CAPTURING SETS OF ORDINALS BY NORMAL ULTRAPOWERS

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ABSTRACT. We investigate the extent to which ultrapowers by normal measures on κ can be correct about powersets $\mathcal{P}(\lambda)$ for $\lambda > \kappa$. We consider two versions of this question, the *capturing property* $\mathrm{CP}(\kappa, \lambda)$ and the *local capturing property* $\mathrm{LCP}(\kappa, \lambda)$. Both of these describe the extent to which subsets of λ appear in ultrapowers by normal measures on κ. After examining the basic properties of these two notions, we identify the exact consistency strength of $\mathrm{LCP}(\kappa, \kappa^+)$. Building on results of Cummings, who determined the exact consistency strength of $\mathrm{CP}(\kappa, \kappa^+)$, and using a forcing due to Apter and Shelah, we show that $\mathrm{CP}(\kappa, \lambda)$ can hold at the least measurable cardinal.

1. INTRODUCTION

It is well known that the ultrapower of the universe by a normal measure on some cardinal κ cannot be very close to V; for example, the measure itself never appears in the ultrapower. It follows that these ultrapowers cannot compute $V_{\kappa+2}$ correctly. In the presence of GCH, this is equivalent to saying that the ultrapower is incorrect about $\mathcal{P}(\kappa^+)$. But if GCH fails, it becomes conceivable that a normal ultrapower could compute additional powersets correctly. This conjecture turns out to be correct: Cummings [4], answering a question of Steel, showed that it is relatively consistent that there is a measurable cardinal κ with a normal measure whose ultrapower computes $\mathcal{P}(\kappa^+)$ correctly; in fact he showed that this situation is equiconsistent with a $(\kappa+2)$-strong cardinal κ. In this paper we will study this *capturing property* and its local variant further.

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Definition 1. Let κ and λ be cardinals. We say that the *local capturing property* $\mathrm{LCP}(\kappa,\lambda)$ holds if, for any $x\subseteq\lambda$, there is a normal measure U_x on κ such that $x\in\mathrm{Ult}(V,U_x)$. We shall say that U_x (or $\mathrm{Ult}(V,U_x)$) *captures* x.

The full capturing property will amount to having a uniform witness for the local version.

Definition 2. Let κ and λ be cardinals. We say that the *capturing property* $\mathrm{CP}(\kappa,\lambda)$ holds if there is a normal measure on κ that captures all subsets of λ; in other words, a normal measure U such that $\mathcal{P}(\lambda)\in\mathrm{Ult}(V,U)$.

Some quick and easy observations: increasing λ clearly gives us stronger properties, $\mathrm{CP}(\kappa,\lambda)$ implies $\mathrm{LCP}(\kappa,\lambda)$, and $\mathrm{CP}(\kappa,\kappa)$ holds for any measurable cardinal κ.

Using this language, we can summarize Cummings' result as follows:

Theorem 3 (Cummings)**.** *If κ is $(\kappa+2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds. Conversely, if $\mathrm{CP}(\kappa,\kappa^+)$ holds, then κ is $(\kappa+2)$-strong in an inner model.*

We should mention that $\mathrm{CP}(\kappa,2^\kappa)$ is provably false: if it held, then some normal ultrapower would contain all families of subsets of κ, in particular the measure from which it arose, which is impossible. Therefore a failure of GCH is necessary for $\mathrm{CP}(\kappa,\kappa^+)$ to hold. By work of Gitik [7], this means that $\mathrm{CP}(\kappa,\kappa^+)$ has consistency strength at least that of a measurable cardinal κ with Mitchell rank $o(\kappa)=\kappa^{++}$, and the actual consistency strength of a $(\kappa+2)$-strong cardinal κ is only slightly beyond that.

The following are the main results of this paper. In section 2 we analyse the consistency strength of $\mathrm{LCP}(\kappa,\kappa^+)$ and show that it is only a small step below the strength of the full capturing property.

Main theorem 1. *Assuming GCH, if $\mathrm{LCP}(\kappa,\kappa^+)$ holds, then $o(\kappa)=\kappa^{++}$. Conversely, if $o(\kappa)\geq\kappa^{++}$, then $\mathrm{LCP}(\kappa,\kappa^+)$ holds in an inner model.*

In section 3 we continue the analysis in the case that GCH fails at κ and show that the first part of the previous theorem, namely that κ has high Mitchell rank, fails dramatically if $2^\kappa>\kappa^+$.

Main theorem 2. *If κ is $(\kappa+2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds and κ is the least measurable cardinal.*

This last theorem is a nontrivial improvement of Cummings' result. Since the forcing he used to achieve $\mathrm{CP}(\kappa,\kappa^+)$ was relatively mild, κ remained quite large in the resulting model; for example, it was still a measurable limit of measurable cardinals. Our theorem shows that, while $\mathrm{CP}(\kappa,\kappa^+)$ has nontrivial consistency strength, it does not directly imply anything about the size of κ in V (beyond κ being measurable).

We will list questions that we have left open wherever appropriate throughout the paper.

2. The local capturing property

Let us begin our analysis of the local capturing property with some simple observations.

Lemma 4. *If* $\mathrm{LCP}(\kappa, \lambda)$ *holds, then it can be witnessed by measures* U *for which* $\mathrm{Ult}(V, U)$ *and* V *agree on cardinals up to and including* λ.

Proof. Using a pairing function we can code a family of bijections $f_\alpha \colon \alpha \to |\alpha|$ for $\alpha \leq \lambda$ as a single subset $y \subseteq \lambda$. If we want to capture $x \subseteq \lambda$ in an ultrapower as in the lemma, we simply capture (a disjoint union of) x and y using $\mathrm{LCP}(\kappa, \lambda)$. □

Proposition 5. $\mathrm{LCP}(\kappa, (2^\kappa)^+)$ *fails for any measurable* κ.

Proof. If $\mathrm{LCP}(\kappa, (2^\kappa)^+)$ held, there would have to be a normal measure ultrapower $j \colon V \to M$ with critical point κ such that M was correct about cardinals up to and including $(2^\kappa)^+$, by Lemma 4. But no such ultrapower can exist, since the ordinals $j(\kappa)$ and $j(\kappa^+)$ are cardinals in M and both have size 2^κ in V. □

The following lemma is quite well known, but it will be key in many of our observations.

Lemma 6. *Suppose that* $j \colon V \to M$ *is an elementary embedding with critical point* κ *and consider the diagram*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i *is the ultrapower by the normal measure on* κ *derived from* j *and* k *is the factor map. Then the critical point of* k *is strictly above* $(2^\kappa)^N$.

Proof. It is clear that the critical point of k is above κ. Consider some ordinal $\alpha \leq (2^\kappa)^N$. Fix a surjective map $f \colon \mathcal{P}(\kappa) \to \alpha$ in N (and note that both N and M compute $\mathcal{P}(\kappa)$ correctly). Since every ordinal up to and including κ is fixed by k, it follows that $k(f) = k \circ f$ is a surjection from $\mathcal{P}(\kappa)$ to $k(\alpha)$ and so $k \restriction \alpha$ is a surjection onto $k(\alpha)$. It follows that we must have $k(\alpha) = \alpha$. □

Using an old argument of Solovay, we can see that the optimal local capturing property automatically holds at sufficiently large cardinals.

Proposition 7. *If a cardinal* κ *is* 2^κ*-supercompact, witnessed by an embedding* $j \colon V \to M$, *then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in both* V *and* M.

Proof. We first show that $\mathrm{LCP}(\kappa, 2^\kappa)$ holds in V. Suppose it fails. Then there is some $x \subseteq 2^\kappa$ which is not captured by any normal measure on κ. The model M agrees that this is the case, since it has all the normal measures on κ and all the functions $f \colon \kappa \to \mathcal{P}(\kappa)$ that could represent x. Let i and k be as in Lemma 6. By that same lemma, the model N computes 2^κ correctly and it also believes that there is some $y \subseteq 2^\kappa$ which is not captured by any normal measure on κ. This y is fixed by k, so M also believes that y is not captured by any normal measure on κ, and V agrees. But this is a contradiction, since y is captured by the ultrapower N. Therefore $\mathrm{LCP}(\kappa, 2^\kappa)$ holds in V.

Observe that $\mathrm{LCP}(\kappa, 2^\kappa)$ only depends on $\mathcal{P}(2^\kappa)$, the normal measures on κ, and the representing functions $\kappa \to \mathcal{P}(\kappa)$. The ultrapower M has all of these objects, therefore M must agree that $\mathrm{LCP}(\kappa, 2^\kappa)$ holds. □

In particular, if κ is 2^κ-supercompact, then there are many $\lambda < \kappa$ for which $\mathrm{LCP}(\lambda, 2^\lambda)$ holds.

The above argument seems to break down if κ is only θ-supercompact for some $\theta < 2^\kappa$, even if we are only aiming to capture subsets of θ; one simply cannot conclude that M has all the necessary measures to correctly judge whether a set is a counterexample to $\mathrm{LCP}(\kappa, \lambda)$ or not. Thus, the following question remains open.

Question 8. *Suppose that κ is θ-supercompact for some $\kappa < \theta < 2^\kappa$. Does it follow that* $\mathrm{LCP}(\kappa, \theta)$ *holds?*

The same conclusion as in Proposition 7 follows even if κ is merely $(\kappa+2)$-strong.

Proposition 9. *If a cardinal κ is $(\kappa + 2)$-strong, witnessed by an embedding $j\colon V \to M$, then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in both V and M.*

Proof. The argument works just like in Proposition 7. Note that M has all the functions $\kappa \to \mathcal{P}(\kappa)$ and all the normal measures on κ. Furthermore, M has all the subsets of 2^κ (use a wellorder of $V_{\kappa+1}$ in $V_{\kappa+2}$ of ordertype 2^κ). It follows that V and M have all the same counterexamples to $\mathrm{LCP}(\kappa, 2^\kappa)$. □

Reflecting back from M to V, this last proposition implies that below a $(\kappa + 2)$-strong cardinal κ there are many cardinals λ satisfying $\mathrm{LCP}(\lambda, 2^\lambda)$. This observation, together with Cummings' Theorem 3, tells us that the consistency strength of $\mathrm{LCP}(\kappa, \kappa^+)$ is strictly lower than that of $\mathrm{CP}(\kappa, \kappa^+)$. Let us determine this consistency strength exactly.

Recall that the *Mitchell order* $\lhd$ on a measurable cardinal κ is a relation on the normal measures on κ, where $U \lhd U'$ if U appears in the ultrapower by U'. It is a standard fact that $\lhd$ is wellfounded, and the *Mitchell rank* of κ is the height $o(\kappa)$ of this order.

Proposition 10. *If* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds, then $o(\kappa) = (2^\kappa)^+$.*

Proof. This is essentially the proof that the large cardinals mentioned in the previous two propositions have maximal Mitchell rank. We shall recursively build a Mitchell-increasing sequence $\langle U_\alpha\,;\, \alpha < (2^\kappa)^+\rangle$ of normal measures on κ. So suppose that $\langle U_\alpha\,;\, \alpha < \delta\rangle$ has been constructed for some $\delta < (2^\kappa)^+$. Using a pairing function we can code each measure U_α as a subset of 2^κ, and then code the entire sequence $\langle U_\alpha\,;\, \alpha < \delta\rangle$ as a subset of 2^κ as well. By $\mathrm{LCP}(\kappa, 2^\kappa)$ there is a normal measure U on κ which captures this subset, and thus the whole sequence of measures. We can then simply let $U_\delta = U$. □

To show that the lower bound from this proposition is sharp we will pass to a suitable inner model. Recall that a *coherent sequence of normal measures $\mathcal{U}$* of length λ (where λ is an ordinal or Ord) is given by a function $o^{\mathcal{U}}\colon \lambda \to \mathrm{Ord}$ and a sequence

$$\mathcal{U} = \langle U_\alpha^\beta\,;\, \alpha < \lambda, \beta < o^{\mathcal{U}}(\alpha)\rangle\,,$$

where each U_α^β is a normal measure on α and for each α, β, if j_α^β is the corresponding ultrapower map, we have

$$j_\alpha^\beta(\mathcal{U}) \restriction \alpha + 1 = \mathcal{U} \restriction (\alpha, \beta)\,.$$

Here $\mathcal{U} \restriction (\alpha, \beta) = \langle U_\gamma^\delta\,;\, (\gamma, \delta) <_{\mathrm{lex}} (\alpha, \beta)\rangle$ and $\mathcal{U} \restriction \alpha = \mathcal{U} \restriction (\alpha, 0)$.

Theorem 11. *Suppose that $V = L[\mathcal{U}]$ where $\mathcal{U}$ is a coherent sequence of normal measures of length $\kappa+1$ with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$. Then* $\mathrm{LCP}(\kappa, \kappa^+)$ *holds.*

Proof. We shall show that, given any $x \subseteq \kappa^+$, there is some $\beta < \kappa^{++}$ such that $x \in L[\mathcal{U} \restriction (\kappa, \beta)]$. The theorem then immediately follows since, given x, we can find a β as described, and the ultrapower by U^β_κ of $L[\mathcal{U}]$ contains $L[\mathcal{U} \restriction (\kappa, \beta)]$, and therefore x.

So fix some $x \subseteq \kappa^+$ and let ρ be a large regular cardinal so that $x \in L_\rho[\mathcal{U}]$. Since GCH holds, we can find an elementary submodel $M \prec L_\rho[\mathcal{U}]$ of size κ^+ such that $x, \mathcal{U} \in M$ and $\kappa^+, \mathcal{P}(\kappa) \subseteq M$. Let $\pi\colon M \to \overline{M}$ be the Mostowski collapse map.

Note that the $\delta = M \cap \kappa^{++} = \pi(\kappa^{++})$ is an ordinal below κ^{++} and that all ordinals below δ are fixed by π. Moreover, π will fix all subsets of κ^+ in M (since these can be described by sequences of ordinals of length $< \delta$), and therefore also all the measures U^β_α for $(\alpha, \beta) <_{\mathrm{lex}} (\kappa, \delta)$ (since each of these can be coded by a subset of κ^+). It follows that $\pi(\mathcal{U})$ is (in $\overline{M}$) a coherent sequence of normal measures of length $\kappa + 1$ with $o^{\pi(\mathcal{U})}(\kappa) = \delta$, and that $\pi(\mathcal{U}) = \mathcal{U} \restriction (\kappa, \delta)$. Therefore $\overline{M} = L_{\bar\rho}[\mathcal{U} \restriction (\kappa, \delta)]$ for some $\bar\rho < \rho$. Since $x \subseteq \kappa^+$ was fixed by π as well, we get $x \in \overline{M} \subseteq L[\mathcal{U} \restriction (\kappa, \delta)]$. □

Even if, starting from a measurable cardinal κ of Mitchell order κ^{++}, one could construct a coherent sequence $\mathcal{U}$ of normal measures with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, it seems to be an open question (according to [16]) whether it is necessarily the case that $\mathcal{U}$ remains coherent in $L[\mathcal{U}]$. We avoid this issue by using a result of Mitchell [14], who showed in ZFC that there is a sequence of filters $\mathcal{F}$ (possibly empty, possibly of length Ord, or anything in between) such that $L[\mathcal{F}]$ satisfies GCH, $\mathcal{F}$ is a coherent sequence of normal measures in $L[\mathcal{F}]$ and $o^{\mathcal{F}}(\alpha) = \min(o(\alpha)^V, (\alpha^{++})^{L[\mathcal{F}]})$. The model we need will be exactly this $L[\mathcal{F}]$.

Corollary 12. *Assume that $o(\kappa) \geq \kappa^{++}$. Then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in a transitive model of GCH.*

Proof. We may assume that κ is the largest measurable cardinal; if not, we can cut off the universe at the next inaccessible in order to achieve this. Let $\mathcal{F}$ be the sequence of filters described above. By Mitchell's results we know that the sequence $\mathcal{F}$ is a coherent sequence of normal measures in $L[\mathcal{F}]$ and $o^{\mathcal{F}}(\kappa) = (\kappa^{++})^{L[\mathcal{F}]}$. Since κ is the largest measurable, the length of $\mathcal{F}$ is $\kappa + 1$, and it follows from Theorem 11 that $\mathrm{LCP}(\kappa, \kappa^+)$ holds in $L[\mathcal{F}]$. □

In fact, these canonical inner models satisfy a strong form of $\mathrm{LCP}(\kappa, \kappa^+)$, where there is a single function which represents any desired subset of κ^+ in an appropriate normal ultrapower.

Definition 13. Let κ be a measurable cardinal. An *$H_{\kappa^{++}}$-guessing Laver function* for κ is a function $\ell\colon \kappa \to V_\kappa$ with the property that for any $x \in H_{\kappa^{++}}$ there is an ultrapower embedding $j\colon V \to M$ by a normal measure on κ such that $j(\ell)(\kappa) = x$.

It is obvious that the existence of an $H_{\kappa^{++}}$-guessing Laver function for κ implies $\mathrm{LCP}(\kappa, \kappa^+)$. The first author [9, Theorem 28] showed that this stronger property holds in appropriate extender models, in particular the one from Corollary 12.

Starting with a cardinal κ of high Mitchell rank, we obtained a model of the local capturing property by passing to an inner model. We are unsure whether one can obtain the local capturing property from the optimal hypothesis via forcing.

Question 14. *Suppose that GCH holds and $o(\kappa) = \kappa^{++}$. Is there a forcing extension in which* $\mathrm{LCP}(\kappa, \kappa^+)$ *holds?*

It is important to note that the hypothesis in Proposition 10 is quite strong: we need to be able to capture all subsets of 2^κ in order to be able to conclude that the Mitchell rank of κ is large. One might wonder whether some strength can be derived even from weaker local capturing properties, for example $\mathrm{LCP}(\kappa, \kappa^+)$ assuming $\kappa^+ < 2^\kappa$. As we shall see in the following section, the answer is an emphatic no.

3. The capturing property at the least measurable cardinal

In this section we will give a proof of our second main theorem. Our argument owes a lot to Cummings' original proof of Theorem 3 and to the forcing machinery introduced by Apter and Shelah. Nevertheless, we shall strive to give a mostly self-contained account, especially with regard to the forcing notions used.

Let us first explain why we cannot simply use the proof from Theorem 3 and afterwards make κ into the least measurable cardinal just by applying the standard methods of destroying measurable cardinals, such as iterated Prikry forcing or adding nonreflecting stationary sets. In his argument, Cummings starts with a (κ, κ^{++})-extender embedding, lifts it through a certain iteration of Cohen forcings (which will, among other things, ensure that $2^\kappa > \kappa^+$, a necessary condition as we explained), and concludes that the lifted embedding $j\colon V[G] \to M[j(G)]$ is in fact equal to the ultrapower by some normal measure on κ and $M[j(G)]$ captures all the subsets of κ^+ in the extension. One would now hope to be able to lift this new embedding further, through any of the usual forcings which would make κ into the least measurable cardinal. However, for this strategy to work, we should somehow ensure that κ is not measurable in $M[j(G)]$. Otherwise lifting the embedding through any of the usual forcing iterations to destroy all the measurables below κ over $V[G]$ would require us to also destroy the measurability of κ over $M[j(G)]$. But if we did that and maintained the capturing property at the same time, there would be enough agreement between the extensions of $V[G]$ and $M[j(G)]$ that κ would necessarily be nonmeasurable in the extension of $V[G]$ as well. All this is to say that, since κ is very much measurable in $M[j(G)]$ after the forcing done by Cummings, a different approach is necessary.

Instead of first forcing the capturing property and then making κ into the least measurable, the solution is to destroy all the measurable cardinals below κ and blow up 2^κ at the same time. The tools to make this approach work are due to Apter and Shelah [1, 2].

3.1. **The forcing notions.** Let us review the particular forcing notions that will go into building our final forcing iteration. The material in this subsection is contained, in some form or another, in sections 1 of [1, 2].

Since we will be discussing the strategic closure of some of these posets, let us fix some terminology. If $\mathbb{P}$ is a poset and α is an ordinal, the *closure game* for $\mathbb{P}$ of length α consists of two players alternately playing conditions $p \in \mathbb{P}$ in a descending sequence of length α, with player II playing at limit steps. Player II loses the game if at any stage she is unable to make a move; otherwise she wins. If $\mathbb{P}$ is a poset and κ is a cardinal, we shall say that:

- $\mathbb{P}$ is $\leq \kappa$-strategically closed if player II has a winning strategy in the closure game for $\mathbb{P}$ of length $\kappa + 1$.
- $\mathbb{P}$ is $\prec \kappa$-strategically closed if player II has a winning strategy in the closure game for $\mathbb{P}$ of length κ.
- $\mathbb{P}$ is $< \kappa$-strategically closed if it is $\leq \lambda$-strategically closed for all $\lambda < \kappa$.

If κ is a cardinal and α is an ordinal, we let $\mathrm{Add}(\kappa, \alpha)$ be the usual forcing notion to add α many Cohen subsets to κ. We think of conditions in $\mathrm{Add}(\kappa, \alpha)$ as filling in an α-by-κ grid with 0s and 1s. Each condition is only allowed to fill in fewer than κ many cells in the grid. Eventually, the generic will fill in the entire grid, and each column of the grid will be a Cohen subset of κ.

If $\delta \geq \omega_2$ is a regular cardinal, we let $\mathbb{S}_\delta$ be the forcing to add a nonreflecting stationary subset of δ, consisting of points of countable cofinality.[1] A condition in $\mathbb{S}_\delta$ is simply a bounded subset of $x \subseteq \delta$, consisting of points of countable cofinality and satisfying the property that $x \cap \alpha$ is nonstationary in α for every limit $\alpha < \delta$ of uncountable cofinality. The conditions in $\mathbb{S}_\delta$ are ordered by end-extension. It is a standard fact that $\mathbb{S}_\delta$ is $\prec \delta$-strategically closed and, if $2^{<\delta} = \delta$, is δ^+-cc (see [5, Section 6] for more details). Note that the generic stationary set added will also be costationary, since it avoids all ordinals of uncountable cofinality.

If $S \subset \delta$ is a costationary set, let $\mathbb{C}(S)$ be the forcing to shoot a club through $\delta \setminus S$; conditions are closed bounded subsets of $\delta \setminus S$. Again, if $2^{<\delta} = \delta$, then $\mathbb{C}(S)$ will be δ^+-cc ([5, Section 6] has more details). In the cases we will be interested in, $\mathbb{C}(S)$ will also be $< \delta$-distributive (see Lemma 16).

Before we continue with the exposition, let us fix some terminology.

Definition 15. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. We say that $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if they have isomorphic dense subsets.

This is not the most general definition of forcing equivalence that has appeared in the literature, but it has the advantage of being obviously upward absolute between transitive models of set theory.

[1] In our argument we could use any other fixed cofinality below the large cardinal in question. We sacrifice a bit of generality in order to avoid carrying an extra parameter with us throughout the proof. The specific choice of countable cofinality also simplifies some arguments.

Lemma 16. *If δ is a cardinal satisfying $\delta^{<\delta} = \delta$ then $\mathbb{S}_\delta * \mathbb{C}(\dot{S})$, where $\dot{S}$ is the name for the generic nonreflecting stationary set added by $\mathbb{S}_\delta$, is forcing equivalent to* $\mathrm{Add}(\delta, 1)$.

Proof. This is standard; the iteration has a dense $<\delta$-closed subset of size δ, which is equivalent to $\mathrm{Add}(\delta, 1)$ by [5, Theorem 14.1]. □

Suppose that γ is a cardinal, δ is an ordinal, $I \subseteq \delta$, and $\vec{X} = \langle x_\alpha \,;\, \alpha \in I\rangle$ is a ladder system (meaning that each $x_\alpha \subseteq \alpha$ is a $\mathrm{cf}(\alpha)$-sequence cofinal in α; the x_α are called *ladders*). The *Apter–Shelah forcing*[2] $\mathbb{A}(\gamma, \delta, \vec{X})$ consists of conditions (p, Z) where

(1) p is a condition in the Cohen forcing $\mathrm{Add}(\gamma, \delta)$, seen as filling in δ many columns of height γ with 0s and 1s. We will denote by $\mathrm{supp}(p) \subseteq \delta$ the set of indices of the nonempty columns of p.
(2) p is a uniform condition, meaning that all of its nonempty columns have the same height.
(3) Z is a set of ladders from the ladder system $\vec{X}$ and each ladder $z \in Z$ is a subset of $\mathrm{supp}(p)$.

The conditions in $\mathbb{A}(\gamma, \delta, \vec{X})$ are ordered by letting $(p', Z') \leq (p, Z)$ if $p' \leq p$ and $Z' \supseteq Z$, and for any $z \in Z$ each new row in $(p' \setminus p)$, restricted to the columns indexed by z, has unboundedly many 0s and 1s. In other words, when strengthening the Cohen part of the condition, the $z \in Z$ are promises that we will not add a row whose values stabilize when restricted z.

The poset $\mathbb{A}(\gamma, \delta, \vec{X})$ is similar to the poset $P^1_{\delta,\lambda}[S]$ defined in [1, Section 4], with some differences which we believe will simplify the poset. For example, our definition permits an arbitrary ladder system, whereas Apter and Shelah work with a very specific one. For our applications, the specific case studied by Apter and Shelah would have sufficed, but the poset can nevertheless be defined more generally. We believe the additional generality will make the role of the side conditions in the arguments more transparent and clarify where additional assumptions on the parameters in the definition of $\mathbb{A}(\gamma, \delta, \vec{X})$ are required.

Some comments are in order regarding the forcing $\mathbb{A}(\gamma, \delta, \vec{X})$. It is similar enough to the Cohen poset $\mathrm{Add}(\gamma, \delta)$ that one would hope that it is just as simple to show that this forcing also adds δ new subsets of γ and so on. But with the addition of the side conditions this is no longer clear. It is not even immediate that generically we will fill out the entire δ-by-γ matrix. On the other hand, if we want to use this forcing as the main part of our construction to destroy many measurable cardinals, then it cannot be too close to plain Cohen forcing after all. This tension between the Apter–Shelah poset and the Cohen poset is controlled by the ladder system $\vec{X}$, so we will have to choose these ladder systems carefully in our proof.

The following facts are parallel to the ones Apter and Shelah give in [1, 2]; we give proofs for the sake of completeness, but the reader familiar with their exposition should expect no surprises.

[2]We chose the letter $\mathbb{A}$ without prejudice against Shelah, but rather to emphasize that the forcing is derived from the Cohen forcing $\mathrm{Add}(\gamma, \delta)$ by adding some side conditions.

Lemma 17. *Suppose γ is inaccessible, δ is an ordinal, and $\vec{X}$ is a ladder system on some subset of δ. Then the forcing $\mathbb{A}(\gamma, \delta, \vec{X})$ is γ^+-Knaster and γ^{++}-Knaster (meaning that any set of γ^+ (or γ^{++}) many conditions has a subset of γ^+ (or γ^{++}) many pairwise compatible conditions).*

Proof. In the described situation, the poset $\mathrm{Add}(\gamma, \delta)$ is both γ^+-Knaster and γ^{++}-Knaster, as can be seen by the standard Δ-system argument. We show that $\mathbb{A}(\gamma, \delta, \vec{X})$ is γ^+-Knaster, and the γ^{++} case works exactly the same way.

Suppose that (p_α, Z_α) for $\alpha < \gamma^+$ are conditions in $\mathbb{A}(\gamma, \delta, \vec{X})$. We may assume that all of the working parts p_α have the same height. Since the poset $\mathrm{Add}(\gamma, \delta)$ is γ^+-Knaster, we can find a subset $J \subseteq \gamma^+$ of size γ^+ such that the conditions p_α for $\alpha \in J$ are pairwise compatible.

Now suppose that we have two full conditions (p_α, Z_α) and (p_β, Z_β) for $\alpha, \beta \in J$. By our choice of J we know that $p = p_\alpha \cup p_\beta$ is a Cohen condition. Observe that p has the same height as p_α and p_β, and that every nonempty column in p was either present already in both p_α and p_β, or else it was present already in p_α and was empty in p_β, or vice versa. If we let $Z = Z_\alpha \cup Z_\beta$, it then follows that (p, Z) is a common strengthening of both (p_α, Z_α) and (p_β, Z_β). This is because, as far as ladders $z \in Z_\alpha$ are concerned, no new rows were added to the Cohen part when it was strengthened from p_α to p, and similarly for Z_β. $\square$

Lemma 18. *Suppose that γ is regular, δ is an ordinal, and $\vec{X}$ is a ladder system on some subset of δ. Then $\mathbb{A}(\gamma, \delta, \vec{X})$ is $< \gamma$-closed.*

The outright closure of the poset is a slight improvement over the presentation that Apter and Shelah chose; they could only guarantee strategic closure, but the difference will not be significant.

Proof. Start with a descending sequence of conditions (p_α, Z_α) for $\alpha < \lambda < \gamma$. We can get a candidate for a lower bound by simply taking unions in each coordinate, letting $p = \bigcup_{\alpha<\lambda} p_\alpha$ and $Z = \bigcup_{\alpha<\lambda} Z_\alpha$, but we need to verify that $(p, Z) \leq (p_\alpha, Z_\alpha)$. Consider any ladder $z \in Z_\alpha$ and look at the restrictions $p_\alpha \restriction z$ and $p \restriction z$. For each new row in $p \restriction z$, we can find a β with $\alpha < \beta < \lambda$ such that that row appears already in $p_\beta \restriction z$. But because we assumed that $(p_\beta, Z_\beta) \leq (p_\alpha, Z_\alpha)$, it must be the case that that row has unboundedly many 0s and 1s. $\square$

Going forward, we will focus particularly on ladder systems supported on very sparse sets, meaning those without any stationary initial segments. The following is essentially [1, Lemma 2]: although they state the result for a very special ladder system, an inspection of their proof shows that the argument works in general.

Lemma 19. *Let γ be inaccessible and δ an ordinal. Suppose that $I \subseteq \delta$ is nonstationary in its supremum and all of its initial segments are nonstationary in their suprema as well. Let $\vec{X}$ be a ladder system on I. Then there are (nonempty) final segments y_α of each $x_\alpha \in \vec{X}$ such that the y_α are pairwise disjoint.*

Lemma 20. *Suppose γ is inaccessible and δ is an ordinal with* $\mathrm{cf}(\delta) \geq \gamma$. *Suppose that $I \subseteq \delta$ and that all of its proper initial segments are nonstationary. Let $\vec{X}$ be a ladder system on I. Then a generic for $\mathbb{A}(\gamma, \delta, \vec{X})$ is a total function on $\delta \times \gamma$ and each of its columns is a new subset of γ.*

Proof. The proof follows the outline in [1, Section 4]. We only need to show that, given a condition (p, Z), we may extend that condition in order to fill any given empty cell with an arbitrary value. For simplicity, let us assume that the height of p is ρ and that we are attempting to add a new bit to row ρ. We start building the stronger Cohen condition by filling in that new cell in the desired way (and potentially the lower cells in that column as well). We still need to make this new Cohen condition uniform, and pay attention to the promises we made regarding the ladders in Z.

Since γ is inaccessible, Z has size less than γ. Since the cofinality of δ is at least γ, the ladders in Z are bounded below δ. It follows that we can apply Lemma 19 to Z (seen as a ladder system) in order to find pairwise disjoint final segments y of each $z \in Z$.

For each $\alpha \in \mathrm{supp}(p)$, there is at most one such final segment y for which $\alpha \in y$. If there is no such y, we can set the bit in row ρ and column α arbitrarily. On the other hand, if such a y exists, we set all the bits in row ρ and columns in y in an alternating pattern to make sure that there are unboundedly many 0s and 1s. The key fact is that these specifications do not contradict each other, since the sets y are pairwise disjoint.

In this way, we extend p to a uniform condition p' of height $\rho + 1$, and $\mathrm{supp}(p')$ contains $\mathrm{supp}(p)$ and at most one additional point. It is now clear that $(p', Z) \leq (p, Z)$, since we made sure to honour the promises in Z $\square$

If δ is a regular cardinal and $S \subseteq \delta$ is stationary, recall that a $\clubsuit_\delta(S)$-sequence is a ladder system $\langle x_\alpha \,;\, \alpha \in S\rangle$ such that for any unbounded $A \subseteq \delta$ there is some $\alpha \in S$ such that $x_\alpha \subseteq A$.[3]

Lemma 21. *Suppose that $\gamma < \delta$ are regular cardinals, with γ inaccessible and $\delta^{<\delta} = \delta$. Let $S \subseteq \delta$ be a nonreflecting stationary set consisting of points of countable cofinality, and let $\vec{X}$ be a $\clubsuit_\delta(S)$-sequence. Then $\mathbb{A}(\gamma, \delta, \vec{X})$ forces that γ is not measurable.*

Proof. The proof follows the strategy of [1, Lemma 3]. We start with a condition (p, Z) and a name $\dot{U}$ for an ultrafilter on γ. For each $i < \delta$, fix a stronger condition $(p_i, Z_i) \leq (p, Z)$ which decides whether the ith column of the generic is in $\dot{U}$ or not. Let's assume without loss of generality that all these conditions (p_i, Z_i) force their corresponding columns to be in $\dot{U}$ and that all the Cohen parts p_i have the same height. We may also assume that $i \in \mathrm{supp}(p_i)$ for all i. Our cardinal arithmetic assumption now allows us to use a Δ-system argument to find an unbounded set $I \subseteq \delta$ such that the conditions (p_i, Z_i) for $i \in I$ are pairwise compatible (and the supports of the p_i actually form a Δ-system).

[3] The principle $\clubsuit_\delta(S)$ is usually stated in the apparently stronger form where there are stationarily many $\alpha \in S$ for which $x_\alpha \subseteq A$. This formulation is equivalent to the one we use; see [15, Observation I.7.2].

We now use the $\clubsuit_\delta(S)$-sequence: there is an $\alpha \in S$ for which $x_\alpha \subseteq I$. If we now let $p^* = \bigcup_{i \in x_\alpha} p_i$ and $Z^* = \bigcup_{i \in x_\alpha} Z_i$, it follows from what we wrote before that (p^*, Z^*) is a strengthening of each (p_i, Z_i) for $i \in x_\alpha$. Now consider the (even stronger) condition $(p^*, Z^* \cup \{x_\alpha\})$; it really is a condition since i was in the support of each p_i, so $x_\alpha \subseteq \operatorname{supp}(p^*)$. This condition forces that the ith columns of the generic, for $i \in x_\alpha$, are in $\dot{U}$, but it also forces that the intersection of these columns is bounded in γ (in fact, bounded by ρ, where ρ is the height of p^*). It follows that $\dot{U}$ cannot be a name for a countably complete nonprincipal ultrafilter on γ, so γ is not measurable in the forcing extension. □

The following lemma is [1, Lemma 1] (and also [2, Lemma 1]); the reader may find the proof there. The argument is much like the proof that $\operatorname{Add}(\omega_1, 1)$ forces $\diamondsuit$.

Lemma 22. *Let δ be a regular cardinal satisfying $\delta^\omega = \delta$. Then $\mathbb{S}_\delta$ forces that $\clubsuit_\delta(S)$ holds, where S is the generic stationary set added.*

Since we now know that $\mathbb{S}_\delta$ adds a $\clubsuit_\delta(S)$-sequence, it makes sense to consider the iteration $\mathbb{S}_\delta * \mathbb{A}(\gamma, \delta, \vec{X})$, where $\vec{X}$ is a $\clubsuit_\delta(S)$-sequence added by the first stage of forcing. Lemma 21 implies that this iteration will definitely make γ nonmeasurable (assuming we start from GCH or a similar hypothesis). The following lemma is a complement to that result and shows that the measurability of γ may be resurrected. It corresponds to [2, Lemma 4].

Lemma 23. *Let $\gamma < \delta$ be regular cardinals with γ inaccessible and δ satisfying $\delta^{<\delta} = \delta$. Then the iteration $\mathbb{S}_\delta * (\mathbb{A}(\gamma, \delta, \vec{X}) \times \mathbb{C}(\dot{S}))$, where $\vec{X}$ is an arbitrary ladder system on S, is equivalent to* $\operatorname{Add}(\delta, 1) \times \operatorname{Add}(\gamma, \delta)$.

Proof. We stick closely to the argument from [2]. Lemma 16 already told us that $\mathbb{S}_\delta * \mathbb{C}(\dot{S})$ is equivalent to $\operatorname{Add}(\delta, 1)$, so it only remains to show that, in the resulting extension $V[S][C]$, $\mathbb{A}(\gamma, \delta, \vec{X})^{V[S]}$ is equivalent to $\operatorname{Add}(\gamma, \delta)^V = \operatorname{Add}(\gamma, \delta)^{V[S][C]}$. Since in $V[S][C]$, the formerly stationary set S is no longer stationary, nor does it have any stationary initial segments, Lemma 19 implies that we can disjointify the ladder system $\vec{X}$ by picking final segments $y_\alpha \subseteq x_\alpha$ for each $\alpha \in S$.

The set δ can now be decomposed into the disjoint union of the y_α plus the remainder $R = \delta \setminus \bigcup_\alpha y_\alpha$. The key realization (as in the proof of Lemma 20) is that we can honour the promises given by a condition $(p, Z) \in \mathbb{A}(\gamma, \delta, \vec{X})^{V[S]}$ by strengthening p carefully on each y_α (and these regions are pairwise disjoint and do not interfere with each other), and strengthening p quite freely on the remainder R. In fact, we can decompose each condition (p, Z) into the sequence of restrictions $(p \restriction y_\alpha, \{y_\alpha\})$ and $(p \restriction R, \emptyset)$. For each particular α, the ordering between the the conditions restricted to y_α is $< \gamma$-closed, and there are γ many of these restricted conditions. The same result quoted in the proof of Lemma 16 now implies that the poset of these restrictions to y_α is equivalent to $\operatorname{Add}(\gamma, 1)$. At the same time, the restrictions to R do not make any reference to the side conditions and therefore behave like $\operatorname{Add}(\gamma, R)$. In this way we produce an equivalence between $\mathbb{A}(\gamma, \delta, \vec{X})^{V[S]}$ and $\operatorname{Add}(\gamma, R) \times \prod_{\alpha \in S} \operatorname{Add}(\gamma, 1)$, where the product at the end is taken with $< \gamma$-support, and we can conclude that $\mathbb{A}(\gamma, \delta, \vec{X})^{V[S]}$ is equivalent to $\mathrm{Add}(\gamma, \delta)$. □

We conclude this list of facts with the following observation relating to the directed closure of $\mathbb{A}(\gamma, \delta, \vec{X})$. While this poset is generally not even countably directed closed, we can recover some amount of directed closure in cases which will be of interest to us.

Lemma 24. *Suppose that $j\colon M \to N$ is an elementary embedding between inner models with critical point γ, which is inaccessible in M. Suppose additionally that g is $\mathbb{A}(\gamma, \delta, \vec{X})^M$-generic over M and that $j[g] \in N$. Then $\bigcup j[g] \in N$ is a condition in $j(\mathbb{A}(\gamma, \delta, \vec{X})^M)$ and, moreover, is a lower bound for the set of conditions $j[g]$.*

Proof. First, consider a condition $(p, Z) \in \mathbb{A}(\gamma, \delta, \vec{X})^M$ and its image $j((p, Z))$. Since $|p|^M < \gamma$ and $|Z|^M < \gamma$ (due to the inaccessibility of γ in M), the condition $j((p, Z))$ is essentially the same as (p, Z), except that the αth nonempty column in p moves to the $j(\alpha)$th column in $j(p)$ and each ladder in $z \in Z$ stretches into $j(z) = j[z]$.

Now consider $(p^*, Z^*) = \bigcup j[g] \in N$. First, let us see that it is really a condition in $j(A(\gamma, \delta, \vec{X})^M)$. The Cohen part p^* is just the union of all the j-images of the Cohen parts of conditions in g. Since each condition in g has a uniform Cohen part (and their heights are cofinal in γ), it follows that p^* is also a uniform Cohen condition of height γ. Secondly, it is clear that every ladder in Z^* arises as the j-image of a ladder in one of the conditions from g, and since all of those were in $\vec{X}$, every ladder in Z^* is in $j(\vec{X})$. The same consideration also tells us that every ladder in Z^* is contained in the support of the condition p^*. Together, this means that (p^*, Z^*) really is a condition.

Finally, let us see that (p^*, Z^*) is a lower bound for $j[g]$. To that end, take some $(p, Z) \in g$. It is clear that $p^* \leq j(p)$ and that $Z^* \supseteq j(Z)$. We only need check that the working parts were extended appropriately with respect to the ladders in $j(Z) = j[Z]$. So pick a ladder $z \in Z$ and consider the restriction $(p^* \setminus j(p)) \restriction j(z)$. Suppose that the ηth row of this restriction is nonempty; we need to see that there are infinitely many 0s and 1s in this row. By genericity, some condition in g has a nonempty ηth row, and, since g is a filter, we may find some $(q, W) \leq (p, Z)$ in g with a nonempty ηth row. By the definition of the ordering in $\mathbb{A}(\gamma, \delta, \vec{X})^M$ it follows that the restriction $(q \setminus p) \restriction z$ has infinitely many 0s and 1s in the ηth row. By elementarity, the restriction $(j(q) \setminus j(p)) \restriction j(z)$ also has infinitely many 0s and 1s in the ηth row (since η is less than γ, the critical point of j). But since p^* extends $j(q)$, our desired conclusion follows. □

3.2. **Some additional facts about forcing and elementary embeddings.** In this subsection we collect some facts about forcing and ultrapowers, some more standard than others, that we will need in throughout our paper. We indicate at each the parallel result from [3] or [5], where proofs are also given.

Fact 25 ([5, Proposition 9.1])**.** *Suppose that M and N are transitive models and $j\colon M \to N$ is an elementary embedding. Let $\mathbb{P} \in M$ be a poset, let G*

be $\mathbb{P}$-generic over M and let H be $j(\mathbb{P})$-generic over N. If $j[G] \subseteq H$ then j can be extended to an elementary embedding $j\colon M[G] \to N[H]$ satisfying $j(G) = H$.

Fact 26 ([3, Section 1.2.2, Fact 3])**.** *With the notation of the previous fact, if $j\colon M \to N$ is a (κ, λ)-extender embedding, then so is the lift $j\colon M[G] \to N[H]$. In particular, if i is the ultrapower by a normal measure on κ, then so is j.*

Recall that, if κ is a cardinal, a poset $\mathbb{P}$ is called $\leq \kappa$*-distributive* (or $< \kappa$*-distributive*) if forcing with $\mathbb{P}$ does not add any new sequences of ordinals of length $\leq \kappa$ (or $< \kappa$). This is equivalent to saying that the intersection of $\leq \kappa$ (or $< \kappa$) many open dense subsets of $\mathbb{P}$ is dense.

Fact 27 ([3, Section 1.2.2, Fact 2])**.** *With the notation of the previous fact, suppose that j is a (κ, λ)-extender embedding and that $\mathbb{P}$ is $\leq \kappa$-distributive in M. Then $j[G]$ generates a $j(P)$-generic filter over N.*

Fact 28 ([3, Section 1.2.3, Fact 1])**.** *Let M be an inner model, let $\mathbb{P} \in M$ be a poset and let κ be a cardinal. Suppose that $\mathbb{P}$ is $< \kappa$-closed (in V) and that the set of maximal antichains of P in M has cardinality at most κ in V. Then there is a P-generic filter G over M in V.*

Fact 29 ([3, Section 1.2.3, Fact 3])**.** *Let M be an inner model, let $\mathbb{P} \in M$ be a poset and let κ be a cardinal. Suppose that ${}^{\kappa}M \subseteq M$ and that $\mathbb{P}$ is κ^+-cc in V. Let G be $\mathbb{P}$-generic over V. Then $M[G]$ is an inner model of $V[G]$ and ${}^{\kappa}M[G] \subseteq M[G]$ in $V[G]$.*

Recall that if $\mathbb{P}$ is a poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, the *term forcing* $\mathrm{Term}(\mathbb{P}, \dot{\mathbb{Q}})$ consists of $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$, ordered by letting $\sigma \leq \tau$ if $\mathbb{P} \Vdash \sigma \leq \tau$. It is easy to see that if $G \subseteq \mathbb{P}$ and $H \subseteq \mathrm{Term}(\mathbb{P}, \dot{\mathbb{Q}})$ are generic over V, then $\{\sigma^G \,;\, \sigma \in H\} \subseteq \dot{\mathbb{Q}}^G$ is generic over $V[G]$ (see [3, Section 1.2.5, Fact 1] for a proof).

Lemma 30 ([3, Section 1.2.5, Fact 2])**.** *Suppose that κ is a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let $\mathbb{P}$ be a κ-cc forcing of size κ. Let $\dot{\mathbb{Q}}_\lambda$ be the $\mathbb{P}$-name for* $\mathrm{Add}(\kappa, \lambda)$ *in the extension. Then* $\mathrm{Term}(\mathbb{P}, \dot{\mathbb{Q}}_\lambda)$ *is forcing equivalent, in V, to* $\mathrm{Add}(\kappa, \lambda)$*.*

Lemma 31. *Let κ be a measurable cardinal satisfying $2^\kappa = \kappa^+$ and let $j\colon V \to M$ be the ultrapower by a normal measure on κ. Given any finite $n \geq 1$, the forcings $j(\mathrm{Add}(\kappa, \kappa^{+n}))$ and $\mathrm{Add}(\kappa^+, \kappa^{+n})$ are equivalent in V.*

Cummings gave a proof of this lemma for $n = 2$ in [3] (attributing the proof to Woodin), and Gitik and Merimovich proved the generalization to all n in [8, Lemma 3.2].

Lemma 32. *Let κ be a regular cardinal, let $\mathbb{P}$ be some $< \kappa$-distributive forcing notion, and let $\mathbb{Q}$ be a κ-cc forcing notion. If $\mathbb{P}$ forces that $\mathbb{Q}$ is κ-cc, then $\mathbb{Q}$ forces that $\mathbb{P}$ is $< \kappa$-distributive.*

Proof. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic over V and consider a sequence $\vec{x}$ of ordinals in $V[H][G]$ of some length less than κ. We wish to see that $\vec{x} \in V[H]$. Since $\vec{x} \in V[H][G] = V[G][H]$ and $\mathbb{Q}$ is κ-cc in $V[G]$, we can find a nice $\mathbb{Q}$-name σ for $\vec{x}$ in $V[G]$ that can also be coded by a sequence of ordinals of length $< \kappa$. Since $\mathbb{P}$ is $< \kappa$-distributive, this name σ is already in V, and so $\vec{x}$ must appear in $V[H]$, as desired. □

The following key observation was already implicit in Cummings' proof of Theorem 3. It shows that, as long as one can arrange the value of 2^κ appropriately, the apparently difficult part of the capturing property tends to follow for free from the construction.

Lemma 33. *Suppose that $j\colon V \to M$ is a (κ, λ)-extender embedding and $2^\kappa \geq \lambda$. Then j is the ultrapower by a normal measure on κ.*

Proof. Let $i\colon V \to N$ be the ultrapower by the normal measure derived from j and let $k\colon N \to M$ be the factor embedding. Consider some $x \in M$. Since j is a (κ, λ)-extender embedding, we can write $x = j(f)(\alpha)$ for some $\alpha < \lambda$ and some function with domain κ. By Lemma 6 the critical point of k is above λ and therefore

$$x = j(f)(\alpha) = k(i(f))(\alpha) = k(i(f)(\alpha))\,,$$

which shows that k is surjective. It follows that k is an isomorphism of transitive structures and thus trivial, so we can conclude that $j = i$. □

3.3. **The proof.** We are now ready to prove the second main theorem. We restate it here for convenience.

Theorem 34. *If κ is $(\kappa + 2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa, \kappa^+)$ holds, $2^\kappa = \kappa^{++}$, and κ is the least measurable.*

This theorem shows that the hypothesis in Proposition 10 is in some sense optimal: if $2^\kappa > \kappa^+$ then $\mathrm{LCP}(\kappa, \kappa^+)$ is not enough to conclude that the Mitchell rank of κ is large. In fact, even $\mathrm{CP}(\kappa, \kappa^+)$ can hold at the least measurable cardinal.

Proof. We make some simplifying assumptions to start with. We may assume that GCH holds and that the $(\kappa + 2)$-strongness of κ is witnessed by a (κ, κ^{++})-extender embedding $j\colon V \to M$. We have the usual diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i is the induced normal ultrapower map. Using the GCH and Lemma 6, we can see that the critical point of k is $(\kappa^{++})^N$. Using the argument from [4], we may also assume that, in V, there is an $i(\mathrm{Add}(\kappa, \kappa^{++}))$-generic filter over N.

The following observation will be important, and we include the straightforward proof.

Claim. *The map k is a $((\kappa^{++})^N, \kappa^{++})$-extender embedding. That is,*

$$M = \{k(g)(\alpha); \alpha < \kappa^{++}, \mathrm{dom}(g) = (\kappa^{++})^N\}\,.$$

Proof. We assumed that we could write M in the form

$$M = \{j(f)(\alpha); \alpha < \kappa^{++}, \mathrm{dom}(f) = \kappa\}\,.$$

Now take an arbitrary element $j(f)(\alpha)$ of M. We can rewrite it as $(k(i(f))(\alpha))$. If we now take $g = i(f) \restriction (\kappa^{++})^N$, it is not hard to see that $j(f)(\alpha) = k(g)(\alpha)$, showing inclusion in one direction.

For the other direction, take an element of the form $k(g)(\alpha)$. The function g itself is of the form $i(F)(\kappa)$ for some function F with domain κ, since N is the ultrapower of V by a normal measure on κ. This means we can write $k(g)(\alpha) = k(i(F)(\kappa))(\alpha) = (j(F)(\kappa))(\alpha)$, since the critical point of k is above κ. From this point, it is not hard to find a function $\tilde{F}$ and an ordinal $\beta < \kappa^{++}$ such that $(j(F)(\kappa))(\alpha) = j(\tilde{F})(\beta)$. ■

We now specify the forcing we will use. Let $\mathbb{P}_\kappa$ be the Easton support iteration of length κ which forces at inaccessible $\gamma < \kappa$ with $\mathbb{S}_{\gamma^{++}} * \mathbb{A}(\gamma, \gamma^{++}, \vec{X})$, where $\vec{X}$ is some $\clubsuit_{\gamma^{++}}(S)$-sequence added by $\mathbb{S}_{\gamma^{++}}$.[4] Let G_κ be $\mathbb{P}_\kappa$-generic over V. We shall try to lift the embeddings i and j through this forcing.

We can factor $j(\mathbb{P}_\kappa)$ as

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_{\kappa^{++}} * \mathbb{A}(\kappa, \kappa^{++}, \vec{Y}) * \mathbb{P}_{\text{tail}}\,,$$

where $\vec{Y}$ is the $\clubsuit_{\kappa^{++}}$-sequence used by the forcing at stage κ in $M[G_\kappa]$ and $\mathbb{P}_{\text{tail}}$ is the remainder of the forcing between κ and $j(\kappa)$. Similarly, we can rewrite $i(\mathbb{P}_\kappa)$ as

$$i(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * (\mathbb{S}_{\kappa^{++}} * \mathbb{A}(\kappa, \kappa^{++}, \vec{Y}'))^{N^{\mathbb{P}_\kappa}} * \mathbb{P}'_{\text{tail}}\,,$$

where $\vec{Y}'$ and $\mathbb{P}'_{\text{tail}}$ are defined analogously. Since G_κ is generic over all of V, it is definitely generic over N and M. The forcing $\mathbb{P}_\kappa$ is below the critical point of the embedding k, so we can easily lift it to $k\colon N[G_\kappa] \to M[G_\kappa]$. Moreover, since $\mathbb{P}_\kappa$ is κ-cc, $N[G_\kappa]$ will be closed under κ-sequences in $V[G_\kappa]$.

We now claim that, in $V[G_\kappa]$, there is an $(\mathbb{S}_{\kappa^{++}})^{N[G_\kappa]}$-generic over $N[G_\kappa]$, and moreover that this generic amounts to a nonstationary subset of $(\kappa^{++})^N$ (which is an ordinal of cofinality κ^+ in V) in $V[G_\kappa]$. This follows from Lemma 16, which tells us that the iteration $\mathbb{S}_{\kappa^{++}} * \mathbb{C}(\dot{S})$ is equivalent to $\mathrm{Add}(\kappa^{++}, 1)$. Since $V[G_\kappa]$ has an $\mathrm{Add}(\kappa^{++}, 1)^{N[G_\kappa]}$-generic over $N[G_\kappa]$ (as this forcing is $\leq \kappa$-closed in $V[G_\kappa]$ and only has κ^+ many dense subsets from $N[G_\kappa]$), we can also extract the generic for $\mathbb{S}_{\kappa^{++}}^{N[G_\kappa]}$. Furthermore, this generic stationary set will be nonstationary in $V[G_\kappa]$, as witnessed by the generic club added by $\mathbb{C}(\dot{S})$.

So let $S' \in V[G_\kappa]$ be $(\mathbb{S}_{\kappa^{++}})^{N[G_\kappa]}$-generic over $N[G_\kappa]$. This means that S' is, in $N[G_\kappa][S']$, a nonreflecting stationary subset of $(\kappa^{++})^{N[G_\kappa]}$. In particular, none of its proper initial segments are stationary in their supremum. This statement is upwards absolute, so $V[G_\kappa] \supseteq N[G_\kappa][S']$ agrees about the nonstationarity of the initial segments of S'. But more than this, S' itself is nonstationary in its supremum $(\kappa^{++})^N$, as we noted in the previous paragraph. Finally, observe that $(\kappa^{++})^N < \kappa^{++}$; this is because $i(\kappa) > (\kappa^{++})^N$ and $i(\kappa)$ has size $2^\kappa = \kappa^+$ in V, since i is the ultrapower by a normal measure on κ. Together, these facts imply that S' is a condition in the real $\mathbb{S}_{\kappa^{++}}$. Let S be some $\mathbb{S}_{\kappa^{++}}$-generic over $V[G_\kappa]$ containing S'. The embedding k lifts

[4] It does not matter much how we pick these $\clubsuit$-sequences. One possible way is to fix in advance a wellordering of some large H_θ and always pick the least appropriate name.

again to $k\colon N[G_\kappa][S'] \to M[G_\kappa][S]$; this is because the critical point of k is $(\kappa^{++})^N$, which means that $k[S'] = S' \subseteq S$ by the choice of S.

Now consider the $\clubsuit_{(\kappa^{++})^N}$-sequence $\vec{Y}'$ used by $i(\mathbb{P}_\kappa)$ at stage κ. Since the critical point of k is $(\kappa^{++})^N$, the sequence $\vec{Y}'$ is simply an initial segment of the sequence $\vec{Y} = k(\vec{Y}')$ used by $j(\mathbb{P}_\kappa)$ at stage κ.[5] It follows that, if we look at the forcing $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$ in $V[G_\kappa][S]$, we can write it as a product

$$\mathbb{A}(\kappa, \kappa^{++}, \vec{Y}) \cong \mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}') \times \mathbb{A}(\kappa, \kappa^{++} \setminus (\kappa^{++})^N, \vec{Y})\,. \tag{1}$$

There is a slight abuse of notation in the second factor: the set $\kappa^{++} \setminus (\kappa^{++})^N$ is not an ordinal, and the ladder system $\vec{Y}$ is defined on a superset of it. Nevertheless, we trust that our meaning is clear. Observe also that, since $\mathbb{S}_{\kappa^{++}}$ does not add bounded subsets to κ^{++}, we know

$$\mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')^{V[G_\kappa][S]} = \mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')^{V[G_\kappa]} = \mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')^{N[G_\kappa][S']}\,.$$

Let g' be $\mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')$-generic over $V[G_\kappa][S]$; in particular, it is also generic over $N[G_\kappa][S']$. Since g' is generic for a forcing that is κ^+-cc in $V[G_\kappa]$, it follows that $N[G_\kappa][S'][g']$ is still closed under κ-sequences in $V[G_\kappa][g']$. This, together with the fact that $\mathbb{P}'_{\mathrm{tail}}$ is $\leq \kappa$-strategically closed in $V[G_\kappa][g']$ and has only κ^+ many dense subsets from this model, allows us to build, using Fact 28 in $V[G_\kappa][g']$, a $\mathbb{P}'_{\mathrm{tail}}$-generic G'_{tail} over $N[G_\kappa][S'][g']$ and lift the embedding i to

$$i\colon V[G_\kappa] \to N[G_\kappa][S'][g'][G'_{\mathrm{tail}}]\,.$$

We can now force over $V[G_\kappa][S]$, using the factorization (1), to complete g' to g which is fully $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$-generic over $V[G_\kappa][S]$. In the extension $V[G_\kappa][S][g]$ we can finally also lift the map k through the last two stages of forcing and obtain

$$k\colon N[G_\kappa][S'][g'][G'_{\mathrm{tail}}] \to M[G_\kappa][S][g][G_{\mathrm{tail}}]\,,$$

where G_{tail} is the filter generated by the pointwise image of G'_{tail}. The lift through g' is straightforward: the critical point of k is $(\kappa^{++})^N$, so $k[g'] = g' \subseteq g$. On the other hand, the forcing $\mathbb{P}'_{\mathrm{tail}}$ is at least $\leq (\kappa^{++})^N$-strategically closed in $N[G_\kappa][S'][g']$, so Fact 27 together with the knowledge that k is a $((\kappa^{++})^N, \kappa^{++})$-extender embedding shows that the pointwise image of G'_{tail} really does generate a generic filter.

Composing the two lifts of i and k gives us a lift of j. The situation is summarized in the following diagram; we should keep in mind that the pictured embeddings exist in $V[G_\kappa][S][g]$.

$$\begin{tikzcd}
V[G_\kappa] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_\kappa][S][g][G_{\mathrm{tail}}] \\
& N[G_\kappa][S'][g'][G'_{\mathrm{tail}}] \arrow[ur, "k"'] &
\end{tikzcd}$$

[5] We could have arranged matters so that $\vec{Y}$ was also a $\clubsuit_{\kappa^{++}}(S)$-sequence in $V[G_\kappa][S]$, but this will not be important for the argument.

As the final act of forcing, let C be $\mathbb{C}(S)^{V[G_\kappa][S]}$-generic over $V[G_\kappa][S][g]$. We claim that $V[G_\kappa][S][g \times C]$ is our desired final extension. Recall that Lemma 23 tells us that we can also write this extension as $V[G_\kappa][H^0 \times H^2]$ for some generic $H^0 \subseteq \mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$ and $H^2 \subseteq \mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$. We will work from now on in this final model, using this alternative representation, and try to lift the embedding j.

By Lemma 30 we know that $\mathrm{Term}(\mathbb{P}_\kappa, \mathrm{Add}(\kappa, \kappa^{++}))$ is forcing equivalent to $\mathrm{Add}(\kappa, \kappa^{++})$ in V. It follows from this by elementarity that the poset $\mathrm{Term}(i(\mathbb{P}_\kappa), i(\mathrm{Add}(\kappa, \kappa^{++})))$ is equivalent to $i(\mathrm{Add}(\kappa, \kappa^{++}))$ in N. Now we return to an assumption we made at the start of the proof. Since V has an $i(\mathrm{Add}(\kappa, \kappa^{++}))$-generic over N, we can use this equivalence to also find a $\mathrm{Term}(i(\mathbb{P}_\kappa), i(\mathrm{Add}(\kappa, \kappa^{++})))$-generic over N. Recalling the key property of term forcing stated just before Fact 30, we can combine this term forcing generic with the $i(\mathbb{P}_\kappa)$-generic $G_\kappa * S' * g' * G'_{\mathrm{tail}}$ to extract an $i(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$-generic K' over $N[G_\kappa][S'][g'][G'_{\mathrm{tail}}]$ in $V[G_\kappa][g']$. Since the forcing $i(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$ is $< i(\kappa)$-distributive in $N[G_\kappa][S'][g'][G'_{\mathrm{tail}}]$, Fact 27 again tells us that the pointwise image $k[K']$ generates a $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$-generic filter $\tilde{K}^0$ over $M[G_\kappa][S][g][G_{\mathrm{tail}}]$. It is not necessarily the case that $j[H^0] \subseteq \tilde{K}^0$, but we can surgically[6] alter $\tilde{K}^0$ to obtain another $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$-generic K^0 over $M[G_\kappa][S][g][G_{\mathrm{tail}}]$ for which this will be the case, and we are able to lift j to

$$j \colon V[G_\kappa][H^0] \to M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0]\,.$$

We can now forget about the maps i and k and focus solely on j. To complete the lift, observe that $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$ remains $\leq \kappa^+$-distributive in $V[G_\kappa][H^0]$ by Easton's lemma, and so Fact 27 implies that the filter $j[H^2]$ generates a generic K^2 over $M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0]$, which gives us our final lift

$$j \colon V[G_\kappa][H^0 \times H^2] \to M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0 \times K^2]\,.$$

Since j was originally a (κ, κ^{++})-extender embedding, the same remains true for the lifted embedding, by Fact 26. Since we clearly have $2^\kappa = \kappa^{++}$ in the final model, Lemma 33 tells us that the lift j is the ultrapower by a normal measure.

Claim. *The embedding j witnesses $\mathrm{CP}(\kappa, \kappa^+)$ in $V[G_\kappa][H^0][H^2]$.*

Proof. Let us write $M^* = M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0][K^2]$. We need to show that every subset x of κ^+ in $V[G_\kappa][S][C][g]$ appears in M^*. To that end, we will first show that x is already in $V[G_\kappa][S][g]$. This follows from Lemma 32: the forcing to add C is $< \kappa^{++}$-distributive in $V[G_\kappa][S]$, and $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})^{V[G_\kappa][S]}$ is κ^+-cc in $V[G_\kappa][S][C]$, since it is equivalent to $\mathrm{Add}(\kappa, \kappa^{++})$ in that model, as we explained in the proof of Lemma 23. Lemma 32 then implies that forcing to add C to $V[G_\kappa][S][g]$ could not have added x, and so x is already in that model.

We next show that x has a name in $M[G_\kappa]$. To start with, let $\sigma \in V[G_\kappa][S]$ be a nice $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$-name for x. Observe that $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$ is actually a

[6]See [5, Theorem 25.1] or [4, Theorem 1, Second step] for fairly detailed examples of this concrete use of the surgery method

subset of $H^{V[G_\kappa]}_{\kappa^{++}}$ (even though it is not an element of $V[G_\kappa]$), so the name σ is as well. Moreover, since $\mathbb{A}(\kappa,\kappa^{++},\vec{Y})$ is κ^+-cc, σ has size κ^+. But as a κ^+-sized subset of $V[G_\kappa]$, the name σ could not have been added by the $\leq\kappa^+$-distributive forcing to add S, and we conclude that $\sigma\in H^{V[G_\kappa]}_{\kappa^{++}}$. Now, since $\mathbb{P}_\kappa$ is κ-cc and $\mathcal{P}(\mathcal{P}(\kappa))\in M$, we know that $H^{M[G_\kappa]}_{\kappa^{++}}=H^{V[G_\kappa]}_{\kappa^{++}}$, so the name σ also appears in $M[G_\kappa]$.

It follows that we can interpret the name σ by the generic filter g in $M[G_\kappa][S][g]$ to find the set x in that model. Finally, we can conclude that $M[G_\kappa][S][g]$ contains all the subsets of κ^+ from $V[G_\kappa][S][C][g]$, and so $M^*\supseteq M[G_\kappa][S][g]$ does as well. ■

We have shown that $\mathrm{CP}(\kappa,\kappa^+)$ holds in $V[G_\kappa][H^0\times H^2]$. To finish the proof we also need to see that κ is the least measurable cardinal in that model. This follows easily from the way we designed the forcing $\mathbb{P}_\kappa$. If $\gamma<\kappa$ were measurable in $V[G_\kappa][H^0\times H^2]$, it must definitely be inaccessible in V. It follows that we did some nontrivial forcing at stage γ in the iteration $\mathbb{P}_\kappa$ and Lemma 21 implies that after the stage γ forcing γ is not measurable. The remaining forcing to get from that model to the model $V[G_\kappa][H^0\times H^2]$ is at least $\leq 2^{2^\gamma}$-strategically closed, which means that it could not have possibly added any measures on γ. We can therefore conclude that γ remains nonmeasurable in $V[G_\kappa][H^0\times H^2]$. □

The iteration we used is essentially the one described in [1, Section 4]. It follows from the results outlined there that, had we additionally assumed in Theorem 34 that κ were κ^+-supercompact, this would remain true in the resulting extension.

Corollary 35. *If GCH holds and κ is κ^+-supercompact, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds, and κ is κ^+-supercompact and the least measurable.*

By starting with a stronger large cardinal hypothesis and modifying the forcing iteration appropriately, we can push up the value of 2^κ beyond just κ^{++} and capture even more powersets. In order to state the results as simply as possible, we make the following definition to add some convenient stages to the hierarchy of strong cardinals.

Definition 36. If X is a set, a cardinal κ is called *X-strong* if there is an elementary embedding $j\colon V\to M$ with critical point κ and M a transitive inner model with $X\in M$.

Theorem 37. *Assume GCH holds and suppose that κ is H_λ-strong for some regular cardinal $\lambda\geq\kappa^{++}$. Then there is a forcing extension in which κ is the least measurable cardinal, $2^\kappa=\lambda$, and $\mathrm{CP}(\kappa,<\lambda)$ holds (meaning that a single normal measure on κ captures every $\mathcal{P}(\mu)$ for $\mu<\lambda$).*

Proof. The argument is much like the proof of Theorem 34, with a handful of changes: we will modify the forcing used slightly, and, more importantly, instead of preparing the model as in [4], we use a result of the second author from [12] and pass to a forcing extension in order to be able to assume that the following hold in V:

(1) $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \lambda$.
(2) κ is H_λ-strong and this is witnessed by a (κ,λ)-extender embedding $j\colon V \to M$; moreover, M is closed under κ-sequences.
(3) There is a function $\ell\colon \kappa \to \kappa$ such that $j(\ell)(\kappa) = \lambda$.
(4) There is in V an M-generic filter for the poset $j(\mathrm{Add}(\kappa,\lambda))$.

Since λ is regular, we may even assume that $\ell(\gamma)$ is regular whenever γ is an inaccessible cardinal. The initial iteration $\mathbb{P}_\kappa$ will now be an Easton-support iteration which forces at inaccessible cardinals $\gamma < \kappa$ with the forcing $\mathbb{S}_{\ell(\gamma)} * \mathbb{A}(\gamma, \ell(\gamma), \vec{X})$, with $\vec{X}$ being an appropriate $\clubsuit$-sequence, *provided that γ is inaccessible in $V^{\mathbb{P}_\gamma}$*.

Note that, since $\mathbb{P}_\kappa$ is κ-cc, there will be nontrivial forcing at stage κ of the iteration $j(\mathbb{P}_\kappa)$ and we can write

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_\lambda * \mathbb{A}(\kappa,\lambda,\vec{Y}) * \mathbb{P}_{\text{tail}}\,.$$

The full forcing that will give us the theorem is then

$$\mathbb{P} = \mathbb{P}_\kappa * \mathbb{S}_\lambda * (\mathbb{A}(\kappa,\lambda,\vec{Y}) \times \mathbb{C}(\dot{S}))\,.$$

The argument now proceeds very much like the proof of Theorem 34, but with some simplifications due to the difference between the preparations from [3] and [12]. We sketch the argument here, referring back to the previously given proof and noting the main differences.

Let $G_\kappa * S * g$ be $\mathbb{P}_\kappa * \mathbb{S}_\lambda * \mathbb{A}(\kappa,\lambda,\vec{Y})$-generic over V. We wish to lift the embedding $j\colon V \to M$ to the extension $V[G_\kappa]$ in the model $V[G_\kappa][S][g]$. Given the factorization of $j(\mathbb{P}_\kappa)$ above, we need to find a $\mathbb{P}_{\text{tail}}$-generic over $M[G_\kappa][S][g]$. Previously we worked with the embeddings i and k, but now we will be able to do without.[7]

Consider any dense open subset D of $\mathbb{P}_{\text{tail}}$ in $M[G_\kappa][S][g]$. Since j was a (κ,λ)-extender embedding, this D has the form $D = j(f)(\alpha)^{G_\kappa * S * g}$ for some $f\colon \kappa \to V_\kappa$ and some $\alpha < \lambda$. For each such f, let

$$\mathcal{D}_f \subseteq \{j(f)(\alpha)^{G_\kappa * S * g}; \alpha < \lambda\} \in M[G_\kappa][S][g]$$

be the set of dense open subsets of $\mathbb{P}_{\text{tail}}$ that can be written in the indicated form. This set has size (at most) λ in $M[G_\kappa][S][g]$. Since the first stage of forcing in $\mathbb{P}_{\text{tail}}$ occurs beyond λ, the forcing $\mathbb{P}_{\text{tail}}$ is $\leq \lambda$-strategically closed in $M[G_\kappa][S][g]$. This means that $\bigcap \mathcal{D}_f \in M[G_\kappa][S][g]$ is a dense subset of $\mathbb{P}_{\text{tail}}$.

Finally, observe that there are $2^\kappa = \kappa^+$ many functions f (counted in V), and therefore only κ^+ many dense sets $\bigcap \mathcal{D}_f$. Since the forcing $\mathbb{P}_\kappa * \mathbb{S}_\lambda * \mathbb{A}(\kappa,\lambda,\vec{Y})$ is composed of a κ^+-cc part, a $<\lambda$-distributive part, and another κ^+-cc part, applying Fact 29 twice allows us to conclude that $M[G_\kappa][S][g]$ is closed under κ-sequences in $V[G_\kappa][S][g]$. It follows that $\mathbb{P}_{\text{tail}}$ remains $\prec \kappa^+$-strategically closed in $V[G_\kappa][S][g]$, which will allow us to line up and meet all the dense sets $\bigcap \mathcal{D}_f$ in turn, and so build a generic G_{tail} for $\mathbb{P}_{\text{tail}}$. This allows us to lift the embedding j to

$$j\colon V[G_\kappa] \to M[G_\kappa][S][g][G_{\text{tail}}]$$

[7] In fact, we could have employed the methods of [12] even in the previous theorem, but we decided to give more details for the specific case $2^\kappa = \kappa^{++}$. The proof we gave will also serve as a template for our proof of Theorem 40.

in $V[G_\kappa][S][g]$.

For the final step of the lift, we use Lemma 23 to see $\mathbb{P}$ as the iteration $\mathbb{P}_\kappa * (\mathrm{Add}(\kappa,\lambda)\times\mathrm{Add}(\lambda,1))$. The lift through the forcing $\mathrm{Add}(\kappa,\lambda)$ proceeds as in the proof of Theorem 34, except that we deal directly with the embedding j instead of passing through i and k as before. We apply Lemma 30 to $j(\mathrm{Term}(\mathbb{P}_\kappa,\mathrm{Add}(\kappa,\lambda)))$ in M and use our starting assumption that we have an M-generic for that poset in V; a surgery argument like the one we alluded to before allows us to build a suitable $j(\mathrm{Add}(\kappa,\lambda))$-generic K^0 over $M[G_\kappa][S][g][G_{\mathrm{tail}}]$ and lift j to

$$j\colon V[G][H^0]\to M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0]\,.$$

The lift through the final forcing $\mathrm{Add}(\lambda,1)^{V[G_\kappa]}$ is handled exactly as in the proof of Theorem 34. The proof that this lifted embedding witnesses $\mathrm{CP}(\kappa,\lambda)$ in the final model and that κ is the least measurable there mirror our previous arguments as well. $\square$

Conversely, we can extend Cummings' argument to show that the large cardinal hypothesis we used above is optimal.

Theorem 38. *Suppose that $\mathrm{CP}(\kappa,<\lambda)$ holds for some regular cardinal $\lambda\geq\kappa^{++}$. Then κ is H_λ-strong in an inner model. Moreover, this inner model satisfies GCH, and so κ is $(\kappa+\alpha)$-strong there, where $\lambda=\kappa^{+\alpha}$.*

Proof. This is essentially standard. Suppose that $j\colon V\to M$ is an ultrapower embedding by a normal measure witnessing $\mathrm{CP}(\kappa,<\lambda)$; it follows that $H_\lambda\in M$.[8] We assume that there is no inner model with a strong cardinal and let K be the core model with the (nonoverlapping) extender sequence $\vec{E}$. It follows that $j\restriction K$ is the result of a normal iteration of $\vec{E}$ and, since the critical point of j is κ, the first extender applied in this iteration must have index (κ,η) for some η. Since $\vec{E}$ is coherent, the sequence $j(\vec{E})$ has no extenders with indices (κ,β) for $\beta\geq\eta$. But since M captured all of H_λ, we must have $K\restriction\lambda=K^M\restriction\lambda$, and so $\vec{E}$ and $j(\vec{E})$ must agree up to λ. It follows that $\eta\geq\lambda$ and so $o(\kappa)\geq\lambda+1$ (and κ is H_λ-strong) in K.

Since K satisfies GCH, $V^K_{\kappa+\alpha}$ is a transitive set of size $\kappa^{+\alpha}=\lambda$ there. It follows that the transitive closure of each element of $V^K_{\kappa+\alpha}$ has size strictly less than λ, so these elements appear in the codomain of the embedding witnessing the H_λ-strongness of κ in K. $\square$

The preparation from [12] works even for singular λ of cofinality strictly above κ (if the cofinality of λ is equal to κ^+, we get $2^{\kappa^+}=\lambda^+$ in (1) above). It is unclear, however, whether Theorem 37 can allow for this weaker hypothesis (in particular, Lemma 23 seems to rely crucially on the second parameter in the Apter–Shelah forcing being regular).

Question 39. *Can Theorem 37 be improved to allow for arbitrary λ of cofinality strictly above κ?*

[8]Recall that $\mathrm{CP}(\kappa,<\lambda)$ implies $2^\kappa\geq\lambda$, so H_λ being in M is weaker than $V_{\kappa+\alpha}$ being in M, where $\lambda=\kappa^{+\alpha}$. In particular, κ might not be $(\kappa+\alpha)$-strong in V.

Another question raised by Theorem 37 is whether $\mathrm{CP}(\kappa, \lambda)$ can fail for the first time at some $\kappa^+ < \lambda < 2^\kappa$. The following theorem shows that the answer is yes.

Theorem 40. *Suppose GCH holds and κ is $H_{\kappa^{+3}}$-strong. Then there is a forcing extension in which κ is the least measurable, $2^\kappa = \kappa^{+3}$, and* $\mathrm{CP}(\kappa, \kappa^+)$ *holds, while* $\mathrm{LCP}(\kappa, \kappa^{++})$ *fails.*

One would expect that it should be possible to force $2^\kappa = \kappa^{+3}$ and $\mathrm{CP}(\kappa, \kappa^+)$ starting from a large cardinal hypothesis weaker than an $H_{\kappa^{+3}}$-strong cardinal κ; an $H_{\kappa^{+2}}$-strong and κ^{+3}-tall cardinal κ likely suffices (recall that κ is λ-*tall* if there is an elementary embedding $j \colon V \to M$ with critical point κ such that M is closed under κ-sequences and $j(\kappa) > \lambda$; see [11]). However, the proof that we are about to give seems to require a stronger hypothesis in order to deduce a connection between the forcings at stage κ over V and over the target model (in particular, the forcings to add a nonreflecting stationary subset to κ^{+3} should look sufficiently similar). It is nevertheless plausible, if unclear, that the required constellation of properties can be forced using a weaker hypothesis.

The proof of this theorem is similar to the proof of Theorem 34, but with some further technical complications. The idea should, nevertheless, be clear: we will attempt to force as in the proof of that theorem, but at the end also adding, in a product manner, a new subset to κ^{++}. We will be able to show that this new subset cannot be captured by any normal ultrapower on κ. For this strategy to work out, we will need to modify the preparatory forcing in order to accommodate this additional forcing at stage κ (and also to obtain $2^\kappa = \kappa^{+3}$ at the end); in fact, we will fold the preparation of Theorem 34 into the main forcing itself.

Proof. As always, let $j \colon V \to M$ be a (κ, κ^{+3})-extender embedding witnessing $H_{\kappa^{+3}}$-strongness, where M is closed under κ-sequences. We draw the usual diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i is the induced normal ultrapower. Note that the critical point of k is $(\kappa^{++})^N$ and that $(\kappa^{+3})^N$ is an ordinal of size κ^+ in V. Let us write $\nu = (\kappa^{+3})^N$.

The forcing iteration we will build will be quite similar to the one used in Theorem 34, with the addition of a Cohen forcing factor at κ^+ that will provide a kind of indestructibility (in Theorem 34, this role was played by the $i(\mathrm{Add}(\kappa, \kappa^{++}))$-generic over N). This insertion of additional forcing will not do significant damage to our arguments for that previous theorem, but some care will still be needed. The second point of departure from our previous arguments is the mismatch between the critical point of k and ν, the ordinal of interest (in our previous argument, we were only pushing 2^κ up to κ^{++} and the critical point of k, being $(\kappa^{++})^N$ matched that). This will complicate some of our arguments, since k will now nontrivially move some of the important objects (that is, subsets of ν) we will be working with.

Let $\mathbb{P}_\kappa$ be an Easton-support iteration which forces at inaccessible cardinals $\gamma < \kappa$ with the forcing $\mathbb{S}_{\gamma^{+3}} * (\mathbb{A}(\gamma, \gamma^{+3}, \vec{X}) \times \mathrm{Add}(\gamma^+, \gamma^{+3}))$. The images of this forcing via j and i can be written as

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_{\kappa^{+3}} * (\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y}) \times \mathrm{Add}(\kappa^+, \kappa^{+3})) * \mathbb{P}_{\mathrm{tail}}$$

and

$$i(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \left(\mathbb{S}_{\kappa^{+3}} * (\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y}') \times \mathrm{Add}(\kappa^+, \kappa^{+3}))\right)^{N^{\mathbb{P}_\kappa}} * \mathbb{P}'_{\mathrm{tail}}\,.$$

Let G_κ be $\mathbb{P}_\kappa$-generic over V. We proceed similarly to the proof of Theorem 34. In $V[G_\kappa]$ there is an unbounded $S' \subseteq \nu$ which is $(\mathbb{S}_{\kappa^{+3}})^{N[G_\kappa]}$-generic over $N[G_\kappa]$. Moreover, S' is nonstationary in ν in $V[G_\kappa]$ for the same reasons as in the proof of Theorem 34. Together, this means that, in $V[G_\kappa]$, the set $S' \subseteq \nu$ is nonstationary, and all of its initial segments are nonstationary in their suprema as well.

We wish to see that the pointwise image $k[S']$ has the same property: it itself and all of its initial segments are nonstationary in their suprema. For suppose that for some ordinal γ the initial segment $k[S'] \cap \gamma$ is stationary in γ. The inverse of k, defined on its range, is an injective map satisfying $k^{-1}(\alpha) \leq \alpha$ for each α. If there were a stationary subset of $k[S']$ on which the above inequality were strict, then Fodor's lemma would imply that k^{-1} must be constant on a further stationary subset, which contradicts its injectivity. On the other hand, if $k^{-1}(\alpha) = \alpha$ for a stationary subset T of $k[S'] \cap \gamma$, then $T \subseteq S'$ is stationary in γ, contradicting the fact that S' has no stationary initial segments.

Since $k[\nu]$ is bounded in κ^{+3}, as ν has size only κ^+, the set $k[S']$ is a condition in $\mathbb{S}_{\kappa^{+3}}$. Let S be a generic for this forcing over $V[G_\kappa]$ such that $k[S']$ is an initial segment of S.

Let $g' \times h'$ be generic for $\mathbb{A}(\kappa, \nu, \vec{Y}') \times \mathrm{Add}(\kappa^+, \nu)$ over $V[G_\kappa][S]$. Observe that $2^\kappa = \kappa^+$ still holds in $V[G_\kappa][S][g' \times h']$ and the model $N[G_\kappa][S'][g' \times h']$ remains closed under κ-sequences in $V[G_\kappa][S'][g' \times h']$, since we obtained this extension by a combination of κ^+-cc forcings, in which case Fact 29 applies, and of $\leq \kappa$-distributive forcings. It follows that we may build a $\mathbb{P}'_{\mathrm{tail}}$-generic G'_{tail} over $N[G_\kappa][S'][g' \times h']$ in $V[G_\kappa][S][g' \times h']$. This already allows us to partially lift the embedding i to

$$i\colon V[G_\kappa] \to N[G_\kappa][S'][g' \times h'][G'_{\mathrm{tail}}]$$

in the model $V[G_\kappa][S][g' \times h']$.

Before we continue lifting the entire diagram, we make a small digression which will be useful later in the argument.

Claim. *The embedding i can be lifted in the model $V[G_\kappa][S][g' \times h']$ to an embedding i^+ defined on this whole model.*

Proof. For the duration of this proof let us write $V^+ = V[G_\kappa]$ and $N^+ = N[G_\kappa][S'][g' \times h'][G'_{\mathrm{tail}}]$. Until now we have built an embedding $i\colon V^+ \to N^+$ in $V^+[S][g' \times h']$. We recall that $2^\kappa = \kappa^+$ holds in $V^+[S][g' \times h']$ and that N^+ is closed under κ-sequences in this model. Since the forcing to add S is $< \kappa^{+3}$-distributive, we can apply Fact 27 to transfer the generic S along i and get a further lift $i\colon V^+[S] \to N^+[i(S)]$. Furthermore, $N^+[i(S)]$ remains closed under κ-sequences in $V^+[S][g' \times h']$.

Consider now the forcing $\mathbb{A}(\kappa, \nu, \vec{Y}')$ that added g'. Its image $i(\mathbb{A}(\kappa, \nu, \vec{Y}'))$ is $i(\kappa^+)$-cc, has size $i(\kappa^+)$ (since ν had size κ^+ in V), and is $< i(\kappa)$-closed in $N^+[i(S)]$.

We pause here to give a brief calculation. Recall that $\nu = (\kappa^{+3})^N$ and that N was closed under κ-sequences in V. We can conclude from this that the cofinality of ν in V is κ^+ (equal to its cardinality). The same is true in $V^+[S]$, since the forcing $\mathbb{P}_\kappa$ was κ-cc and the forcing to add S was $< \kappa^{+3}$-distributive. It follows by elementarity that the cofinality of $i(\nu)$ in $N^+[i(S)]$ is $i(\kappa^+)$.

Let us return to the forcing $i(\mathbb{A}(\kappa, \nu, \vec{Y}'))$. Using the fact that $2^\kappa = \kappa^+$ in $V^+[S][g' \times h']$, we can enumerate all of the maximal antichains of this poset from $N^+[i(S)]$ as A_α for $\alpha < \kappa^+$. Moreover, each of these antichains only contains $i(\kappa)$ many conditions, each of which in turn has support of size less than $i(\kappa)$. It follows that the support of each maximal antichain A_α has size at most $i(\kappa)$ and, by the cofinality calculation in the previous paragraph, this support must therefore be bounded in $i(\nu)$.

The embedding i arose as a normal ultrapower of V by a measure on κ. It follows that i, and all of its lifts, are continuous at ν (and all ordinals of cofinality not equal to κ), which means that $i(\nu) = \sup_{\alpha<\nu} i(\alpha)$. This observation lets us conclude that, since the support of each antichain A_α is bounded in $i(\nu)$, it must be bounded by some ordinal $i(\beta_\alpha)$ for $\beta_\alpha < \nu$. In other words, A_α is contained in $i(\mathbb{A}(\kappa, \beta_\alpha, \vec{Y}'))$.[9] By increasing the bound as necessary, we may even assume that the β_α form an increasing sequence converging to ν. We shall use these maximal antichains to build an appropriate generic object over $N^+[i(S)]$. The argument we use is similar, if simpler, to the one used in the construction of the generic $H^{6,0}_{\alpha_0}$ in [2, Lemma 9], although the general strategy goes back to Magidor [13].

As a notational device, if $(p, Z) \in \mathbb{A}(\kappa, \nu, \vec{Y}')$ is a condition and $\beta < \nu$ is an ordinal, we will consider the restriction $(p, Z) \restriction \beta = (p \restriction (\beta \times \kappa), Z \cap \mathcal{P}(\beta))$. Similarly, if S is a set of conditions, we write $S \restriction \beta$ for the set of restrictions of the conditions in S. It should be clear that $(p, Z) \restriction \beta$ is a condition in $\mathbb{A}(\kappa, \beta, \vec{Y}')$.

Let us start with (p_0, Z_0) being the trivial condition. As $N^+[i(S)]$ is closed under κ-sequences, we have $i[g' \restriction \beta_0] \in N^+[i(S)]$. It follows from Lemma 24 that $\bigcup i[g' \restriction \beta_0]$ is actually a condition in $i(\mathbb{A}(\kappa, \beta_0, \vec{Y}'))$. Let $(q_0, W_0) \le (p_0, Z_0)$ be the union of this condition with (p_0, Z_0) and then let $(p_1, Z_1) \le (q_0, W_0)$ be some condition in $i(\mathbb{A}(\kappa, \beta_0, \vec{Y}'))$ deciding the maximal antichain A_0.

The next step works much the same: $(r, U) = \bigcup i[g' \restriction \beta_1]$ is a condition in $i(\mathbb{A}(\kappa, \beta_1, \vec{Y}'))$. It is easy to see that the Cohen conditions r and p_1 are compatible; this is because r matches q_0 up to $i(\beta_0)$ and p_1 is an extension of that restriction. It follows by an argument as in the proof of Lemma 20, that the full conditions (r, U) and (p_1, Z_1) are compatible; let (q_1, W_1) be a common lower bound in $i(\mathbb{A}(\kappa, \beta_1, \vec{Y}'))$. To finish the step, let (p_2, Z_2) be an extension of (q_1, W_1) in this poset that decides the maximal antichain A_1.

[9] There is a small abuse of notation here, since $\vec{Y}'$ is not a ladder system on β_α but rather on ν; we trust that this will not cause much confusion.

We can continue in this way, building a descending sequence of conditions (p_α, Z_α) for κ^+ many steps. The key point is that we are meeting more and more of the maximal antichains A_α, while also ensuring that our conditions conform to $i[g]$. At limit steps γ we use the $\leq\kappa$-closure of $i(\mathbb{A}(\kappa,\kappa^+,\vec{Y}))$ in $V^+[S][g'\times h']$ to first find a lower bound of the sequence of conditions we've built so far, and then extend that lower bound as in the previous steps in order to meet the antichain A_γ.

Let g^+ be the filter generated by the descending sequence of conditions (p_α, Z_α). We ensured that g^+ is generic over $N^+[i(S)]$, and, since we fed information about $i[g']$ into the conditions during the construction, we get $i[g']\subseteq g^+$. It follows from Fact 25 that we may lift the embedding i to $i\colon V^+[S][g']\to N^+[i(S)][g^+]$.

To finish the proof, we simply observe that the forcing to add h' over $V^+[S][g']$ is $\leq\kappa$-distributive, which means that we can use Fact 27 to lift i again by simply transferring the generic h'. This final lift

$$i^+\colon V^+[S][g'][h']\to N^+[i(S)][g^+][i^+(h')]$$

is the lift we required. ■

Let us now complete the generics g' and h' to fully fledged generics. First, observe that, since $k[S']$ is an initial segment of S, the only ladders appearing in $\vec{Y}$ below $\sup k[\nu]$ are the pointwise images of the ladders in $\vec{Y}'$, and we may assume that none of the other ladders in $\vec{Y}$ have points below $\sup k[\nu]$. It follows that we may factor the forcing $\mathbb{A}(\kappa,\kappa^{+3},\vec{Y})$ as

$$\mathbb{A}(\kappa,\kappa^{+3},\vec{Y})=\mathbb{A}(\kappa,k[\nu],k[\vec{Y}'])\times\mathbb{A}(\kappa,\kappa^{+3}\setminus k[\nu],\vec{Y})\,.$$

This factorization is analogous to (1) in the proof of Theorem 34 and abuses notation in similar ways: neither $k[\nu]$ nor $\kappa^{+3}\setminus k[\nu]$ are ordinals, nor is $\vec{Y}$ defined exactly on $\kappa^{+3}\setminus k[\nu]$, but these details aren't significant to our arguments.

The restriction $k\restriction\nu$ is a ν-sequence of ordinals less than $k(\nu)=(\kappa^{+3})^M=\kappa^{+3}$. Since ν has size κ^+ in V, the map $k\restriction\nu$ appears in $H_{\kappa^{+3}}$, and therefore also in M. It follows that the poset $\mathbb{A}(\kappa,k[\nu],k[\vec{Y}'])$ appears in both $V[G_\kappa][S]$ and $M[G_\kappa][S]$. Moreover, this restriction of k induces an isomorphism between the posets $\mathbb{A}(\kappa,\nu,\vec{Y}')$ and $\mathbb{A}(\kappa,k[\nu],k[\vec{Y}'])$ by taking a condition (p,Z) to $(k(p),k(Z))$. Since g' was $\mathbb{A}(\kappa,\nu,\vec{Y}')$-generic over $V[G_\kappa][S]$, its isomorphic image $k[g']$ will be $\mathbb{A}(\kappa,k[\nu],k[\vec{Y}'])$-generic over the same model. Notice, though, that $k[g']\in V[G_\kappa][S][g']$.

If we now let g'' be $V[G_\kappa][S][g'\times h']$-generic for the second factor above, we get a $\mathbb{A}(\kappa,\kappa^{+3},\vec{Y})$-generic $g=k[g']\times g''$ over $V[G_\kappa][S]$.

In a similar fashion we can use the factorization

$$\mathrm{Add}(\kappa^+,\kappa^{+3})=\mathrm{Add}(\kappa^+,k[\nu])\times\mathrm{Add}(\kappa^+,\kappa^{+3}\setminus k[\nu])$$

and the isomorphism between $\mathrm{Add}(\kappa^+,\nu)$ and $\mathrm{Add}(\kappa^+,k[\nu])$ induced by k to force over $V[G_\kappa][S][g\times h']$ with $\mathrm{Add}(\kappa^+,\kappa^{+3}\setminus k[\nu])^{V[G_\kappa][S]}$, adding a generic h'', and completing $k[h']$ to an $\mathrm{Add}(\kappa^+,\kappa^{+3})^{V[G_\kappa][S]}$-generic $h=k[h']\times h''$ over $V[G_\kappa][S][g]$.

With all these generics in hand, and using k to transfer the generic G'_{tail} to a generic G_{tail} for $\mathbb{P}_{\mathrm{tail}}$ (note that the forcing $\mathbb{P}'_{\mathrm{tail}}$ is $\leq\nu$-distributive in $N[G_\kappa][S'][g' \times h']$ and Fact 27 applies), we can lift the entire diagram in $V[G_\kappa][S][g \times h]$.

$$\begin{tikzcd}
V[G_\kappa] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_\kappa][S][g \times h][G_{\text{tail}}] \\
& N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}] \arrow[ur, "k"'] &
\end{tikzcd}$$

Now force over the model $V[G_\kappa][S][g \times h]$ to add a $\mathbb{C}(S)^{V[G_\kappa][S]}$-generic C. By Lemma 23 we can rewrite the resulting extension $V[G_\kappa][S][g \times h \times C]$ as $V[G_\kappa][H^0 \times h \times H^3]$ where H^0 is $\mathrm{Add}(\kappa, \kappa^{+3})$-generic and H^3 is $\mathrm{Add}(\kappa^{+3}, 1)$-generic.

We first work to find an $i(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$. This is where we are going to make use of the additional forcing we folded into the forcing iteration, and the claim about the embedding i^+ we just proved. Note that the forcing $i(\mathrm{Add}(\kappa, \kappa^{+3}))$ is the same as $i^+(\mathrm{Add}(\kappa, \kappa^{+3}))$. Fact 26 implies that the embedding i^+ arises from a normal ultrapower on κ (since the original embedding $i \colon V \to N$ was such), after which Lemma 31 tells us that the forcing $i^+(\mathrm{Add}(\kappa, \kappa^{+3}))$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g']}$.

What we have available is the $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S]}$-generic h'', but these two posets do not quite match up. However, all is not lost. Notice that the forcing $\mathbb{A}(\kappa, \nu, \vec{Y}')$ is the same, whether defined in the model $V[G_\kappa][S]$ or $V[G_\kappa][S][h']$, since the forcing that adds h' does not add any sets of ordinals of size κ. It follows that $\mathbb{A}(\kappa, \nu, \vec{Y}')$ must therefore be κ^+-cc in both of these models.

Lemma 30 now tells us that the term forcing $\mathrm{Term}(\mathbb{A}(\kappa, \nu, \vec{Y}'), \mathrm{Add}(\kappa^+, \kappa^{+3}))$ in the model $V[G_\kappa][S][h']$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{+3})$ in that model. We actually already have an $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][h']}$-generic over $V[G_\kappa][S][h']$ available in $V[G_\kappa][S][h]$: it is exactly h'' (since the forcing to add h' does not change the poset $\mathrm{Add}(\kappa^+, \kappa^{+3})$). Using the mentioned forcing equivalence, we can find, in $V[G_\kappa][S][h]$, a $\mathrm{Term}(\mathbb{A}(\kappa, \nu, \vec{Y}'), \mathrm{Add}(\kappa^+, \kappa^{+3}))^{V[G_\kappa][S][h']}$-generic over $V[G_\kappa][S][h']$. Moreover, using the basic property of term forcing mentioned just before Lemma 30, we can interpret this term-forcing generic via the generic g' to produce, in $V[G_\kappa][S][g \times h]$, an $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g' \times h']}$-generic over $V[G_\kappa][S][g' \times h']$.

Again, the poset $\mathrm{Add}(\kappa^+, \kappa^{+3})$ is the same, whether defined in $V[G_\kappa][S][g']$ or $V[G_\kappa][S][g' \times h']$, so we have actually produced an $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g']}$-generic over $V[G_\kappa][S][g' \times h']$. Thinking back to the start of this journey, we originally wanted an $i(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $N[G_\kappa][S'][g' \times h']$. But since we explained how this forcing is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g']}$, and we know that $N[G_\kappa][S'][g' \times h']$ is a submodel of $V[G_\kappa][S][g' \times h']$, we've achieved our goal.

This $i(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$ we've just built can be transferred along k (using Fact 27, since the forcing is $< i(\kappa)$-distributive) to give a $j(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $M[G_\kappa][S][g \times h][G_{\text{tail}}]$. Let K^0 be the result of a surgical modification to this generic to ensure that

$j[H^0] \subseteq K^0$. This allows us to lift j to

$$j\colon V[G_\kappa][H^0] \to M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0]\,.$$

At this point we can argue much like in the conclusion of the proof of Theorem 34. The forcing to add $h \times H^3$ was $\leq \kappa$-distributive, so Fact 27 implies that the generic can simply be transferred along j to yield $K^1 \times K^3$ and a final lift

$$j\colon V[G_\kappa][H^0 \times h \times H^3] \to M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0 \times K^1 \times K^3]\,.$$

The iteration $\mathbb{P}_\kappa$ destroyed the measurability of all $\gamma < \kappa$, so κ is now the least measurable cardinal. Furthermore, since we clearly have $2^\kappa = \kappa^{+3}$ in this final extension, the lifted j is a normal ultrapower by Lemma 33.

The embedding j witnesses $\mathrm{CP}(\kappa, \kappa^{++})$, which can be seen using Lemma 32, just as in the final claim of the proof of Theorem 34. Just like in that argument we can show that any subset $x \subseteq \kappa^{++}$ in the model $V[G_\kappa][S][C \times g \times h]$ has a name $\sigma \in H^{V[G_\kappa]}_{\kappa^{+3}}$ such that $x = \sigma^{g \times h}$. But since $V[G_\kappa]$ and $M[G_\kappa]$ agree on $H_{\kappa^{+3}}$, the name σ must appear in $M[G_\kappa]$ as well and therefore the set $x = \sigma^{g \times h}$ appears in $M[G_\kappa][S][g \times h]$, and also in $M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0 \times K^1 \times K^3]$. In other words, the embedding j (or rather, its codomain) captures x.

Finally, let H^2 be $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$-generic over this final model. This forcing remains $\leq \kappa$-distributive in $V[G_\kappa][H^0 \times h \times H^3]$, so we can apply Fact 27 yet again and transfer H^2 along j to obtain another generic K^2 and lift j again to

$$j\colon V[G_\kappa][H^0 \times h \times H^2 \times H^3] \to M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0 \times K^1 \times K^2 \times K^3]$$

Adding H^2 did not add any new subsets to κ^+, so κ is still the least measurable cardinal. For the same reason, the target model $M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0 \times K^1 \times K^2 \times K^3]$ still captures all the subsets of κ^+. Since j was the ultrapower by a normal measure on κ before this final lift, Fact 26 implies that the lifted j is such an ultrapower as well, and we can conclude that the lifted j still witnesses $\mathrm{CP}(\kappa, \kappa^+)$.

Claim. $\mathrm{LCP}(\kappa, \kappa^{++})$ *fails in the model* $V[G_\kappa][H^0 \times h \times H^2 \times H^3]$.

Proof. Let us write $H = H^0 \times h \times H^2 \times H^3$ and let $\mathbb{P}$ be the entire forcing to add $G_\kappa * H$ over V. Assume that $\mathrm{LCP}(\kappa, \kappa^{++})$ holds. Then there is a normal ultrapower $j^*\colon V[G_\kappa][H] \to M^*[G^*][H^*]$ on κ, with G^* being $j^*(\mathbb{P}_\kappa)$-generic over M^* and H^* being generic over $M^*[G^*]$ for the forcing at stage $j^*(\kappa)$, and this ultrapower embedding captures H^2 and $\mathcal{P}(\kappa^+)^V$. In particular, this implies that $M^*[G^*][H^*]$ computes κ^{++} correctly.

Let us write $G^* = G_\kappa * (S^* * (g^* \times h^*)) * G^*_{\text{tail}}$; note that G_κ really is an initial segment of G^*, since we necessarily have $j^*(G_\kappa) = G^*$ and thus $p = j^*(p) \in G^*$ for any $p \in G_\kappa$. Let γ_0 be the least inaccessible cardinal in V. First, observe that we can factor $\mathbb{P}$ as $\mathbb{P} = \mathbb{Q}_0 * \mathbb{Q}^0$ where $\mathbb{Q}_0$ is nontrivial of size less than γ_0^{+5} and $\mathbb{Q}^0$ is $\leq \gamma_0^{+5}$-strategically closed (for example, we may take $\mathbb{Q}_0$ to be the first step of the iteration taking place at γ_0, and $\mathbb{Q}^0$ to be the tail of the iteration, which starts at the next inaccessible after γ_0 and is therefore clearly $\leq \gamma_0^{+5}$-strategically closed). It follows from Hamkins' gap forcing theorem [10] that j^* restricts to an elementary embedding $j^*\colon V \to$

M^* and that $M^* = V \cap M^*[G^*][H^*]$ (which also means that M^* is an inner model of V). Since we assumed that $\mathcal{P}(\kappa^+)^V \in M^*[G^*][H^*]$, and this power set is clearly in V, this implies that $\mathcal{P}(\kappa^+)^V \in M^*$ as well. It also follows that $\mathcal{P}(\kappa^+)^{V[G_\kappa]} \in M^*[G_\kappa]$; this is because the forcing $\mathbb{P}_\kappa$ is κ-cc and has size κ, so each subset of κ^+ in $V[G_\kappa]$ has a nice name of size κ^+ which may be coded by an element of $\mathcal{P}(\kappa^+)^V$, and all of these codes appear in M^* as we noted. Moreover, it means that $H^2 \notin M^*[G_\kappa]$, since H^2 is generic over $V[G_\kappa] \supseteq M^*[G_\kappa]$. Additionally, the further extension $M^*[G_\kappa][S^*]$ does not add any new subsets of κ^{++}, so H^2 does not appear there either.

Since we assumed that GCH holds in V, the existence of the restricted elementary embedding $j^*\colon V \to M^*$ tells us that GCH holds in M^* as well. From this, it is clear that GCH holds in $M^*[G_\kappa][S^*]$ at κ and above. Working in $M^*[G_\kappa][S^*]$, Lemma 17 tells us that $\mathbb{A}(\kappa, \kappa^{+3}, \vec{X})$ is κ^{++}-Knaster. Since $\mathrm{Add}(\kappa^+, \kappa^{+3})$ is also κ^{++}-Knaster, their product must be as well. It follows from this that the square of the product $\mathbb{A}(\kappa, \kappa^{+3}, \vec{X}) \times \mathrm{Add}(\kappa^+, \kappa^{+3})$, computed in $M^*[G_\kappa][S^*]$, is κ^{++}-cc.

Unger [17, Lemma 2.4] showed that any poset whose square was λ-cc for some regular λ has the λ-approximation property, which states that any set of ordinals in the extension, all of whose pieces of size less than λ are in the ground model, must itself be in the ground model. As a special case, such forcings cannot add fresh subsets of λ (recall that a set of ordinals is *fresh* over a model if it is not in that model but all of its initial segments are). Applying this to our situation, we can conclude that passing from $M^*[G_\kappa][S^*]$ to $M^*[G_\kappa][S^*][g^* \times h^*]$ does not add any new fresh subsets of κ^{++}. Of course, H^2 is a fresh subset of κ^{++} over $V[G_\kappa]$, and since $V[G_\kappa]$ and $M^*[G_\kappa][S^*]$ have the same bounded subsets of κ^{++}, it is also fresh over $M^*[G_\kappa][S^*]$. Therefore H^2 does not appear in $M^*[G_\kappa][S^*][g^* \times h^*]$. To conclude the proof, notice that the remainder of the forcing to go from $M^*[G_\kappa][S^*][g^* \times h^*]$ to $M^*[G^*][H^*]$ does not add any subsets of κ^{++}, so it definitely cannot add H^2. But this contradicts our assumption that $M^*[G^*][H^*]$ captured H^2. ■

To summarize, the model $V[G_\kappa][H^0 \times h \times H^2 \times H^3]$ satisfies $2^\kappa = \kappa^{+3}$ and the lifted embedding j witnesses $\mathrm{CP}(\kappa, \kappa^+)$, but, as the last claim shows, $\mathrm{LCP}(\kappa, \kappa^{++})$ fails. This finishes the proof of the theorem. □

At the end of the paper, let us give another example of the power of Lemma 33 in showing that $\mathrm{CP}(\kappa, \kappa^+)$ holds in known forcing extensions. As we have seen, $\mathrm{CP}(\kappa, \kappa^+)$ does not have any implications for the outright size of κ, since it may consistently hold at the least measurable cardinal κ. But one might try to measure its effects slightly differently. While the capturing property says that there is a normal measure on κ which is quite "fat", in the sense that it captures all subsets of κ^+, perhaps κ must inevitably also carry some, or many, "thin" measures which do not capture much at all. In other words, perhaps $\mathrm{CP}(\kappa, \kappa^+)$ has some implications about the number of normal measures on κ. A combination of Lemma 33 and a theorem of Friedman and Magidor will show us that this is not the case.

Theorem 41. *If V is the minimal extender model with a $(\kappa + 2)$-strong cardinal κ and $\lambda \leq \kappa^{++}$ is a cardinal, then there is a forcing extension in which κ carries exactly λ many normal measures and each of them witnesses*

$\mathrm{CP}(\kappa,\kappa^+)$. *In particular, it is consistent that* κ *has a unique normal measure and* $\mathrm{CP}(\kappa,\kappa^+)$ *holds.*

Proof. The hard part of the proof was done by Friedman and Magidor [6, Theorem 19], who showed that, starting from the listed hypotheses, there is a forcing extension $V[G]$ satisfying $2^\kappa = \kappa^{++}$ in which κ carries exactly λ many normal measures. They also show that each of these normal measures is derived from a lift of the ground model extender embedding $j\colon V \to M$ witnessing the $(\kappa+2)$-strongness of κ. However, Lemma 33 implies that these lifts are themselves already ultrapowers by a normal measure on κ. Finally, an analysis of their proof shows that the forcing used to obtain the model $V[G]$ can be written as $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P} \subseteq H_{\kappa^{++}}$ is a κ^{++}-cc poset which is regularly embedded in $j(\mathbb{P})$, and $\dot{\mathbb{Q}}$ is forced to be $\leq \kappa^+$-distributive. It follows that every subset of κ^+ in $V[G]$ has a nice name in $H^V_{\kappa^{++}} \in M$ and therefore appears in $M[j(G)]$. □

It is unclear whether one can obtain similar results at the least measurable cardinal κ. It seems likely that, to do so, it would be necessary to adapt the Apter–Shelah forcing to incorporate the Sacks forcing machinery that Friedman and Magidor used in their arguments.

Question 42. *Is it consistent that the least measurable cardinal* κ *carries a unique normal measure and* $\mathrm{CP}(\kappa,\kappa^+)$ *holds?*

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