $\tilde{g}^*$ (unlike [3] which, roughly speaking, uses g of Lemma 2.4 instead).

Remark 2.9 If we only wished to preserve the weaker property of κ being μ-tall (see Footnote 2), we can entirely omit P^1_κ from the definition of P^1, i.e. we can identify P^1 with $i(\mathrm{Add}(\kappa, \mu))$. Corollary 2.7 would still hold, but without the claim that $H(\mu)$ is included in M.

2.3 The final model V^*

We start with the model V from Corollary 2.7 and define a reverse Easton iteration of length κ which will provide the desired model V^* in which the $H(\mu)$-hypermeasurability of κ is indestructible by $\mathrm{Add}(\kappa, \gamma)$ for any $0 \leq \gamma \leq \mu$, where we construe $\mathrm{Add}(\kappa, 0)$ as the trivial forcing.

Let $\mathcal{I}$ denote the set of inaccessible cardinals below κ which are in $C(\mathfrak{f})$ (the closure points of $\mathfrak{f}$).

Definition 2.10 *Define*

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha \in \mathcal{I} \rangle$$

as a reverse Easton iteration such that at each $\alpha \in \mathcal{I}$,

$$\dot{Q}_\alpha = \bigoplus \{\mathrm{Add}(\alpha, \beta)^{V[P_\alpha]} \,|\, 0 \leq \beta \leq \mu_\alpha\}.$$

Let G_κ be P_κ-generic over V, and denote $V^* = V[G_\kappa]$.

Theorem 2.11 *Assume μ and $j : V \to M$ are as in Corollary 2.7. Then V^* is the desired model where the hypermeasurability of κ is indestructible under the Cohen forcing at κ of any length up to μ: For any $0 \leq \gamma \leq \mu$, κ remains $H(\mu)$-hypermeasurable in the generic extension of V^* by $\mathrm{Add}(\kappa, \gamma)^{V^*}$.*

PROOF. Let us fix γ, and let g_γ be $\mathrm{Add}(\kappa,\gamma)^{V^*}$-generic over V^*. We will show that j lifts to $V[G_\kappa][g_\gamma]$.

We follow closely the argument in Cummings [3] but with the important simplification that we do not use a measure ultrapower embedding obtained from j and a generic filter over the measure ultrapower, but use the generic object $\tilde{g}^*$ directly.

Using standard methods, lift j to

$$j^* : V[G_\kappa] \to M[G_\kappa][g_\gamma][H]$$

as follows:

- At stage κ of $j(P_\kappa)$ work below a condition which chooses in the lottery the forcing $\mathrm{Add}(\kappa,\gamma)$.
- The generic object H is constructed using the extender representation of M: the dense open sets in the forcing $j(P_\kappa)$ in the interval $(\kappa, j(\kappa))$ can be grouped into κ^+-many groups each of size μ in M; these groups are of the form $\{j(f)(\alpha) \mid \alpha < \mu\}$, where f is a function from κ to $H(\kappa^+)$. The intersection of each group is a dense set because the forcing $j(P_\kappa)$ in the interval $(\kappa, j(\kappa))$ is μ^+-closed in M. Since there are only κ^+-many of these groups, a generic H can be constructed in $V[G_\kappa][g_\gamma]$ which meets them all.

It remains to find a generic filter for the j^*-image of g_γ. We use at this crucial step the generic object $\tilde{g}^*$ which we prepared in V. Let us denote by $\tilde{g}^*_\gamma$ the truncation of $\tilde{g}^*$ to $j(\gamma)$. Notice that $\tilde{g}^*_\gamma$ is generic for the wrong forcing: it is $j(\mathrm{Add}(\kappa,\gamma)^V)$-generic over M, but we need a generic object for $j^*(\mathrm{Add}(\kappa,\gamma)^{V^*})$ over $M[G_\kappa][g_\gamma][H]$. Fact 2.6 comes to the rescue again: By elementarity and the fact that $j(P_\kappa)$ has size $j(\kappa)$ and is $j(\kappa)$-cc in M, and $j(\kappa)^{<j(\kappa)} = j(\kappa)$, Fact 2.6 implies that in $V[G_\kappa][g_\gamma]$, $\tilde{g}^*_\gamma$ yields a generic object g^*_γ over $M[G_\kappa][g_\gamma][H]$ for $j^*(\mathrm{Add}(\kappa,\gamma)^{V^*})$.

The object g^*_γ is still not good enough to lift j^* to g_γ because it may not contain the pointwise image $j^*“g_\gamma$. Using the method of surgery (see [3]), we modify g^*_γ to g^{**}_γ which is still $j^*(\mathrm{Add}(\kappa,\gamma)^{V^*})$-generic, but in addition contains the pointwise image $j^*“g_\gamma$. It follows we can lift to

$$j^{**} : V[G_\kappa][g_\gamma] \to M[G_\kappa][g_\gamma][H][g^{**}_\gamma].$$

The last lifting shows that κ remains measurable, in fact $H(\mu)$-hypermeasurable, in the model $V[G_\kappa][g_\gamma]$ as desired. □

Remark 2.12 We have proved Theorem 2.11 under the assumption $\mu \geq \kappa^{++}$. Strictly speaking, our argument does not apply to the case $\mu = \kappa^+$ since the measure U is no longer present in M (since M can be identified with a normal measure ultrapower in this case), and the lifting argument for P^1 immediately below (2.6) will not work. However, a much easier argument is available. Suppose $j : V \to M$ is a normal measure embedding generated by a normal measure at κ; it suffices to lift the embedding (the inclusion of $H(\kappa^+)$ in the target model is then automatic). There is no need for the preparation forcing P^1 since it is possible to construct the generic object for $j(\mathrm{Add}(\kappa,\kappa^+))$ – which is used in Theorem 2.11 – directly in the current universe: There are only $j(\kappa^+)$-many maximal antichains of $j(\mathrm{Add}(\kappa,\kappa^+))$ which we need to meet, and $j(\kappa^+)$ has size just κ^+ in V.

3 Open questions

3.1 The forcing $i(\mathrm{Add}(\kappa,\mu))$ in V

In the presentation in [3], the use of the lottery in Definition 2.2 is avoided as one can show that $i(\mathrm{Add}(\kappa,\kappa^{++}))$ (in the notation of Definition 2.2) is in V isomorphic to $\mathrm{Add}(\kappa^+,\kappa^{++})$, and therefore in [3] the preparation is carried out with $\mathrm{Add}(\alpha^+,\alpha^{++})$ at inaccessible $\alpha<\kappa$. Gitik and Merimovich [11] generalised this result to $i(\mathrm{Add}(\kappa,\kappa^{+n}))$ for any $1\le n<\omega$, showing that $i(\mathrm{Add}(\kappa,\kappa^{+n}))$ is isomorphic to $\mathrm{Add}(\kappa^+,\kappa^{+n})$. It is likely that this may be extended to $i(\mathrm{Add}(\kappa,\mu))$ for any μ which is below the first singular cardinal of cofinality κ above κ (i.e. below $\kappa^{+\kappa}$; compare with **Q3**). But the general case is not clear:

Q1 Suppose $i:V\to N$ is an elementary embedding with critical point κ generated by a normal measure at κ, $2^\kappa=\kappa^+$ and $\mu>\kappa$ is a cardinal. Is $i(\mathrm{Add}(\kappa,\mu))$ isomorphic in V to $\mathrm{Add}(\kappa^+,i(\mu))$?

The positive answer would eliminate the use of lottery in Definition 2.2.

3.2 Generalization to more forcing notions

It is natural to ask whether the indestructibility can be extended to include more forcing notions.

Q2 Is there a comprehensive and well-defined class of forcing notions which add new subsets of κ for which the hypermeasurability of κ can be made indestructible?

We give two examples related to **Q2**.

3.2.1 Mitchell forcing

Recall the definition of the Mitchell forcing $\mathbb{M}(\kappa,\mu)$ as introduced in [20] and further analysed in [1]. If GCH holds, κ is regular and μ is a weakly compact cardinal above κ, then $\mathbb{M}(\kappa,\mu)$ forces $2^\kappa=\mu=\kappa^{++}$ and the tree property at μ. By an analysis in [1], $\mathbb{M}(\kappa,\mu)$ is equivalent to $\mathrm{Add}(\kappa,\mu)*\dot{Q}_{\mathbb{M}}$, for some $\dot{Q}_{\mathbb{M}}$ which is forced to be κ^+-distributive.

With a slight modification of the preparation in Definition 2.10, one can ensure that the hypermeasurability of κ is preserved in a generic extension via $\mathbb{M}(\kappa,\mu)$, where μ is the least weakly compact above κ in V' of Corollary 2.7. Working in $V=V'[P^1]$ of Corollary 2.7, let $\mathcal{I}$ be the set of measurable cardinals below κ, and let $\mathfrak{f} : \kappa \to \kappa$ satisfy $j(\mathfrak{f})(\kappa) = \mu$; let us denote $\mathfrak{f}(\alpha)$ by μ_α:[6]

Definition 3.1 *Work in $V = V'[P^1]$ of Corollary 2.7. Let us define:*

$$P^{\mathbb{M}}_\kappa = \langle (P^{\mathbb{M}}_\alpha, \dot{Q}_\alpha) \mid \alpha \in \mathcal{I} \rangle$$

as a reverse Easton iteration such that at each $\alpha \in \mathcal{I}$,

$$\dot{Q}_\alpha = \mathbb{M}(\alpha, \mu_\alpha)^{V[P^{\mathbb{M}}_\alpha]}.$$

Corollary 3.2 *Assume that κ, μ and j' are in V' as in Corollary 2.7 and μ is the least weakly compact cardinal above κ in V'. Then κ is still $H(\mu)$-hypermeasurable in*

$$V'[P^1][P^{\mathbb{M}}_\kappa][\mathbb{M}(\kappa, \mu)]. \tag{3.7}$$

Proof. Let us assume that j is the lifting of j' to $j : V \to M$, as in Corollary 2.7. We follow the proof of Theorem 2.11. Let $G_\kappa * g_\kappa$ be a generic filter over V for $P^{\mathbb{M}}_\kappa * \mathbb{M}(\kappa, \mu)$; we wish to show that κ is still $H(\mu)$-hypermeasurable in $V[G_\kappa][g_\kappa]$. As in the proof of Theorem 2.11, we can lift in $V[G_\kappa][g_\kappa]$ the embedding j to

$$j^* : V[G_\kappa] \to M[G_\kappa][g_\kappa][H], \tag{3.8}$$

for some H. We now use the fact that $\mathbb{M}(\kappa, \mu)$ can be written as $\mathrm{Add}(\kappa, \mu) * \dot{Q}_{\mathbb{M}}$, where $\dot{Q}_{\mathbb{M}}$ is forced by $\mathrm{Add}(\kappa, \mu)$ to be κ^+-distributive; let us write g_κ as $g^0_\kappa * g^1_\kappa$, where $g^0_\kappa * g^1_\kappa$ is $\mathrm{Add}(\kappa, \mu) * \dot{Q}_{\mathbb{M}}$-generic. Using the argument in the proof of Theorem 2.11 applied with $P^{\mathbb{M}}_\kappa$ ($P^{\mathbb{M}}_\kappa$ has size κ and is κ-cc), we first lift j^* to

$$j^{**} : V[G_\kappa][g^0_\kappa] \to V[G_\kappa][g_\kappa][H][g^{**0}_\kappa], \tag{3.9}$$

for some g^{**0}_κ which contains the pointwise image of g^0_κ. Finally, we lift j^{**} further to j^{***} using the standard fact that the pointwise image $j^{**}{}'' g^1_\kappa$ generates a generic filter g^{**1}_κ for $j^{**}((\dot{Q}_{\mathbb{M}})^{g^0_\kappa})$ over $M[G_\kappa][g_\kappa][H][g^{**0}_\kappa]$:

$$j^{***} : V[G_\kappa][g_\kappa] \to M[G_\kappa][g_\kappa][H][g^{**0}_\kappa][g^{**1}_\kappa]. \tag{3.10}$$

The embedding j^{***} witnesses that κ is still $H(\mu)$-hypermeasurable in $V[G_\kappa][g_\kappa]$ as required. □

Remark 3.3 By including a lottery in the definition of $P^{\mathbb{M}}_\kappa$ (as in Definition 2.10) in which we generically choose either the trivial forcing or $\mathbb{M}(\alpha, \mu_\alpha)$ at $\alpha \in \mathcal{I}$, we can ensure that κ is hypermeasurable in $V'[P^1][P^{\mathbb{M}}_\kappa]$ as well, if required.

[6]Notice that μ is the least weakly compact cardinal above κ in V', but not in $V = V'[P^1]$ since μ lost its strong limitness in V. However we can still target μ using $\mathfrak{f}$.

Note that the lifting through the Mitchell forcing at κ in Corollary 3.2 (but without the preservation of hypermeasurability) is straightforward if the Mitchell forcing is used only at κ because – as we noted above – it decomposes into $\mathrm{Add}(\kappa,\mu)$ followed by the κ^+-distributive forcing notion $Q_{\mathbb{M}}$. However, in many applications (such as [7] or [13]) it is necessary to include in $P_\kappa^{\mathbb{M}}$ the Mitchell forcing below κ as well. As it turns out, the detailed argument in this case hinges on the omission of k in the argument in Theorem 2.11 (see Remark 2.8) – the point being that k moves non-trivially the collapsing part of the Mitchell forcing at κ and it is not clear how to lift k at this stage.

3.2.2 Forcings with fusion

Recall that the Sacks forcing at κ of length $\mu > 0$, $\mathrm{Sacks}(\kappa,\mu)$,[7] adds new subsets of κ, yet all decreasing sequences of length κ of a certain type have a lower bound; we call these sequences *fusion sequences* (see [17] for more details about this forcing). There are many forcing notions which add subsets of κ and are closed under suitably defined fusion sequences (see [15] or [8]).

For these forcing notions, it is much easier to show the indestructibility as in Theorem 2.11 – we can completely omit the preparation P^1. For instance if κ,μ and j are in V' as in Corollary 2.7, then if we modify P_κ from Definition 2.10 so that $\dot{Q}_\alpha$ is equal to $\mathrm{Sacks}(\alpha,\mu_\alpha)$, then in $V'[P_\kappa]$, the hypermeasurability of κ is indestructible under $\mathrm{Sacks}(\kappa,\mu)$.

The proof is based on methods introduced in [9], and developed in [6] and [8].

Remark 3.4 Note that for small μ, an analogous situation is actually true for the Cohen forcing $\mathrm{Add}(\kappa,\mu)$ as well; see Section 3.3.

3.3 Is the preparation P^1 in fact necessary?

Recently, starting with GCH, Shalom showed in [21] that for $\kappa < \mu < \kappa^{+\kappa}$, Woodin's preparation is not necessary: Suppose $0 < \gamma < \kappa$ is a successor ordinal and let S be a reverse Easton iteration which forces with $\mathrm{Add}(\alpha,\alpha^{+\gamma})$ for every inaccessible $\alpha \le \kappa$. Then if $j : V \to M$ is an embedding with critical point κ which satisfies ${}^\kappa M \subseteq M$, and $\kappa^{+\gamma} = (\kappa^{+\gamma})^M$, j lifts to $V[S]$.

In particular, Theorem 2.11 holds for all μ, $\kappa^+ < \mu < \kappa^{+\kappa}$,[8] even if we omit the preparation P^1 (2.4).

Q3 Is the preparation P^1 necessary for Theorem 2.11 with $\mu > \kappa^{+\kappa}$? By [21], we know it is not necessary for $\kappa^+ < \mu < \kappa^{+\kappa}$.

Acknowledgements. The author wishes to thank to Sy Friedman and Š. Stejskalová for

[7] The forcing may be a product or an iteration, what we say applies to both.

[8] Notice that μ in Theorem 2.11 is required to have cofinality greater than κ, so only the successor cardinals μ between κ^+ and $\kappa^{+\kappa}$ are legitimate choices.

helpful discussions regarding this paper. In particular, the proof of Lemma 2.3(ii) was suggested by Sy Friedman.

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