FORCING OVER A FREE SUSLIN TREE

JOHN KRUEGER AND ŠÁRKA STEJSKALOVÁ

ABSTRACT. We introduce an abstract framework for forcing over a free Suslin tree with suborders of products of forcings which add some structure to the tree using countable approximations. The main ideas of this framework are consistency, separation, and the Key Property. We give three applications of this framework: specializing derived trees of a free Suslin tree, adding uncountable almost disjoint subtrees of a free Suslin tree, and adding almost disjoint automorphisms of a free Suslin tree. Using the automorphism forcing, we construct a model in which there is an almost Kurepa Suslin tree and a non-saturated Aronszajn tree, and there does not exist a Kurepa tree. This model solves open problems due to Bilaniuk, Moore, and Jin and Shelah.

CONTENTS

2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E40.
Key words and phrases. Free Suslin tree, separation, consistency, Key Property, almost Kurepa Suslin tree.

1. Introduction

A classical and fundamental result in mathematics states that any non-empty linearly ordered set without endpoints which is dense, complete, and separable is isomorphic to the real number line. Suslin asked whether the same conclusion follows if the assumption of separability is replaced by the countable chain condition, which means that every pairwise disjoint family of open intervals is countable ([Sus20]). Equivalently, is every linearly ordered set with the countable chain condition separable? A counter-example to this statement is called a *Suslin line*. *Suslin's hypothesis* (SH) is the statement that there does not exist a Suslin line. A number of authors independently discovered that SH can be characterized in terms of trees ([Kur36], [Mil43], [Sie48]). SH is equivalent to the non-existence of a *Suslin tree*, which is an uncountable tree which has no uncountable chain or uncountable antichain.

The first systematic study of trees appeared in the dissertation of Kurepa ([Kur35]). It includes a construction of an *Aronszajn tree*, which is an ω_1-tree with no uncountable chains, whose existence had been proved by Aronszajn in 1934 (by an ω_1-tree, we mean a tree of height ω_1 with countable levels). Shortly after, Kurepa proved the existence of a *special* Aronszajn tree (a tree is special if it is a union of countably many antichains) ([Kur38]). Later, Kurepa investigated the question of how many cofinal branches exist in ω_1-trees ([Kur42]). The statement that there exists an ω_1-tree with more than ω_1-many cofinal branches became known as *Kurepa's Hypothesis* (KH), and such a tree is called a *Kurepa tree*.

The resolution of SH and KH came with the advent of modern methods of set theory, namely, constructibility, forcing, and large cardinals ([Göd40], [Coh66]). The consistency of $\neg$SH was established independently by Jech and Tennenbaum, who defined forcings which add a Suslin tree ([Jec67], [Ten68]). The consistency of KH was observed to follow from an inaccessible cardinal by Bukovský and Rowbottom ([Buk66], [Row64]). Namely, the Lévy collapse of an inaccessible cardinal to become ω_1 forces the existence of a Kurepa tree. Later, KH was shown to be consistent without large cardinals by Stewart using a direct forcing construction ([Ste66]). It was also shown that in the constructible universe L, there exists a Suslin tree and there exists a Kurepa tree. Namely, Jensen proved that $\diamondsuit$ implies the existence of a Suslin tree and $\diamondsuit^+$ implies the existence of a Kurepa tree, and these diamond principles hold in L.

For the other direction, Silver proved the consistency of $\neg$KH by showing that after forcing with the Lévy collapse to turn an inaccessible cardinal into ω_2, there does not exist a Kurepa tree ([Sil71]). Solovay proved that Silver's use of an inaccessible cardinal is necessary, because if ω_2 is not an inaccessible cardinal in L then there exists a Kurepa tree. Solovay and Tennenbaum proved the consistency of SH using their newly developed technique of finite support iterations of c.c.c. forcings to construct a model of Martin's axiom together with the negation of the Continuum Hypothesis (CH) ([ST71]; also see [MS70]). Baumgartner isolated a statement about trees which implies SH and follows from Martin's axiom + $\neg$CH, namely, that all Aronszajn trees are special ([Bau70], [BMR70]). The consistency of SH together with CH was proved by Jensen ([DJ74]). Jensen's proof motivated Shelah's invention of proper forcing, and Shelah gave an alternative proof of the consistency of SH + CH as an application of his general technique for iterating proper forcing while not adding reals ([She82]). Both Jensen's and Shelah's models satisfy the stronger statement that all Aronszajn trees are special.

Among the earliest topics studied about Suslin trees after their existence was shown to be consistent are rigidity and homogeneity ([DJ74], [Jec72]). Jensen proved that $\diamondsuit$ implies the existence of both a rigid Suslin tree and a homogeneous Suslin tree with exactly ω_1-many automorphisms. And Jensen proved that $\diamondsuit^+$ implies the existence of a homogeneous Suslin tree with at least ω_2-many automorphisms. Reviewing the construction of this last tree from $\diamondsuit^+$, it is easy to verify that it is an example of an *almost Kurepa Suslin tree*, which is a Suslin tree which becomes a Kurepa tree after forcing with it. Motivated by Jensen's results, Jech proved that if CH holds and κ is a cardinal such that $2^\omega \leq \kappa \leq 2^{\omega_1}$ and $\kappa^\omega = \kappa$, then there exists a forcing which adds a Suslin tree with exactly κ-many automorphisms ([Jec72]). In the case that $\kappa \geq \omega_2$, Jech's forcing gives another example of an almost Kurepa Suslin tree.

Jensen's constructions of a rigid Suslin tree and a homogeneous Suslin tree from $\diamondsuit$ identified two important types of trees: free Suslin trees and uniformly coherent Suslin trees.[1] For any positive $n < \omega$, a Suslin tree T is *n-free* if for any distinct elements $x_0, \ldots, x_{n-1}$ of the same level of T, the product tree $T_{x_0} \otimes \cdots \otimes T_{x_{n-1}}$ (called a *derived tree with dimension* n) is Suslin. And T is *free* if it is n-free for all positive $n < \omega$. The idea of a free Suslin tree is due to Jensen, and the rigid Suslin tree he had constructed earlier from $\diamondsuit$ is free. The homogeneous Suslin tree constructed by Jensen from $\diamondsuit$ is an example of a *uniformly coherent Suslin tree*, which means a Suslin tree consisting of countable sequences of natural numbers, downwards closed and closed under finite modifications, such that any two elements of the tree disagree on at most finitely many elements of their domain. Any uniformly coherent Suslin tree is homogeneous. Uniformly coherent Suslin trees have proven useful in a variety of contexts, including in $\mathbb{P}_{\max}$-style constructions involving a Suslin tree, consistency results, and in forcing axioms ([Lar99], [SZ99], [Woo99], [LT02], [Tod]).

Free Suslin trees satisfy some remarkable properties. Freeness is the strongest known form of rigidity for Suslin trees. Free Suslin trees have the *unique branch property*, which means that forcing with a free Suslin tree introduces exactly one cofinal branch to it ([FH09]). A free Suslin tree is *forcing minimal* in the sense that after forcing with it, there are no intermediate models strictly between the ground model and the generic extension. Any strictly increasing and level preserving map from a free Suslin tree into any Aronszajn tree is injective on a club of levels. A Suslin tree is free if and only if it satisfies a property which is essentially a translation of the definition of an entangled set of reals into the context of trees ([Kru20], [AS81]). Another noteworthy fact about free Suslin trees is their ubiquitousness. The generic Suslin trees of both Jech and Tennenbaum are free. Larson proved that if there exists a uniformly coherent Suslin tree, then there exists a free Suslin tree ([Lar99]). Since forcing a Cohen real adds a uniformly coherent Suslin tree, it also adds a free Suslin tree ([Tod87]). In fact, it is an open problem due to Shelah and Zapletal whether the existence of a Suslin tree implies the existence of a free Suslin tree ([SZ99]). In other words, it could be the case that SH is actually equivalent to the non-existence of a free Suslin tree.

In order to motivate the problems which this article addresses, we review some of the early forcings for adding ω_1-trees with different properties. Jech's forcing for adding a Suslin tree consists of conditions which are countable infinitely splitting downwards closed normal subtrees of the tree $({}^{<\omega_1}\omega, \subset)$, ordered by end-extension ([Jec67]). Jech's forcing is countably closed, ω_2-c.c. assuming CH, and adds a free Suslin tree. Tennenbaum's forcing for adding a Suslin tree consists of conditions which are finite trees whose elements are in ω_1 and whose tree ordering is consistent with the ordinal ordering, ordered by end-extension ([Ten68]). Tennenbaum's forcing is c.c.c. and adds a free Suslin tree. Both Jech's and Tennenbaum's forcings serve as a foundation on which other forcing posets for adding ω_1-trees are based. A variation of Jech's poset, in which any two elements of a condition differ on a finite set and conditions are closed under finite modifications, adds a uniformly coherent Suslin tree. Another variation adds a Suslin tree together with any number of automorphisms of it ([Jec72]). Stewart's forcing for adding a Kurepa tree consists of conditions of the form (T, f), where T is a condition in Jech's forcing with successor height and f is an injective function from a countable subset of ω_2 into the top level of T. Conditions are ordered by letting $(U, g) \leq (T, f)$ if

[1]The concept of a free Suslin tree goes by different names in the literature. Free Suslin trees were originally introduced by Jensen as *full Suslin trees* ([Jenb]). Abraham and Shelah refer to free Suslin trees as *Suslin trees all of whose derived trees are Suslin* ([AS85], [AS93]). The phrase *free Suslin tree* was used by Larson and Shelah-Zapletal ([Lar99], [SZ99]).

U end-extends T, $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, and for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \leq_U g(\alpha)$. Stewart's forcing is countably closed, and assuming CH, is ω_2-c.c. and adds an ω_1-tree together with ω_2-many cofinal branches of it.

In light of Stewart's forcing for adding a Kurepa tree based on Jech's forcing, a natural question is whether there exists a c.c.c. forcing for adding a Kurepa tree based on Tennenbaum's forcing. Jensen introduced the *generic Kurepa hypothesis* (GKH), which states that there exists a Kurepa tree in some c.c.c. forcing extension ([Jena]). Jensen and Schlechta proved that GKH is not a theorem of ZFC: if κ is a Mahlo cardinal, then after forcing with the Lévy collapse to turn κ into ω_2, any c.c.c. forcing fails to add a Kurepa tree ([JS90]). On the other hand, Jensen proved that $\Box_{\omega_1}$ implies the existence of a c.c.c. forcing for adding a Kurepa tree ([Jena]). Since the failure of $\Box_{\omega_1}$ is equiconsistent with a Mahlo cardinal, so is the statement $\neg\mathsf{GKH}$. Later, Veličković defined a c.c.c. forcing for adding a Kurepa tree which is simpler than Jensen's forcing; it is based on Tennenbaum's forcing and uses the function ρ of Todorčević derived from a $\Box_{\omega_1}$-sequence ([Vel92]).

The forcings of Stewart and Veličković for adding a Kurepa tree have size at least ω_2, due to the fact that the conditions in these forcing posets approximate both an ω_1-tree and a sequence of ω_2-many cofinal branches of the tree. Jin and Shelah asked whether it is possible to force the existence of a Kurepa tree using a forcing of size at most ω_1, especially in the context of CH ([JS97]). This question was motivated in part by the fact that there exists a forcing of size ω which adds a Suslin tree, namely, the forcing for adding one Cohen real ([She84]). The main result of Jin and Shelah [JS97] is that assuming the existence of an inaccessible cardinal κ, there exists a forcing which preserves ω_1, collapses κ to become ω_2, forces that there does not exist a Kurepa tree, and introduces a countably distributive Aronszajn tree which when you force with it produces a Kurepa tree. Jin and Shelah asked whether it is possible to obtain such a model where the Aronszajn tree is replaced by some c.c.c. forcing of size at most ω_1.

Problem 1 (Jin and Shelah [JS97])**.** *Is it consistent that CH holds, there does not exist a Kurepa tree, and there exists a c.c.c. forcing of size at most ω_1 which forces the existence of a Kurepa tree?*

As previously mentioned, Jensen proved that $\Diamond^+$ implies the existence of a Kurepa tree and a Suslin tree with at least ω_2-many automorphisms. In his dissertation written under the supervision of Baumgartner, Bilaniuk proved that if $\Diamond$ holds and there exists a Kurepa tree, then there exists a Suslin tree with at least ω_2-many automorphisms [Bil89].

Problem 2 (Bilaniuk [Bil89])**.** *Is it consistent that $\Diamond$ holds, there does not exist a Kurepa tree, and there exists a Suslin tree with at least ω_2-many automorphisms?*

In the Jin-Shelah model, forcing with the Aronszajn tree introduces *another* tree which is a Kurepa tree. On the other hand, an *almost Kurepa Suslin tree* is a Suslin tree which *itself* becomes a Kurepa tree after forcing with it. The following problem is closely related to both Problems 1 and 2 and has been worked on by a number of set theorists since Bilaniuk's dissertation.

Problem 3 (Folklore)**.** *Is it consistent that there exists an almost Kurepa Suslin tree and there does not exist a Kurepa tree?*[2]

Baumgartner introduced the idea of a *subtree base* for an ω_1-tree T, which is a collection $\mathcal{B}$ of uncountable downwards closed subtrees of T such that every uncountable downwards closed subtree of T contains some member of $\mathcal{B}$ ([Bau85]). He proved that after forcing with the Lévy collapse

[2]Concerning the relationship between Problems 2 and 3, start with a model with a Mahlo cardinal κ and a uniformly coherent Suslin tree T. After forcing with $\mathrm{Col}(\omega_1, {<}\kappa) * \mathrm{Add}(\omega, \omega_2)$, T is a Suslin tree with ω_2-many automorphisms. But T is not an almost Kurepa Suslin tree, since by the result of Jensen and Schlechta, no c.c.c. forcing can introduce a Kurepa tree in this model.

$\mathrm{Col}(\omega_1, <\kappa)$, where κ is an inaccessible cardinal, every Aronszajn tree has a base of cardinality ω_1. A related idea called *Aronszajn tree saturation* was introduced by König, Moore, Larson, and Veličković in the context of attempting to reduce the large cardinal assumption used to produce a model with a five element basis for the class of uncountable linear orders ([KLMV08], [Moo06]). An Aronszajn tree T is *saturated* if every almost disjoint family of uncountable downwards closed subtrees of T has cardinality at most ω_1. Note that if T has a subtree base of size ω_1, then T is saturated. So after forcing with the Lévy collapse $\mathrm{Col}(\omega_1, <\kappa)$, where κ is an inaccessible cardinal, every Aronszajn tree is saturated, and by Silver's result, there does not exist a Kurepa tree. On the other hand, Baumgartner and Todorčević proved that if there exists a Kurepa tree, then there exists a special Aronszajn tree which is not saturated ([Bau85]). These facts lead to the following natural question of Moore.

Problem 4 (Moore [Moo08])**.** *Is it consistent that there exists a non-saturated Aronszajn tree and there does not exist a Kurepa tree?*[3]

In this article, we provide solutions to Problems 1, 2, 3, and 4. Our main result is as follows:

Main Theorem. *Suppose that there exists an inaccessible cardinal κ and there exists an infinitely splitting normal free Suslin tree T. Then there exists a forcing poset $\mathbb{P}$ satisfying that the product forcing* $\mathrm{Col}(\omega_1, <\kappa) \times \mathbb{P}$ *forces:*

1. $\kappa = \omega_2$;
2. *GCH holds;*
3. *T is a Suslin tree;*
4. *there exists an almost disjoint family $\{f_\tau : \tau < \omega_2\}$ of automorphisms of T;*
5. *there does not exist a Kurepa tree.*

If b is a generic branch obtained by forcing with the Suslin tree T over a generic extension by $\mathrm{Col}(\omega_1, <\kappa) \times \mathbb{P}$, then $\{f_\tau[b] : \tau < \omega_2\}$ is a family of ω_2-many cofinal branches of T. Thus, in this generic extension T is a c.c.c. forcing of size ω_1 which forces the existence of a Kurepa tree. Starting with a model with an inaccessible cardinal and forcing the existence of an infinitely splitting normal free Suslin tree (for example, by Jech's forcing), we get the following corollary which solves Problems 1 and 3.

Corollary. *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, CH holds, there exists an almost Kurepa Suslin tree, and there does not exist a Kurepa tree.*

Concerning Problem 2, it suffices to find a generic extension as described in the Main Theorem which satisfies $\diamondsuit$. Start with a model V in which there exists an inaccessible cardinal κ and $\diamondsuit$ holds. Let $\mathbb{Q}$ be Jech's forcing in V for adding a Suslin tree. Let $\dot{\mathbb{P}}$ be a $\mathbb{Q}$-name for the forcing described in the Main Theorem using the generic Suslin tree. Since $\mathbb{Q}$ is ω_1-closed, the forcings $\mathbb{Q} * (\mathrm{Col}(\omega_1, <\kappa)^{V^{\mathbb{Q}}} \times \dot{\mathbb{P}})$ and $(\mathbb{Q} * \dot{\mathbb{P}}) \times \mathrm{Col}(\omega_1, <\kappa)$ are forcing equivalent. We will show in Section 6 that the two-step iteration $\mathbb{Q} * \dot{\mathbb{P}}$ is forcing equivalent to some ω_1-closed forcing, and consequently so is $(\mathbb{Q} * \dot{\mathbb{P}}) \times \mathrm{Col}(\omega_1, <\kappa)$. But ω_1-closed forcings preserve $\diamondsuit$, so $\diamondsuit$ holds in the generic extension of $V^{\mathbb{Q}}$ described in the Main Theorem. Since we can force $\diamondsuit$, we have the following corollary.

Corollary. *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, $\diamondsuit$ holds, there exists a normal Suslin tree with ω_2-many automorphisms, and there does not exist a Kurepa tree.*

---

[3]According to Moore, this question is implicit in [Moo08] (see the comments after Question 9.2).

Concerning Problem 4, working in the generic extension by $\mathrm{Col}(\omega_1, <\kappa) \times \mathbb{P}$, for all $\tau < \kappa$ let $U_\tau = \{(x, f_\tau(x)) : x \in T\}$. Then each U_τ is an uncountable downwards closed subtree of the Aronszajn tree $T \otimes T$, and any two such subtrees have countable intersection.[4] We thus get the following corollary which answers Problem 4.

Corollary. *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, there exists a non-saturated Aronszajn tree, and there does not exist a Kurepa tree.*

In their study of rigidity properties of Suslin trees, Fuchs and Hamkins asked, for any positive $n < \omega$, whether a Suslin tree being n-free implies the apparently stronger property of being $(n+1)$-free ([FH09]). This problem was solved by Scharfenberger-Fabian, who proved that if there exists a uniformly coherent Suslin tree then for each positive $n < \omega$, there exists a Suslin tree which is n-free but not $(n+1)$-free ([SF10]). Scharfenberger-Fabian suggested the possibility of having a Suslin tree which is n-free and *n-self specializing* in the sense that forcing with the tree n-many times specializes the part of the tree outside of the n-many generic branches ([SF10]). The first author achieved this possibility by defining a c.c.c. forcing which specializes all derived trees of a free Suslin tree with dimension $n+1$ while preserving the fact that the tree is n-free ([Kru]). In this article, we prove the following theorem which provides the first example of such a forcing which does not add reals.

Theorem. *Let $n < \omega$ be positive and assume that T is an infinitely splitting normal free Suslin tree. Then there exists a forcing which is totally proper, has size ω_1 assuming CH, preserves the fact that T is n-free, and specializes all derived trees of T with dimension $n+1$.*

We now outline the contents of the article in more detail. We introduce a general framework for forcing over a free Suslin tree. The goal is to add some structure to a free Suslin tree using a forcing poset with countable conditions which is totally proper and satisfies other nice properties, such as preserving the Suslinness of T and its derived trees and not adding new cofinal branches of ω_1-trees. The main technique we will use for proving such properties is building total master conditions over countable elementary substructures and related constructions. We apply this general framework to find forcings which add three fundamental types of structures to a free Suslin tree: specializing functions, subtrees, and automorphisms.

For the entirety of the paper, we fix a normal infinitely splitting ω_1-tree T. Most of the ideas we develop do not require any other properties of T. But in order to prove the strongest properties of the forcings, at some points we will need to assume that T is a free Suslin tree. In Section 2 we present our abstract framework for forcing over a free Suslin tree T. We isolate three important ideas on which this framework is based: *consistency*, *separation*, and the *Key Property* which describes the interplay between consistency and separation. In Sections 3, 4, and 5 we give three applications of this framework, with increasing levels of complexity. These three sections are self-contained and can be read independently of each other.

In Section 3, we define for each positive $n < \omega$ a forcing which specializes all derived trees of the free Suslin tree T with dimension $n+1$ while preserving the fact that T is n-free. The preservation of the n-freeness of T uses a generalization of the Key Property which we call the

[4]More generally, Justin Moore has pointed out that if T is a normal almost Kurepa Suslin tree, then the Aronszajn tree $T \otimes T$ is non-saturated. For suppose that $\langle \dot{b}_\tau : \tau < \omega_2 \rangle$ is a sequence of T-names for distinct cofinal branches of T. For each $\tau < \omega_2$, let U_τ be the downward closure of the set of $(x, y) \in T \otimes T$ such that $x \Vdash_T y \in \dot{b}_\tau$. Using the Suslinness of T, one can show that each U_τ is uncountable and any two such subtrees have countable intersection. So the family $\{U_\tau : \tau < \omega_2\}$ witnesses that $T \otimes T$ is not saturated.

n-Key Property. In Section 4, we fix a non-zero ordinal κ and define a forcing which adds an almost disjoint κ-sequence of uncountable downwards closed normal subtrees of T. Assuming that T is a free Suslin tree, this forcing is totally proper, and also assuming CH, is ω_2-c.c. and does not add any new cofinal branches of ω_1-trees which are in the ground model. In particular, if $\kappa \geq \omega_2$, then under these assumptions it is forced that T is a non-saturated Aronszajn tree. The material in this section utilizes many of the main ideas and types of arguments of the article, but in a form which is simpler and easier to understand than the much more complex automorphism forcing.

Section 5 contains the most substantial results of the article. We fix a non-zero ordinal κ and develop a forcing for adding an almost disjoint κ-sequence of automorphisms of T. Assuming that T is a free Suslin tree, we show that the forcing is totally proper, and assuming CH and $\kappa \geq \omega_2$, forces that T is an almost Kurepa Suslin tree. Subsections 5.6-5.9, which contain the most intricate arguments in the article, are devoted to proving that the automorphism forcing does not add new cofinal branches of ω_1-trees appearing in intermediate models of its generic extensions. The main idea used for this result is the concept of a *nice condition*, which is a total master condition over a countable elementary substructure for some regular suborder of the automorphism forcing which has a universality-type property with respect to quotient forcings in intermediate extensions. In Section 6 we prove the main theorem of the article.

Background and preliminaries: The prerequisites for this article are a graduate level background in combinatorial set theory and forcing which includes the basics of ω_1-trees, product forcing, and proper forcing.

An *ω_1-tree* is a tree with height ω_1 whose levels are countable. Let T be an ω_1-tree. For any $x \in T$, we let $\mathrm{ht}_T(x)$ denote the height of x in T. For each $\alpha < \omega_1$, $T_\alpha = \{x \in T : \mathrm{ht}_T(x) = \alpha\}$ is *level α of T*, and $T \upharpoonright \alpha = \{x \in T : \mathrm{ht}_T(x) < \alpha\}$. For all $x \in T$ and $\alpha \leq \mathrm{ht}_T(x)$, $x \upharpoonright \alpha$ denotes the unique $y \leq_T x$ with height α. If $X \subseteq T_\beta$ and $\alpha < \beta$, $X \upharpoonright \alpha$ denotes the set $\{x \upharpoonright \alpha : x \in X\}$. For $\alpha < \beta < \omega_1$ and $X \subseteq T_\beta$, we say that X has *unique drop-downs to* α if the function $x \mapsto x \upharpoonright \alpha$ is injective on X; similar language is used for finite tuples of elements of T_β.

A *branch* of T is a maximal chain, and a branch is *cofinal* if it meets every level of the tree. If b is a branch and α is an ordinal less than its order type, we will write $b(\alpha)$ for the unique element of b of height α. An *antichain* of T is a set of incomparable elements of T. A *subtree* of T is any subset of T considered as a tree with the order inherited from T. The tree T is *infinitely splitting* if every element of T has infinitely many immediate successors. The tree T is *normal* if it has a root, every element of T has at least two immediate successors, every element of T has some element above it at any higher level, and any two distinct elements of the same limit height do not have same set of elements below them.

An *Aronszajn tree* is an ω_1-tree with no cofinal branch. A tree T of height ω_1 is *special* if it is a union of countably many antichains, or equivalently, there exists a *specializing function* $f : T \to \mathbb{Q}$, which means that $x <_T y$ implies that $f(x) < f(y)$. A *Kurepa tree* is an ω_1-tree with at least ω_2-many cofinal branches. A *Suslin tree* is an uncountable tree with no uncountable chain or uncountable antichain. Suslin trees are ω_1-trees. A normal ω_1-tree is a Suslin tree if and only if it has no uncountable antichain.

Any ω_1-tree T can be considered as a forcing poset, where we let y be stronger than x in the forcing if $x \leq_T y$, that is, with the order reversed. A normal ω_1-tree T is Suslin if and only if the forcing poset T is c.c.c. When we use forcing language such as "dense" and "open" when talking about an ω_1-tree T, we mean with regards to T considered as a forcing poset as just discussed. We highlight the following important fact because we will use it almost every time we invoke the Suslin property: *A normal ω_1-tree T is Suslin if and only if whenever D is a dense open subset of T, there* *exists some* $\gamma < \omega_1$ *such that* $T_\gamma \subseteq D$. Note that since D is open, $T_\gamma \subseteq D$ implies that $T_\xi \subseteq D$ for all $\gamma \leq \xi < \omega_1$.

Given finitely many ω_1-trees $T_0, \ldots, T_{n-1}$, the product $T_0 \otimes \cdots \otimes T_{n-1}$ is the partial order, ordered componentwise, consisting of all tuples $(a_0, \ldots, a_{n-1})$ such that for some $\alpha < \omega_1$, $a_k \in (T_k)_\alpha$ for all $k < n$. This product is a tree, and if each factor is normal, then so is the product. Let T be an ω_1-tree. For any positive $n < \omega$, we will write T^n for the product of n-many copies of T. If $\vec{a} = (a_0, \ldots, a_{n-1})$ and $\vec{b} = (b_0, \ldots, b_{n-1})$ are in T^n, we will write $\vec{a} < \vec{b}$ to mean that $a_i <_T b_i$ for all $i < n$, and similarly for $\vec{a} \leq \vec{b}$.

For every $a \in T$, define T_a as the subtree $\{b \in T : a \leq_T b\}$. For any positive $n < \omega$ and n-tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ consisting of distinct elements of T of the same height, define $T_{\vec{a}}$ as the product $T_{a_0} \otimes \cdots \otimes T_{a_{n-1}}$, which is called a *derived tree of* T *with dimension* n. The tree T is said to be *n-free* if all of its derived trees with dimension n are Suslin, and is *free* if it is n-free for all positive $n < \omega$. Note that by the fact we highlighted in the previous paragraph, if T is n-free and $T_{\vec{a}}$ is a derived tree of T with dimension n, then for any dense open subset D of $T_{\vec{a}}$, there exists some $\gamma < \omega_1$ such that every member of $T_{\vec{a}}$ whose elements have height at least γ is in D.

A function $f : T \to U$ between trees is *strictly increasing* if $x <_T y$ implies $f(x) <_U f(y)$, is an *embedding* if $x <_T y$ iff $f(x) <_U f(y)$, is *level preserving* if $\mathrm{ht}_T(x) = \mathrm{ht}_U(f(x))$ for all $x \in T$, is an *isomorphism* if it is a bijective embedding, and is an *automorphism* if it is an isomorphism and $T = U$. We will use the basic fact that a strictly increasing and level preserving map $f : T \to U$ is an embedding if and only if it is injective, and therefore is an isomorphism if and only if it is a bijection. If f is an automorphism of T, we will write f^1 for f and f^{-1} for the inverse of f. An ω_1-tree T is *rigid* if there does not exist any automorphism of T other than the identity function, and is *homogeneous* if for all a and b in T with the same height, there exists an automorphism $f : T \to T$ such that $f(a) = b$. For an ω_1-tree T, $\sigma(T)$ denotes the cardinality of the set of all automorphisms of T.

When we say that a family of sets (or sequence of sets) is *almost disjoint*, we mean that the intersection of any two sets in the family (or in the sequence) is countable. An *almost Kurepa Suslin tree* is a Suslin tree such that when you force with it, it becomes a Kurepa tree. A sufficient condition for a Suslin tree T to be an almost Kurepa Suslin tree is that there exists an almost disjoint family $\{f_\tau : \tau < \omega_2\}$ of automorphisms of T. For in that case, if b is a cofinal branch of T, then $\{f_\tau[b] : \tau < \omega_2\}$ is a family of ω_2-many cofinal branches of T. An *antichain of subtrees* of an Aronszajn tree T is an almost disjoint family of uncountable downwards closed subtrees of T. An Aronszajn tree T is *saturated* if every antichain of subtrees of T has size at most ω_1, and otherwise is *non-saturated*.

When we say that a regular cardinal λ is *large enough*, we mean that it is large enough so that all of the sets under discussion are members of $H(\lambda)$. For a forcing poset $\mathbb{P}$ and a countable elementary substructure $N \prec H(\lambda)$ with $\mathbb{P} \in N$, a condition $q \in \mathbb{P}$ is a *total master condition over* N if for every dense open subset D of $\mathbb{P}$ which is a member of N, there exists some $s \in D \cap N$ such that $q \leq s$. A forcing poset $\mathbb{P}$ is *totally proper* if for all large enough regular cardinals λ and for any countable elementary substructure $N \prec H(\lambda)$, for all $p \in N \cap \mathbb{P}$ there exists some $q \leq p$ such that q is a total master condition over N. Clearly, totally proper forcings are proper and countably distributive. A separative forcing is totally proper if and only if it is proper and does not add reals. The Lévy collapse of an inaccessible cardinal κ to become ω_2, denoted by $\mathrm{Col}(\omega_1, <\kappa)$, is the forcing poset consisting of all countable partial functions p from $\kappa \times \omega_1$ into κ such that for all $(\alpha, \xi) \in \mathrm{dom}(p)$, $p(\alpha, \xi) < \alpha$, ordered by reverse inclusion. The Lévy collapse is ω_1-closed and κ-c.c. Finally, we note that ω_1-closed forcings do not add new cofinal branches of ω_1-trees in the ground model ([She98, Chapter V §8]).

2. Abstract Framework for Forcing Over a Free Suslin Tree

In this section we present the basic framework for forcing over a free Suslin tree. We fix several objects satisfying some abstract properties, and then derive important consequences from those assumptions. Assume for the remainder of the article that T is a fixed ω_1-tree which is normal and infinitely splitting, and κ is a fixed non-zero ordinal. Additional assumptions about T and κ will be made on occasion, most notably, that T is a free Suslin tree.

We assume that the following objects and properties are given:

(1) For each $\tau < \kappa$, $(\mathbb{Q}_\tau, \leq_\tau)$ is a forcing poset.
(2) For each $\tau < \kappa$, associated to any condition $q \in \mathbb{Q}_\tau$ is a countable ordinal which we call the *top level of* q.
(3) $\mathbb{P}$ is a forcing poset satisfying that for all $p \in \mathbb{P}$, p is a partial function whose domain is a countable subset of κ, and there is a fixed countable ordinal α, called the *top level of* p, such that for all $\tau \in \mathrm{dom}(p)$, $p(\tau) \in \mathbb{Q}_\tau$ and $p(\tau)$ has top level α. Also, $q \leq_\mathbb{P} p$ implies that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\tau \in \mathrm{dom}(q)$, $q(\tau) \leq_\tau p(\tau)$.
(4) For all $\tau < \kappa$, for all $q \in \mathbb{Q}_\tau$, and for all positive $n < \omega$, we have a fixed relation between members $\vec{a} < \vec{b}$ of T^n which we call $\vec{a}$ and $\vec{b}$ being *q-consistent*.
(5) For any $p \in \mathbb{P}$, for any finite set $A \subseteq \mathrm{dom}(p)$, and for all $\vec{a} \in T^{<\omega}$, we have a fixed property which we call $\{p(\tau) : \tau \in A\}$ being *separated on* $\vec{a}$.

We make the following assumptions about the above objects and properties:

(A) If $q \leq_\mathbb{P} p$ then the top level of q is greater than or equal to the top level of p.
(B) (Transitivity) Suppose that $\vec{a} < \vec{b} < \vec{c}$, $\tau < \kappa$, $r \leq_\tau q$, and the heights of $\vec{b}$ and $\vec{c}$ are equal to the top levels of q and r respectively. If $\vec{a}$ and $\vec{b}$ are q-consistent and $\vec{b}$ and $\vec{c}$ are r-consistent, then $\vec{a}$ and $\vec{c}$ are r-consistent.
(C) (Persistence) Let $\alpha < \beta < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α and $A \subseteq \mathrm{dom}(p)$ is finite. Let $\vec{a} < \vec{b}$ have heights α and β respectively. If $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$, then for any $q \leq_\mathbb{P} p$ with top level β, $\{q(\tau) : \tau \in A\}$ is separated on $\vec{b}$.
(D) (Extension) Let $\alpha < \beta < \omega_1$. Assume that $\vec{a} < \vec{b}$ have heights α and β respectively, $p \in \mathbb{P}$ has top level α, and $A \subseteq \mathrm{dom}(p)$ is finite. Then there exists some $q \leq_\mathbb{P} p$ with top level β and with the same domain as p such that for all $\tau \in A$, $\vec{a}$ and $\vec{b}$ are $q(\tau)$-consistent.
(E) (Key Property) Let $\alpha < \beta < \omega_1$. Suppose that $\vec{a}$ has height α, p is a condition with top level α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$. Then for any $q \leq_\mathbb{P} p$ with top level β and any finite set $t \subseteq T_\beta$, there exists some $\vec{b}$ of height β such that $\vec{a} < \vec{b}$, the elements of $\vec{b}$ are not in t, and for all $\tau \in A$, $\vec{a}$ and $\vec{b}$ are $q(\tau)$-consistent.

While the above properties are described in terms of tuples, it is often the case that the properties are independent of the order in which a tuple lists its elements. For the applications in this article, this independence always holds for the consistency relations. But the definition of separation for the automorphism forcing of Section 5 depends on the order of a tuple. In any case, it is a simple matter to translate the results of this section for tuples into analogous results for finite sets.

Lemma 2.1. *Suppose that D is a dense open subset of $\mathbb{P}$ and $p \in \mathbb{P}$ has top level ξ. Let $A \subseteq \mathrm{dom}(p)$ be finite. Consider a tuple $\vec{a}$ of height ξ and assume that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$. Let $\mathcal{X}$ be the set of all $\vec{c}$ in the derived tree $T_{\vec{a}}$ for which there exists some $q \leq_\mathbb{P} p$ in D whose top level equals the height of $\vec{c}$ such that for all $\tau \in A$, $\vec{a}$ and $\vec{c}$ are $q(\tau)$-consistent. Then $\mathcal{X}$ is a dense open subset of $T_{\vec{a}}$.*

Proof. For openness, consider a tuple $\vec{b} \in \mathcal{X}$ as witnessed by a condition $q \leq_{\mathbb{P}} p$. Let $\vec{c}$ be a tuple such that $\vec{b} < \vec{c}$ and $\vec{c}$ has height ζ. By (D) (Extension), find $r \leq_{\mathbb{P}} q$ with top level ζ such that for all $\tau \in A$, $\vec{b}$ and $\vec{c}$ are $r(\tau)$-consistent. Since D is open, $r \in D$. By (B) (Transitivity), for all $\tau \in A$, $\vec{a}$ and $\vec{c}$ are $r(\tau)$-consistent. So r witnesses that $\vec{c} \in \mathcal{X}$.

To show that $\mathcal{X}$ is dense, consider $\vec{b} \in T_{\vec{a}}$ with height $\delta > \xi$. By (D) (Extension), fix $q \leq_{\mathbb{P}} p$ with top level δ such that for all $\tau \in A$, $\vec{a}$ and $\vec{b}$ are $q(\tau)$-consistent. By (C) (Persistence), $\{q(\tau) : \tau \in A\}$ is separated on $\vec{b}$. Let E be the set of condition $s \in \mathbb{P}$ such that s has top level greater than δ. By (D) (Extension), E is dense, and by (A), E is open. So $D \cap E$ is dense open. Fix $r \leq_{\mathbb{P}} q$ in $D \cap E$ and let ρ be the top level of r. Then $\rho > \delta$. Since $r \leq q$ and $\{q(\tau) : \tau \in A\}$ is separated on $\vec{b}$, by (E) (Key Property) we can find some $\vec{c}$ with height ρ such that $\vec{b} < \vec{c}$ and for all $\tau \in A$, $\vec{b}$ and $\vec{c}$ are $r(\tau)$-consistent. By (B) (Transitivity), for all $\tau \in A$, $\vec{a}$ and $\vec{c}$ are $r(\tau)$-consistent. So r is a witness that $\vec{c} \in \mathcal{X}$. □

The next proposition will be used to prove that the forcings introduced in this article are totally proper.

Proposition 2.2 (Consistent Extensions Into Dense Sets)**.** *Suppose that T is a free Suslin tree. Let λ be a large enough regular cardinal and let N be a countable elementary substructure of $H(\lambda)$ containing as members T, κ, $\langle \mathbb{Q}_\tau : \tau < \kappa \rangle$, and $\mathbb{P}$. Let $\delta = N \cap \omega_1$. Assume that $D \in N$ is a dense open subset of $\mathbb{P}$, $p \in N \cap \mathbb{P}$ has top level β, and $A \subseteq \mathrm{dom}(p)$ is finite. Let $\vec{a}$ have height δ with unique drop-downs to β such that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a} \restriction \beta$. Then there exists some $q \leq_{\mathbb{P}} p$ in $D \cap N$ whose top level is some ordinal $\gamma < \delta$ such that for all $\tau \in A$, $\vec{a} \restriction \beta$ and $\vec{a} \restriction \gamma$ are $q(\tau)$-consistent.*

Proof. Let $\mathcal{X}$ be the set of all tuples $\vec{b}$ in the derived tree $T_{\vec{a} \restriction \beta}$ satisfying that for some $q \leq_{\mathbb{P}} p$ in D, for all $\tau \in A$, $\vec{a} \restriction \beta$ and $\vec{b}$ are $q(\tau)$-consistent. Note that $\mathcal{X} \in N$ by elementarity, and $\mathcal{X}$ is dense open in $T_{\vec{a} \restriction \beta}$ by Lemma 2.1. Since $\mathcal{X}$ is dense open, fix $\gamma > \beta$ such that any member of $T_{\vec{a} \restriction \beta}$ whose elements have height at least γ is in $\mathcal{X}$. By elementarity, we can choose $\gamma \in N \cap \omega_1 = \delta$. Then $\vec{a} \restriction \gamma \in \mathcal{X}$, which by elementarity can be witnessed by some $q \in D \cap N$. Clearly, q is as required. □

We now develop a higher dimensional variant of the Key Property which will be used for preserving Suslin trees.

Definition 2.3 (n-Key Property)**.** *Let $n < \omega$. The forcing poset $\mathbb{P}$ satisfies the n-*Key Property *if the following statement holds. Assume that:*

- *$\alpha < \beta < \omega_1$;*
- *$\vec{a} = (a_0, \ldots, a_{l-1})$ is an injective tuple consisting of elements of T_α, where $l \geq n$;*
- *$p \in \mathbb{P}$ has top level α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$;*
- *$i_0, \ldots, i_{n-1} < l$ are distinct and for each $k < n$, $c_k \in T_\beta$ is above a_{i_k}.*

Then for any $q \leq_{\mathbb{P}} p$ with top level β, there exists some $\vec{b} = (b_0, \ldots, b_{l-1})$ above $\vec{a}$ consisting of elements of T_β such that:

(1) *$b_{i_k} = c_k$ for all $k < n$;*
(2) *for all $\tau \in A$, $\vec{a}$ and $\vec{b}$ are $q(\tau)$-consistent.*

Proposition 2.4 (Consistent Extensions for Sealing)**.** *Suppose that T is a free Suslin tree. Assume that $\mathbb{P}$ has the n-Key Property, where $n < \omega$ is positive. Let λ be a large enough regular cardinal and let N be a countable elementary substructure of $H(\lambda)$ containing T, κ, $\langle \mathbb{Q}_\tau : \tau < \kappa \rangle$, and $\mathbb{P}$.*

Let $\delta = N \cap \omega_1$. Let $\beta < \delta$ and let $\vec{x} = (x_0, \dots, x_{n-1})$ have height β. Suppose that $\dot{E} \in N$ is a $\mathbb{P}$-name for a dense open subset of the derived tree $T_{\vec{x}}$.

Let $p \in N \cap \mathbb{P}$ have top level γ, where $\gamma \geq \beta$. Suppose that $\vec{a} = (a_0, \dots, a_{l-1})$ has height δ, where $l \geq n$, $i_0, \dots, i_{n-1} < l$ are distinct, and $x_k <_T a_{i_k}$ for all $k < n$. Assume that $A \subseteq \mathrm{dom}(p)$ is finite and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a} \restriction \gamma$. Then there exists some $r \leq_{\mathbb{P}} p$ in N with some top level ξ such that $r \Vdash_{\mathbb{P}} (a_{i_0}, \dots, a_{i_{n-1}}) \in \dot{E}$ and for all $\tau \in A$, $\vec{a} \restriction \beta$ and $\vec{a} \restriction \xi$ are $r(\tau)$-consistent.

Proof. Let $\mathcal{X}$ be the set of all $\vec{d} = (d_0, \dots, d_{l-1})$ in the derived tree $T_{\vec{a} \restriction \gamma}$ satisfying that for some $r \leq_{\mathbb{P}} p$ with top level equal to the height of $\vec{d}$,

$$r \Vdash_{\mathbb{P}} (d_{i_0}, \dots, d_{i_{n-1}}) \in \dot{E},$$

and for all $\tau \in A$, $\vec{a} \restriction \gamma$ and $\vec{d}$ are $r(\tau)$-consistent. Note that $\mathcal{X} \in N$ by elementarity.

We claim that $\mathcal{X}$ is dense open in $T_{\vec{a} \restriction \gamma}$. To show that $\mathcal{X}$ is dense, consider $\vec{b} = (b_0, \dots, b_{l-1})$ in $T_{\vec{a} \restriction \gamma}$ whose elements have height $\zeta > \gamma$. By (D) (Extension), find $q \leq_{\mathbb{P}} p$ with top level ζ such that for all $\tau \in A$, $\vec{a} \restriction \gamma$ and $\vec{b}$ are $q(\tau)$-consistent. As $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a} \restriction \gamma$, (C) (Persistence) implies that $\{q(\tau) : \tau \in A\}$ is separated on $\vec{b}$.

Since $\dot{E}$ is forced to be dense open in $T_{\vec{x}}$, fix $r \leq_{\mathbb{P}} q$ with some top level ξ and $\vec{c} = (c_0, \dots, c_{n-1})$ above $(b_{i_0}, \dots, b_{i_{n-1}})$ with some height ρ such that $r \Vdash (c_0, \dots, c_{n-1}) \in \dot{E}$. By extending further if necessary using (D) (Extension) and using the fact that $\dot{E}$ is forced to be open, we may assume without loss of generality that $\xi = \rho > \zeta$. Since $\mathbb{P}$ satisfies the n-Key Property, we can find $\vec{d} = (d_0, \dots, d_{l-1})$ above $\vec{b}$ such that $d_{i_k} = c_k$ for all $k < n$ and for all $\tau \in A$, $\vec{b}$ and $\vec{d}$ are $r(\tau)$-consistent. By (B) (Transitivity), for all $\tau \in A$, $\vec{a} \restriction \gamma$ and $\vec{d}$ are $r(\tau)$-consistent. So $\vec{b} < \vec{d} \in \mathcal{X}$.

To show that $\mathcal{X}$ is open, suppose that $\vec{d} = (d_0, \dots, d_{l-1})$ is in $\mathcal{X}$ as witnessed by r, and let $\vec{e} = (e_0, \dots, e_{l-1})$ be above $\vec{d}$ of height ξ. We will show that $\vec{e} \in \mathcal{X}$. By (D) (Extension), find $s \leq_{\mathbb{P}} r$ with top level ξ such that for all $\tau \in A$, $\vec{d}$ and $\vec{e}$ are $s(\tau)$-consistent. Since $\dot{E}$ is forced to be open and $r \Vdash_{\mathbb{P}} (d_{i_0}, \dots, d_{i_{n-1}}) \in \dot{E}$, it follows that $s \Vdash_{\mathbb{P}} (e_{i_0}, \dots, e_{i_{n-1}}) \in \dot{E}$. By (B) (Transitivity), for all $\tau \in A$, $\vec{a} \restriction \gamma$ and $\vec{e}$ are $s(\tau)$-consistent. So $\vec{e} \in \mathcal{X}$.

Since $\mathcal{X}$ is dense open, we can fix some $\xi > \gamma$ such that any member of $T_{\vec{a} \restriction \gamma}$ whose elements have height at least ξ is in $\mathcal{X}$. By elementarity, we can choose ξ in $N \cap \omega_1 = \delta$. So $\vec{a} \restriction \xi \in \mathcal{X}$, which by elementarity can be witnessed by some $r \in N$. So $r \Vdash_{\mathbb{P}} (a_{i_0}, \dots, a_{i_{n-1}}) \restriction \xi \in \dot{E}$ and for all $\tau \in A$, $\vec{a} \restriction \gamma$ and $\vec{a} \restriction \xi$ are $r(\tau)$-consistent. Since $\dot{E}$ is forced to be open, $r \Vdash_{\mathbb{P}} (a_{i_0}, \dots, a_{i_{n-1}}) \in \dot{E}$. □

In comparing the Key Property with the n-Key Property, note the absence in the latter of the set t appearing in the former. Merging the two properties we get a natural strengthening of the n-Key Property.

Definition 2.5 (Strong n-Key Property)**.** *Let $n < \omega$. The forcing poset $\mathbb{P}$ satisfies the n-*Key Property *if the following statement holds. Assume that:*

- *$\alpha < \beta < \omega_1$;*
- *$t \subseteq T_\beta$ finite;*
- *$\vec{a} = (a_0, \dots, a_{l-1})$ is an injective tuple consisting of elements of T_α, where $l \geq n$;*
- *$p \in \mathbb{P}$ has top level α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$;*
- *$i_0, \dots, i_{n-1} < l$ are distinct and for each $k < n$, $c_k \in T_\beta$ is above a_{i_k}.*

Then for any $q \leq_{\mathbb{P}} p$ with top level β, there exists some $\vec{b} = (b_0, \dots, b_{l-1})$ above $\vec{a}$ consisting of elements of $T_\beta \setminus t$ such that:

1. *$b_{i_k} = c_k$ for all $k < n$;*
2. *for all $\tau \in A$, $\vec{a}$ and $\vec{b}$ are $q(\tau)$-consistent.*

Observe that the Key Property is equivalent to the Strong 0-Key Property, and the 0-Key Property is weaker than the Key Property. We note that the automorphism forcing of Section 5 satisfies the 1-Key Property but not the Strong 1-Key Property.

The following lemma is immediate.

Lemma 2.6. *For any $m \leq n < \omega$, if $\mathbb{P}$ satisfies the n-Key Property then $\mathbb{P}$ satisfies the m-Key Property, and if $\mathbb{P}$ satisfies the Strong n-Key Property then $\mathbb{P}$ satisfies the Strong m-Key Property and the Key Property.*

3. Application I: Specializing Derived Trees of a Free Suslin Tree

For the remainder of this section fix a natural number $n \geq 1$. We will develop a forcing poset which, assuming that T is a free Suslin tree, specializes all derived trees of T with dimension $n+1$ while preserving the fact that T is n-free. In the notation from Section 2, let $\kappa = \omega_1$.

3.1. Suitable Families of Specializing Functions.

Definition 3.1. *An injective tuple $\vec{a} \in T^{n+1}$ of height α is* minimal *if for all $\beta < \alpha$, $\vec{a} \upharpoonright \beta$ is not injective.*

Since T is normal, the height of any minimal tuple is a successor ordinal. So a tuple $\vec{a} = (a_0, \ldots, a_n) \in T^{n+1}$ is minimal if and only if it is injective, has some successor height $\beta + 1$, and there exist $i < j < n$ such that $a_i \upharpoonright \beta = a_j \upharpoonright \beta$.

Clearly, there are uncountably many minimal tuples in T^{n+1}. So the next lemma can be thought of as a generalization of Kurepa's theorem that $T \otimes T$ is not Suslin.

Lemma 3.2. *Let $\vec{a} = (a_0, \ldots, a_n)$ and $\vec{b} = (b_0, \ldots, b_n)$ be distinct injective tuples in T^{n+1} which are both minimal. Then the derived trees $T_{\vec{a}}$ and $T_{\vec{b}}$ are disjoint.*

Proof. Let $\vec{a}$ have height α and let $\vec{b}$ have height β. If $\vec{c}$ is a member of both derived trees, then $\vec{c} \upharpoonright \alpha = \vec{a}$ and $\vec{c} \upharpoonright \beta = \vec{b}$. Since $\vec{a} \neq \vec{b}$, $\alpha \neq \beta$. Without loss of generality, assume that $\alpha < \beta$. Then $\vec{b} \upharpoonright \alpha = (\vec{c} \upharpoonright \beta) \upharpoonright \alpha = \vec{c} \upharpoonright \alpha = \vec{a}$. This contradicts the minimality of $\vec{b}$ since $\vec{a}$ is injective. □

Lemma 3.3. *Any derived tree of T with dimension $n+1$ is a subtree of some derived tree $T_{\vec{a}}$, where $\vec{a}$ is minimal.*

Proof. Let $\vec{b}$ be an injective tuple in T^{n+1}. Let α be the least ordinal such that $\vec{b} \upharpoonright \alpha$ is injective. Since T has a root, $\alpha > 0$, and since T is normal, α is equal to some successor ordinal $\gamma + 1$. Then $\vec{b} \upharpoonright (\gamma + 1)$ is minimal and $T_{\vec{b}} \subseteq T_{\vec{b} \upharpoonright (\gamma+1)}$. □

So in order to specialize all derived trees of T with dimension $n+1$, it suffices to specialize all derived trees of the form $T_{\vec{a}}$, where $\vec{a} \in T^{n+1}$ is minimal. For the remainder of the section, fix an enumeration $\langle \vec{a}^\tau : \tau < \omega_1 \rangle$ of all injective tuples in T^{n+1} which are minimal. For each $\tau < \omega_1$, let $T^\tau = T_{\vec{a}_\tau}$ and let

$$U^\tau = T^\tau \cup \{\vec{c} \in T^{n+1} : \vec{c} < \vec{a}_\tau\}.$$

Note that the height of a tuple in the tree U^τ coincides with the heights in T of its elements. So we will use U^τ in the definition of our poset for specializing T^τ to provide some simplifications in notation.

Definition 3.4 (Specializing Functions)**.** *Let $\tau < \omega_1$ and $\beta < \omega_1$. A* specializing function *on $U^\tau \upharpoonright (\beta + 1)$ is any function $f : U^\tau \upharpoonright (\beta+1) \to \mathbb{Q}$ satisfying:*

(1) *for all $\vec{a} \in \mathrm{dom}(f)$, if $\vec{a} < \vec{a}_\tau$ then $f(\vec{a}) = -1$, and if $\vec{a} \in T^\tau$ then $f(\vec{a}) > 0$;*

(2) *for all $\vec{b}$ and $\vec{c}$ in $T^\tau \cap \mathrm{dom}(f)$, if $\vec{b} < \vec{c}$ then $f(\vec{b}) < f(\vec{c})$.*

In the above, we refer to β as the top level of f.

Let $\alpha < \beta < \omega_1$ and $\tau < \omega_1$. If g is a specializing function on $U^\tau \upharpoonright (\beta+1)$, then we will write $g \upharpoonright (\alpha+1)$ for $g \upharpoonright (U^\tau \upharpoonright (\alpha+1))$, which is easily seen to be a specializing function on $U^\tau \upharpoonright (\alpha+1)$. If $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family, where $I \subseteq \omega_1$ and each g_τ is a specializing function on $U^\tau \upharpoonright (\beta+1)$, we will write $\mathcal{G} \upharpoonright (\alpha+1)$ for the indexed family $\{g_\tau \upharpoonright (\alpha+1) : \tau \in I\}$.

Instead of having a single consistency relation, as is the case in the applications of Sections 4 and 5, for the specializing forcing we will have a consistency relation for each finite set of positive rational numbers. Let $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$.

Definition 3.5 (Consistency)**.** *Let $\alpha < \beta < \omega_1$ and $\tau < \omega_1$. Let $Q \subseteq \mathbb{Q}^+$ be finite. Suppose that f is a specializing function on $U^\tau \upharpoonright (\beta+1)$.*

(1) *Let $X \subseteq T_\beta$ be finite with unique drop-downs to α. Then $X \upharpoonright \alpha$ and X are (f, Q)-*consistent *if for all $q \in Q$, for any $\vec{c} \in T^\tau \cap X^{n+1}$, if $f(\vec{c} \upharpoonright \alpha) < q$ then $f(\vec{c}) < q$.*

(2) *Let $\vec{a} = (a_0, \dots, a_{m-1})$ be an injective tuple consisting of elements of T_β. Then $\vec{a} \upharpoonright \beta$ and $\vec{a}$ are (f, Q)-*consistent *if $\{a_0, \dots, a_{m-1}\} \upharpoonright \alpha$ and $\{a_0, \dots, a_{m-1}\}$ are (f, Q)-consistent.*

Note that if the set X in (1) has size less than $n+1$, then consistency holds vacuously. Also, observe that in (1), if consistency holds for X, then it also holds for any $Y \subseteq X$.

The following lemma is easy to check.

Lemma 3.6 (Transitivity)**.** *Let $\alpha < \beta < \gamma < \omega_1$ and $\tau < \omega_1$. Let $Q \subseteq \mathbb{Q}^+$ be finite and let $X \subseteq T_\gamma$ be finite with unique drop-downs to α. Suppose that f is a specializing function on $U^\tau \upharpoonright (\gamma+1)$. If $X \upharpoonright \alpha$ and $X \upharpoonright \beta$ are $((f \upharpoonright (\beta+1)), Q)$-consistent and $X \upharpoonright \beta$ and X are (f, Q)-consistent, then $X \upharpoonright \alpha$ and X are (f, Q)-consistent.*

In contrast to the other applications in Sections 4 and 5, we will not need any notion of separation for the specializing forcing. So when we apply the results of Section 2, we will assume that the separation assumptions made there hold automatically.

Definition 3.7 (Specializing Families)**.** *Let $\beta < \omega_1$. An indexed family $\{f_\tau : \tau \in I\}$, where $I \subseteq \omega_1$ is countable and each f_τ is a specializing function on $U^\tau \upharpoonright (\beta+1)$, is called a* specializing family with top level β.

Definition 3.8 (Suitable Specializing Families)**.** *Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ is a specializing family with top level γ. We say that $\mathcal{G}$ is* suitable *if the following holds. Assume that:*

- *$\alpha < \beta \leq \gamma$;*
- *$B \subseteq I$, $R \subseteq \mathbb{Q}^+$, and $t \subseteq T_\beta$ are finite sets;*
- *$a_0, \dots, a_{l-1}$ are distinct elements of T_α, where $l \geq n$;*
- *$i_0, \dots, i_{n-1} < l$ are distinct and for each $k < n$, $c_k \in T_\beta \setminus t$ is above a_{i_k}.*

Then there exist $b_0, \dots, b_{l-1}$ in $T_\beta \setminus t$ such that:

(1) *$a_i <_T b_i$ for all $i < l$;*

(2) *$b_{i_k} = c_k$ for all $k < n$;*

(3) *for all $\tau \in B$, $\{a_0, \dots, a_{l-1}\}$ and $\{b_0, \dots, b_{l-1}\}$ are $(g_\tau \upharpoonright (\beta+1), R)$-consistent.*

Clearly, if $\mathcal{G}$ is a suitable specializing family with top level γ and $\xi < \gamma$, then $\mathcal{G} \upharpoonright (\xi+1)$ is a suitable specializing family.

3.2. **Constructing and Extending Suitable Families.**

Lemma 3.9. *Assume the following:*

- *$\gamma < \omega_1$;*
- *$\{f_\tau : \tau \in I\}$ is a suitable specializing family with top level γ;*
- *$Q \subseteq \mathbb{Q}^+$ is finite, $A \subseteq I$ is finite, and $X \subseteq T_{\gamma+1}$ is finite and has unique drop-downs to γ.*

Then there exists a suitable specializing family $\{g_\tau : \tau \in I\}$ with top level $\gamma + 1$ satisfying:

(1) *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
(2) *for all $\tau \in A$, $X \restriction \gamma$ and X are (g_τ, Q)-consistent.*

Proof. For each $\tau \in I$ define $g_\tau \restriction (\gamma + 1) = f_\tau$. We define the values of g_τ on $(U^\tau)_{\gamma+1}$, for all $\tau \in I$, in ω-many stages, where at any stage we will have defined only finitely many values of g_τ for finitely many $\tau \in I$. We also define finite sets $X_m \subseteq T_{\gamma+1}$, $Q_m \subseteq \mathbb{Q}^+$, and $A_m \subseteq I$ for each $m < \omega$ so that $\bigcup_m X_m = T_{\gamma+1}$, $\bigcup_m Q_m = \mathbb{Q}^+$, and $\bigcup_m A_m = I$. Our inductive hypothesis is that for all $m < \omega$, for all $\tau \in I$, and for all $\vec{a} \in (U^\tau)_{\gamma+1}$, if $g_\tau(\vec{a})$ was defined by stage m, then $\tau \in A_m$ and $\vec{a} \in X_m^{n+1}$.

Fix an enumeration $\langle z_m : m < \omega \rangle$ of $T_{\gamma+1}$, an enumeration $\langle q_m : m < \omega \rangle$ of $\mathbb{Q}^+$, and an enumeration $\langle \tau_m : m < \omega \rangle$ of I (with repetitions if I is finite).

Stage 0: Consider $\tau \in A$ and $\vec{a} \in (U^\tau)_{\gamma+1} \cap X^{n+1}$, and we define $g_\tau(\vec{a})$. If $\vec{a} < \vec{a}_\tau$, then let $g_\tau(\vec{a}) = -1$. Suppose that $\vec{a} \in T^\tau$. Choose some positive rational number q such that $f_\tau(\vec{a} \restriction \gamma) < q$, and for all $r \in Q$, if $f_\tau(\vec{a} \restriction \gamma) < r$, then $q < r$. This is possible since Q is finite. Now define $g_\tau(\vec{a}) = q$. Let $X_0 = X$, $Q_0 = Q$, and $A_0 = A$. The inductive hypothesis clearly holds.

Stage $m + 1$: Let $m < \omega$ and assume that we have completed stage m. In particular, we have defined X_m, A_m, and Q_m. Define $X_{m+1} = X_m \cup \{z_m\}$, $A_{m+1} = A_m \cup \{\tau_m\}$, and $Q_{m+1} = Q_m \cup \{q_m\}$. Consider $\tau \in A_{m+1}$ and $\vec{a} \in (U^\tau)_{\gamma+1} \cap X_{m+1}^{n+1}$. Assuming that $g_\tau(\vec{a})$ has not already been defined, we will specify its value now. If $\vec{a} < \vec{a}_\tau$, let $g_\tau(\vec{a}) = -1$. Suppose that $\vec{a} \in T^\tau$. Choose some positive rational number q such that $f_\tau(\vec{a} \restriction \gamma) < q$, and for all $r \in Q_{m+1}$, if $f_\tau(\vec{a} \restriction \gamma) < r$, then $q < r$. Now define $g_\tau(\vec{a}) = q$. This completes stage $m + 1$. Clearly, the inductive hypothesis is maintained.

This completes the construction. Each g_τ is a specializing function on $U^\tau \restriction (\gamma + 2)$ such that $f_\tau \subseteq g_\tau$. By what we did at stage 0, for all $\tau \in A$, $X \restriction \gamma$ and X are (g_τ, Q)-consistent. We claim that $\{g_\tau : \tau \in I\}$ is suitable. So let $B \subseteq I$, $R \subseteq \mathbb{Q}^+$, and $t \subseteq T_{\gamma+1}$ be finite sets and let $\alpha < \gamma + 1$. Suppose that $l \geq n$, $a_0, \ldots, a_{l-1}$ are distinct elements of T_α, $i_0, \ldots, i_{n-1} < l$ are distinct, and for each $k < n$, $c_k \in T_{\gamma+1} \setminus t$ is above a_{i_k}.

If $\alpha < \gamma$, then applying the fact that $\{f_\tau : \tau \in I\}$ is suitable, fix $b_0, \ldots, b_{l-1}$ in T_γ such that $a_i <_T b_i$ for all $i < l$, $b_{i_k} = c_k \restriction \gamma$ for all $k < n$, and for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are (f_τ, R)-consistent. On the other hand, if $\alpha = \gamma$, then let $b_i = a_i$ for all $i < l$. Choose $m < \omega$ large enough so that $R \subseteq Q_{m+1}$. Since T is infinitely splitting, we can choose $d_i >_T b_i$ in $T_{\gamma+1} \setminus (X_m \cup t)$ for each $i \in l \setminus \{i_0, \ldots, i_{n-1}\}$. For all $k < n$, let $d_{i_k} = c_k$. Then for all $i < l$, $d_i \in T_{\gamma+1} \setminus t$.

We claim that for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{d_0, \ldots, d_{l-1}\}$ are (g_τ, R)-consistent, which finishes the proof. Consider $r \in R$ and $\vec{e} \in T^\tau \cap \{d_0, \ldots, d_{l-1}\}^{n+1}$ such that $g_\tau(\vec{e} \restriction \alpha) < r$. We will show that $g_\tau(\vec{e}) < r$. Now $g_\tau(\vec{e} \restriction \gamma) < r$ holds, trivially if $\alpha = \gamma$, and because $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are (f_τ, R)-consistent in the case that $\alpha < \gamma$. Since $\vec{e}$ has $(n + 1)$-many elements, it must contain at least one element not in $\{c_k : k < n\}$. So $\vec{e}$ contains d_i for some $i \in l \setminus \{i_0, \ldots, i_{n-1}\}$. Hence, $\vec{e}$ contains some element not in X_m. By the inductive hypothesis, we did not define $g_\tau(\vec{e})$ until some stage $m' > m$. Since $r \in Q_{m+1} \subseteq Q_{m'}$, at stage m' we defined $g_\tau(\vec{e})$ so that, if $f_\tau(\vec{e} \restriction \gamma) < r$, then $g_\tau(\vec{e}) < r$. But $f_\tau(\vec{e} \restriction \gamma) = g_\tau(\vec{e} \restriction \gamma) < r$, so indeed $g_\tau(\vec{e}) < r$. □

Proposition 3.10. *Assume the following:*

- *$\gamma < \delta < \omega_1$;*

- *$\{f_\tau : \tau \in I\}$ is a suitable specializing family with top level γ;*
- *$Q \subseteq \mathbb{Q}^+$, $A \subseteq I$, and $X \subseteq T_\delta$ are finite;*
- *X has unique drop-downs to γ.*

Then there exists a suitable specializing family $\{g_\tau : \tau \in I\}$ with top level δ satisfying:

(1) *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
(2) *for all $\tau \in A$, $X \restriction \gamma$ and X are (g_τ, Q)-consistent.*

Proof. The proof is by induction on δ, where the base case and the successor case follow easily from Lemma 3.9 and the inductive hypothesis. Assume that δ is a limit ordinal and the statement holds for all β such that $\gamma < \beta < \delta$. We will prove that the statement holds for δ.

We fix several objects in order to help with our construction. Fix an enumeration $\langle q_m : m < \omega\rangle$ of $\mathbb{Q}^+$, an enumeration $\langle z_m : m < \omega\rangle$ of T_δ, and an enumeration $\langle \tau_m : m < \omega\rangle$ of I (with repetitions if I is finite). Fix an increasing sequence $\langle \gamma_m : m < \omega\rangle$ of ordinals cofinal in δ with $\gamma_0 = \gamma$. Fix a surjection h from ω onto the set of all tuples of the form $(t, B, R, \vec{a}, \vec{i}, \vec{c})$, where:

- $t \subseteq T_\delta$, $B \subseteq I$, and $R \subseteq \mathbb{Q}^+$ are finite;
- $\vec{a} = (a_0, \dots, a_{l-1})$ is an injective tuple, where $l \geq n$, consisting of elements of T_α for some $\alpha < \delta$;
- $\vec{i} = (i_0, \dots, i_{n-1})$ is an injective tuple consisting of numbers less than l;
- $\vec{c} = (c_0, \dots, c_{n-1})$ is a tuple consisting of elements of $T_\delta \setminus t$ such that $a_{i_k} <_T c_k$ for all $k < n$.

We will define by induction in ω-many stages the following objects:

- a subset-increasing sequence $\langle X_m : m < \omega\rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_m : m < \omega\rangle$ of finite subsets of I with union equal to I;
- a subset-increasing sequence $\langle Q_m : m < \omega\rangle$ of finite subset of $\mathbb{Q}^+$ whose union is equal to $\mathbb{Q}^+$;
- a strictly increasing sequence $\langle \delta_m : m < \omega\rangle$ of ordinals cofinal in δ;
- for each $m < \omega$, a suitable specializing family $\{f^m_\tau : \tau \in I\}$ which has top level δ_m and satisfies that $f_\tau \subseteq f^m_\tau \subseteq f^{m'}_\tau$ for all $\tau \in I$ and $m' \geq m$;
- functions $h^m_\tau : (X_m)^{n+1} \cap (U^\tau)_\delta \to \mathbb{Q}$ for all $m < \omega$ and $\tau \in A_m$.

The following inductive hypotheses will be maintained for all $m < \omega$:

(a) X_m has unique drop-downs to δ_m;
(b) for all $\tau \in A_m$, $X_m \restriction \delta_m$ and $X_m \restriction \delta_{m+1}$ are (f^{m+1}_τ, Q_m)-consistent;
(c) for all $\tau \in A_m$ and $\vec{a} \in (X_m)^{n+1} \cap (U^\tau)_\delta$, if $\vec{a} < \vec{a}_\tau$ then $h^m_\tau(\vec{a}) = -1$, and if $\vec{a} \in T^\tau$ then $f^m_\tau(\vec{a} \restriction \delta_m) < h^m_\tau(\vec{a})$;
(d) for all $\tau \in A_m$ and $\vec{a} \in (X_m)^{n+1} \cap (U^\tau)_\delta$, $h^{m+1}_\tau(\vec{a}) = h^m_\tau(\vec{a})$.

<u>Stage 0</u>: Let $X_0 = X$, $A_0 = A$, $Q_0 = Q$, and $\delta_0 = \gamma$. For each $\tau \in I$, let $f^0_\tau = f_\tau$. Consider $\tau \in A$ and $\vec{a} \in X^{n+1} \cap (U^\tau)_\delta$, and we define $h^0_\tau(\vec{a})$. If $\vec{a} < \vec{a}_\tau$, then let $h^0_\tau(\vec{a}) = -1$. Suppose that $\vec{a} \in T^\tau$. Define $h^0_\tau(\vec{a})$ to be some positive rational number r such that $f_\tau(\vec{a} \restriction \gamma) < r$ and for all $q \in Q$, if $f_\tau(\vec{a} \restriction \gamma) < q$ then $r < q$. This is possible since Q is finite.

<u>Stage $m + 1$</u>: Let $m < \omega$ and assume that we have completed stage m. In particular, we have defined $X_m, A_m, Q_m, \delta_m, \{f^m_\tau : \tau \in I\}$, and $\{h^m_\tau : \tau \in A_m\}$ satisfying the required properties.

Let $h(m) = (t, B, R, \vec{a}, \vec{i}, \vec{c})$, where $\vec{a} = (a_0, \dots, a_{l-1})$ consists of elements of T_α, $l \geq n$, $\vec{i} = (i_0, \dots, i_{n-1})$, and $\vec{c} = (c_0, \dots, c_{n-1})$. Fix $\delta_{m+1} < \delta$ greater than δ_m, γ_{m+1}, and α such that the set $X_m \cup \{z_m\} \cup \{c_k : k < n\}$ has unique drop-downs to δ_{m+1}. It follows that $c_0 \restriction \delta_{m+1}, \dots, c_{n-1} \restriction \delta_{m+1}$ are not in $((X_m \cup \{z_m\}) \setminus \{c_k : k < n\}) \restriction \delta_{m+1}$. Apply the inductive hypothesis to find a suitable specializing family $\{f^{m+1}_\tau : \tau \in I\}$ with top level δ_{m+1} satisfying:

- $f^m_\tau \subseteq f^{m+1}_\tau$ for all $\tau \in I$;
- for all $\tau \in A_m$, $X_m \upharpoonright \delta_m$ and $X_m \upharpoonright \delta_{m+1}$ are $(f^{m+1}_\tau, Q_m \cup \bigcup\{\mathrm{ran}(h^m_\sigma) : \sigma \in A_m\})$-consistent.

Applying the fact that $\{f^{m+1}_\tau : \tau \in I\}$ is suitable, fix $b_0, \ldots, b_{l-1}$ in

$$T_{\delta_{m+1}} \setminus (((X_m \cup \{z_m\}) \setminus \{c_k : k < n\}) \upharpoonright \delta_{m+1})$$

such that $a_i <_T b_i$ for all $i < l$, $b_{i_k} = c_k \upharpoonright \delta_{m+1}$ for all $k < n$, and for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are (f^{m+1}_τ, R)-consistent. For each $i \in l \setminus \{i_k : k < n\}$ pick some $d_i \in T_\delta \setminus (X_m \cup t)$ above b_i. Also, let $d_{i_k} = c_k$ for all $k < n$. Define $X_{m+1} = X_m \cup \{d_k : k < l\} \cup \{z_m\}$, $Q_{m+1} = Q_m \cup R \cup \{q_m\}$, and $A_{m+1} = A_m \cup B \cup \{\tau_m\}$. Note that X_{m+1} has unique drop-downs to δ_{m+1}.

Consider $\tau \in A_{m+1}$ and $\vec{a} \in X^{n+1}_{m+1} \cap (U^\tau)_\delta$, and we define $h^{m+1}_\tau(\vec{a})$. If $\tau \in A_m$ and $\vec{a} \in X^{n+1}_m$, then $h^m_\tau(\vec{a})$ is already defined, and we let $h^{m+1}_\tau(\vec{a}) = h^m_\tau(\vec{a})$. Assume that either $\tau \notin A_m$ or $\vec{a} \notin X^{n+1}_m$. If $\vec{a} < \vec{a}_\tau$, then let $h^{m+1}_\tau(\vec{a}) = -1$. Suppose that $\vec{a} \in T^\tau$. Define $h^{m+1}_\tau(\vec{a})$ to be some positive rational number r such that $f^{m+1}_\tau(\vec{a} \upharpoonright \delta_{m+1}) < r$ and for all $q \in Q_{m+1}$, if $f^{m+1}_\tau(\vec{a} \upharpoonright \delta_{m+1}) < q$ then $r < q$. This is possible since Q_{m+1} is finite.

Suppose that $\vec{e} \in \{d_0, \ldots, d_{l-1}\}^{n+1} \cap T^\tau$. Since $\vec{e}$ has $(n+1)$-many elements and $\{d_0, \ldots, d_{l-1}\} \cap X_m \subseteq \{c_k : k < n\}$, $\vec{e}$ contains a member which is not in X_m. By construction, for all $\tau \in B$ and for all $q \in R$, if $f^{m+1}_\tau(\vec{e} \upharpoonright \delta_{m+1}) < q$ then $h^{m+1}_\tau(\vec{e}) < q$. Also, for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are (f^{m+1}_τ, R)-consistent. So for all $\tau \in B$ and for all $q \in R$, if $f^{m+1}_\tau(\vec{e} \upharpoonright \alpha) < q$ then $h^{m+1}_\tau(\vec{e}) < q$.

This completes stage $m + 1$. It is routine to check that the required properties hold.

This completes the construction. For all $\tau \in I$, let

$$h_\tau = \bigcup\{h^m_\tau : m < \omega,\ \tau \in A_m\}.$$

Then h_τ is a function from $(U^\tau)_\delta$ to $\mathbb{Q}$ since $\bigcup_m X_m = T_\delta$, $\bigcup_m A_m = I$, and by inductive hypothesis (d). Let

$$g_\tau = \bigcup\{f^m_\tau : m < \omega,\ \tau \in A_m\} \cup h_\tau.$$

Inductive hypothesis (c) easily implies that for all $\tau \in I$, g_τ is a specializing function on $U^\tau \upharpoonright (\delta + 1)$. Obviously, $f_\tau \subseteq g_\tau$ for all $\tau \in I$. By what we did at stage 0, for all $\tau \in A$, $X \upharpoonright \gamma$ and X are (g_τ, Q)-consistent.

It remains to prove that $\{g_\tau : \tau \in I\}$ is suitable. Let $\alpha < \delta$, and suppose that $B \subseteq I$, $R \subseteq \mathbb{Q}^+$, and $t \subseteq T_\delta$ are finite. Assume that $l \geq n$, $\vec{a} = (a_0, \ldots, a_{l-1})$ is an injective tuple consisting of elements of T_α, and $i_0, \ldots, i_{n-1} < l$ are distinct. For each $k < n$, let $c_k \in T_\delta \setminus t$ above a_{i_k}. Fix $m < \omega$ such that $h(m) = (t, B, R, \vec{a}, \vec{i}, \vec{c})$. Reviewing what we did in case $m + 1$, there are $d_0, \ldots, d_{l-1}$ in $T_\delta \setminus t$ such that $a_i <_T d_i$ for all $i < l$, $d_{i_k} = c_k$ for all $k < n$, and for all $\tau \in B$ and for all $q \in R$, if $g_\tau(\vec{e} \upharpoonright \alpha) = f^{m+1}_\tau(\vec{e} \upharpoonright \alpha) < q$ then $g_\tau(\vec{e}) = h^{m+1}_\tau(\vec{e}) < q$. In other words, for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{d_0, \ldots, d_{l-1}\}$ are (g_τ, R)-consistent. □

Proposition 3.11. *Assume the following:*

- $\gamma < \delta < \omega_1$;
- $\{g_\tau : \tau \in I\}$ *is a suitable specializing family with top level* δ;
- $\sigma \in \omega_1 \setminus I$ *and* f_σ *is a specializing function with top level* γ;
- $\{f_\sigma\} \cup \{g_\tau \upharpoonright (\gamma + 1) : \tau \in I\}$ *is suitable;*
- $Q \subseteq \mathbb{Q}^+$ *and* $X \subseteq T_\delta$ *are finite and* X *has unique drop-downs to* γ.

Then there exists a specializing function g_σ *with top level* δ *such that:*

(1) $f_\sigma \subseteq g_\sigma$;

(2) *$\{g_\tau : \tau \in I \cup \{\sigma\}\}$ is suitable;*
(3) *$X \upharpoonright \gamma$ and X are (g_σ, Q)-consistent.*

The proof is a variation of the proofs of Lemma 3.9 and Proposition 3.10. We leave it as an exercise for the interested reader.

3.3. **The Forcing Poset for Specializing Derived Trees.**

Definition 3.12. *For each $\tau < \omega_1$, let $\mathbb{Q}_\tau$ be the forcing poset whose conditions are all specializing functions on $U^\tau \upharpoonright (\alpha + 1)$, for some $\alpha < \omega_1$, ordered by $q \leq_{\mathbb{Q}_\tau} p$ if $p \subseteq q$. If $p \in \mathbb{Q}_\tau$ is a specializing function on $U^\tau \upharpoonright (\alpha + 1)$, then we refer to α as the* top level *of p.*

Definition 3.13. *Let $\mathbb{P}$ be the forcing poset whose conditions are all functions p satisfying:*

(1) *the domain of p is a countable subset of ω_1;*
(2) *there exists an ordinal $\alpha < \omega_1$, which we call the* top level *of p, such that for all $\tau \in \mathrm{dom}(p)$, $p(\tau)$ is a specializing function on $U^\tau \upharpoonright (\alpha + 1)$;*
(3) *the family $\{p(\tau) : \tau \in \mathrm{dom}(p)\}$ is suitable.*

Let $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\tau \in \mathrm{dom}(p)$, $p(\tau) \subseteq q(\tau)$.

Definition 3.14 (Consistency)**.** *Let $\alpha < \beta < \omega_1$ and $\tau < \omega_1$. Let $p \in \mathbb{Q}_\tau$ have top level β. Let $Q \subseteq \mathbb{Q}^+$ be finite.*

(1) *Let $X \subseteq T_\beta$ be finite with unique drop-downs to α. We say that $X \upharpoonright \alpha$ and X are (p, Q)-*consistent *if for all $r \in Q$, for any $\vec{c} \in T^\tau \cap X^{n+1}$, if $p(\vec{c} \upharpoonright \alpha) < r$ then $p(\vec{c}) < r$.*
(2) *Let $\vec{a} = (a_0, \ldots, a_{l-1})$ be an injective tuple consisting of elements of T_β. We say that $\vec{a} \upharpoonright \alpha$ and $\vec{a}$ are (p, Q)-*consistent *if $\{a_0, \ldots, a_{l-1}\} \upharpoonright \alpha$ and $\{a_0, \ldots, a_{l-1}\}$ are (p, Q)-consistent.*

We have now defined for the subtree forcing the objects and properties described in Section 2, where we can consider each statement of Section 2 about consistency as relative to a fixed finite set $Q \subseteq \mathbb{Q}^+$.

We now work towards verifying properties (A)-(E) of Section 2. (A) is clear. (B) (Transitivity) follows from Lemma 3.6 and (C) (Persistence) is automatically true. The next lemma implies (D) (Extension).

Lemma 3.15 (Extension)**.** *Let $\alpha < \beta < \omega_1$, let $Q \subseteq \mathbb{Q}^+$ be finite, and let $X \subseteq T_\beta$ be finite with unique drop-downs to α. Suppose that $p \in \mathbb{P}$ has top level α and $A \subseteq \mathrm{dom}(p)$ finite. Then there exists some $q \leq p$ with top level β and with the same domain as p such that for all $\tau \in A$, $X \upharpoonright \alpha$ and X are $(q(\tau), Q)$-consistent.*

Proof. Immediate from Proposition 3.10. □

Finally, (E) (Key Property) holds by the next proposition.

Proposition 3.16 (Strong n-Key Property)**.** *Let $\alpha < \beta < \omega_1$, let $t \subseteq T_\beta$ be finite, and let $p \in \mathbb{P}$ have top level α. Suppose that $a_0, \ldots, a_{l-1}$ are distinct elements of T_α, where $l \geq n$. Let $Q \subseteq \mathbb{Q}^+$ and $A \subseteq \mathrm{dom}(p)$ be finite sets. Assume that $i_0, \ldots, i_{n-1} < l$ are distinct, and for each $k < n$, $c_k \in T_\beta \setminus t$ is above a_{i_k}. Then for any $q \leq p$ with top level β, there exist $b_0, \ldots, b_{l-1}$ in $T_\beta \setminus t$ such that $a_i <_T b_i$ for all $i < l$, $b_{i_k} = c_k$ for all $k < n$, and for all $\tau \in A$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are $(q(\tau), Q)$-consistent.*

Proof. Immediate from Definition 3.8 letting $\mathcal{G} = \{q(\tau) : \tau \in \mathrm{dom}(q)\}$, which is suitable by the definition of $\mathbb{P}$. □

In other words, $\mathbb{P}$ satisfies the Strong n-Key Property essentially by definition. While this may seem like cheating, it is justified by the constructions of Subsection 3.2.

The next lemma follows immediately from Propositions 3.10 and 3.11.

Lemma 3.17. *For any $\tau < \omega_1$, $\rho < \omega_1$, and $p \in \mathbb{P}$, there exists some $q \leq p$ with $\tau \in \mathrm{dom}(q)$ and q has top level at least ρ.*

3.4. **Basic Properties of the Specializing Forcing.** In this subsection we will prove, assuming that T is a free Suslin tree, that the forcing poset $\mathbb{P}$ is totally proper and forces that T is n-free. Note that under CH, $\mathbb{P}$ has cardinality ω_1, so it preserves all cardinals. Also, by a density argument using Lemma 3.17, $\mathbb{P}$ specializes all derived trees of T with dimension $n + 1$.

Theorem 3.18. *Suppose that T is a free Suslin tree. Then the forcing poset $\mathbb{P}$ is totally proper and forces that T is n-free.*

Proof. Let $\vec{x}$ be an injective tuple in T^n and let $\dot{E}$ be a $\mathbb{P}$-name for a dense open subset of the derived tree $T_{\vec{x}}$. Let λ be a large enough regular cardinal. Suppose that N is a countable elementary substructure of $H(\lambda)$ containing as members T, $\langle \mathbb{Q}_\tau : \tau < \omega_1 \rangle$, $\mathbb{P}$, and $\vec{x}$. Let $\delta = N \cap \omega_1$. We will prove that for any $p \in N \cap \mathbb{P}$, there exists a total master condition $r \leq p$ over N such that r forces that every tuple in $T_{\vec{x}}$ whose members have height δ is in $\dot{E}$. It easily follows that $\mathbb{P}$ is totally proper and forces that every derived tree of T with dimension n is Suslin.

So let N, δ, and p be given as above, and let β be the top level of p. To help with our construction, we fix the following objects:

- an increasing sequence $\langle \gamma_m : m < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \beta$;
- an enumeration $\langle z_m : m < \omega \rangle$ of T_δ;
- an enumeration $\langle \vec{y}^m : m < \omega \rangle$ of all of the tuples in $T_{\vec{x}}$ whose members have height δ;
- an enumeration $\{\tau_m : m < \omega\}$ of δ;
- an enumeration $\langle D_m : m < \omega \rangle$ of all dense open subsets of $\mathbb{P}$ which lie in N.

Fix a surjection h from ω onto the set of all tuples of the form $(t, B, R, \vec{a}, \vec{i}, \vec{c})$, where:

- $t \subseteq T_\delta$, $B \subseteq \delta$, and $R \subseteq \mathbb{Q}^+$ are finite;
- $\vec{a} = (a_0, \ldots, a_{l-1})$ is an injective tuple, where $l \geq n$, consisting of elements of T_α for some $\alpha < \delta$;
- $\vec{i} = (i_0, \ldots, i_{n-1})$ is an injective tuple consisting of numbers less than l;
- $\vec{c} = (c_0, \ldots, c_{n-1})$ is a tuple consisting of elements of T_δ, where $a_{i_k} <_T c_k$ for all $k < n$.

We will define by induction in ω-many stages the following objects:

- a subset-increasing sequence $\langle X_m : m < \omega \rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_m : m < \omega \rangle$ of finite subsets of δ with union equal to δ;
- a subset-increasing sequence $\langle Q_m : m < \omega \rangle$ of finite subset of $\mathbb{Q}^+$ with union equal to $\mathbb{Q}^+$;
- a strictly increasing sequence $\langle \delta_m : m < \omega \rangle$ of ordinals cofinal in δ;
- a descending sequence $\langle p_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}$, where $p_0 = p$ and each p_n has top level δ_n;
- functions $h^m_\tau : (X_m)^{n+1} \cap (U^\tau)_\delta \to \mathbb{Q}$ for all $m < \omega$ and $\tau \in A_m$.

The following inductive hypotheses will be maintained for each $m < \omega$:

(a) X_m has unique drop-downs to δ_m and $A_m \subseteq \mathrm{dom}(p_m)$;
(b) for all $\tau \in A_m$, $X_m \restriction \delta_m$ and $X_m \restriction \delta_{m+1}$ are $(p_{m+1}(\tau), Q_m)$-consistent;
(c) for all $\tau \in A_m$ and for all $\vec{a} \in (X_m)^{n+1} \cap (U^\tau)_\delta$, if $\vec{a} < \vec{a}_\tau$ then $h^m_\tau(\vec{a}) = -1$, and if $\vec{a} \in T^\tau$ then $p_m(\tau)(a \restriction \delta_m) < h^m_\tau(\vec{a})$;
(d) for all $\tau \in A_m$ and for all $\vec{a} \in (X_m)^{n+1} \cap (U^\tau)_\delta$, $h^{m+1}_\tau(\vec{a}) = h^m_\tau(\vec{a})$.

Stage 0: Let $X_0 = \emptyset$, $A_0 = \emptyset$, $Q_0 = \emptyset$, $\delta_0 = \beta$, and $p_0 = p$. The required properties clearly hold.

Stage $m+1$: Let $m < \omega$ and assume that we have completed stage m. In particular, we have defined X_m, A_m, Q_m, δ_m, p_m, and h^m_τ for all $\tau \in A_m$ satisfying the required properties. Let $h(m) = (t, B, R, \vec{a}, \vec{i}, \vec{c})$, where $\vec{a} = (a_0, \ldots, a_{l-1})$ consists of elements of T_α, $l \geq n$, $\vec{i} = (i_0, \ldots, i_{n-1})$, and $\vec{c} = (c_0, \ldots, c_{n-1})$. Fix $\rho < \delta$ larger than δ_m, γ_{m+1}, and α such that

$$X'_m = X_m \cup \{z_m, c_0, \ldots, c_{n-1}\} \cup t$$

has unique drop-downs to ρ. Stage $m+1$ will consist of three steps.

Step 0: Apply Proposition 2.2 (Consistent Extension Into Dense Sets) and Lemma 3.17 to fix some $p_{m,0} \leq p_m$ in $N \cap D_m$ with some top level $\delta_{m,0}$ which is greater than ρ such that $B \cup \{\tau_m\} \subseteq \mathrm{dom}(p_{m,0})$ and for all $\tau \in A_m$, $X_m \restriction \delta_m$ and $X_m \restriction \delta_{m,0}$ are $(p_{m,0}(\tau), Q_m \cup \bigcup\{\mathrm{ran}(h^m_\sigma) : \sigma \in A_m\})$-consistent.

Step 1: Apply Proposition 2.4 (Consistent Extensions for Sealing) to fix some $p_{m+1} \leq p_{m,0}$ in $N \cap \mathbb{P}$ with some top level δ_{m+1} such that $p_{m+1} \Vdash_{\mathbb{P}} \vec{y}^m \in \dot{E}$ and for all $\tau \in A_m$, $X_m \restriction \delta_{m,0}$ and $X_m \restriction \delta_{m+1}$ are $(p_{m+1}(\tau), Q_m \cup \bigcup\{\mathrm{ran}(h^m_\sigma) : \sigma \in A_m\})$-consistent.

Step 2: Note that by unique drop-downs, $c_0 \restriction \delta_{m+1}, \ldots, c_{n-1} \restriction \delta_{m+1}$ are not in

$$(X'_m \setminus \{c_0, \ldots, c_{n-1}\}) \restriction \delta_{m+1}.$$

Applying the fact that $\{p_{m+1}(\tau) : \tau \in \mathrm{dom}(p_{m+1})\}$ is suitable, fix $b_0, \ldots, b_{l-1}$ in

$$T_{\delta_{m+1}} \setminus ((X'_m \setminus \{c_0, \ldots, c_{n-1}\}) \restriction \delta_{m+1})$$

such that $a_i <_T b_i$ for all $i < l$, $b_{i_k} = c_k \restriction \delta_{m+1}$ for all $k < n$, and for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are $(p_{m+1}(\tau), R)$-consistent. For each $i \in l \setminus \{i_k : k < n\}$ pick some $d_i \in T_\delta \setminus X'_m$ above b_i. Also, let $d_{i_k} = c_k$ for all $k < n$. Note that for all $i < l$, $d_i \notin t$. Define $X_{m+1} = X'_m \cup \{d_i : i < l\}$, $A_{m+1} = A_m \cup B \cup \{\tau_m\}$, and $Q_{m+1} = Q_m \cup R \cup \{q_m\}$. Note that X_{m+1} has unique drop-downs to δ_{m+1}.

Consider $\tau \in A_{m+1}$ and $\vec{a} \in X^{n+1}_{m+1} \cap (U^\tau)_\delta$, and we define $h^{m+1}_\tau(\vec{a})$. If $\tau \in A_m$ and $\vec{a} \in X^{n+1}_m$, then $h^m_\tau(\vec{a})$ is already defined, and we let $h^{m+1}_\tau(\vec{a}) = h^m_\tau(\vec{a})$. Assume that either $\tau \notin A_m$ or $\vec{a} \notin X^{n+1}_m$. If $\vec{a} < \vec{a}_\tau$ then let $h^{m+1}_\tau(\vec{a}) = -1$. Suppose that $\vec{a} \in T^\tau$. Define $h^{m+1}_\tau(\vec{a})$ to be some positive rational number r such that $p_{m+1}(\tau)(\vec{a} \restriction \delta_{m+1}) < r$ and for all $q \in Q_{m+1}$, if $p_{m+1}(\tau)(\vec{a} \restriction \delta_{m+1}) < q$ then $r < q$. This is possible since Q_{m+1} is finite.

Suppose that $\vec{e} \in \{d_0, \ldots, d_{l-1}\}^{n+1} \cap T^\tau$. Since $\vec{e}$ has $(n+1)$-many elements and the intersection of $\{d_0, \ldots, d_{l-1}\}$ and X_m is a subset of $\{c_k : k < n\}$, $\vec{e}$ contains a member which is not in X_m. By construction, for all $\tau \in B$ and for all $q \in R$, if $p_{m+1}(\tau)(\vec{e} \restriction \delta_{m+1}) < q$ then $h^{m+1}_\tau(\vec{e}) < q$. Also, for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{b_0, \ldots, b_{l-1}\}$ are $(p_{m+1}(\tau), R)$-consistent. So for all $\tau \in B$ and for all $q \in R$, if $p_{m+1}(\tau)(\vec{e} \restriction \alpha) < q$ then $h^{m+1}_\tau(\vec{e}) < q$.

This completes stage m. It is routine to check that the required properties hold.

This completes the construction. We define a condition r with top level δ and domain δ as follows. For all $\tau \in \delta$, let

$$h_\tau = \bigcup\{h^m_\tau : m < \omega,\ \tau \in A_m\}.$$

Then h_τ is a function from $(U^\tau)_\delta$ to $\mathbb{Q}$ since $\bigcup_m X_m = T_\delta$, $\bigcup_m A_m = \delta$, and by inductive hypothesis (d). Now let

$$r(\tau) = \bigcup\{p_m(\tau) : m < \omega,\ \tau \in \mathrm{dom}(p_m)\} \cup h_\tau.$$

Using inductive hypothesis (c), it is easy to check that each $r(\tau)$ is a specializing function on $(U^\tau) \restriction (\delta + 1)$.

In order to prove that r is a condition, we need to show that $\{r(\tau) : \tau \in \delta\}$ is suitable. Let $B \subseteq \delta$, $R \subseteq \mathbb{Q}^+$, and $t \subseteq T_\delta$ be finite. Suppose that $\alpha < \delta$, $l \geq n$, $a_0, \ldots, a_{l-1}$ are distinct elements of T_α, $i_0, \ldots, i_{n-1} < l$ are distinct, and for each $k < n$, $c_k \in T_\delta \setminus t$ is above a_{i_k}. Fix m such that $h(m) = (B, R, t, \vec{a}, \vec{i}, \vec{c})$, where $\vec{a} = (a_0, \ldots, a_{l-1})$, $\vec{i} = (i_0, \ldots, i_{n-1})$, and $\vec{c} = (c_0, \ldots, c_{n-1})$. Reviewing what we did in case $m+1$, there are $d_0, \ldots, d_{l-1}$ in $T_\delta \setminus t$ such that $a_i <_T d_i$ for all $i < l$, $d_{i_k} = c_k$ for all $k < n$, and for all $\tau \in B$ and for all $q \in R$, if $r(\tau)(\vec{e} \upharpoonright \alpha) = p_{m+1}(\tau)(\vec{e} \upharpoonright \alpha) < q$ then $r(\tau)(\vec{e}) = h_\tau^{m+1}(\vec{e}) < q$. In other words, for all $\tau \in B$, $\{a_0, \ldots, a_{l-1}\}$ and $\{d_0, \ldots, d_{l-1}\}$ are $r(\tau)$-consistent.

So $r \in \mathbb{P}$ and clearly $r \leq p_n$ for all $n < \omega$. By our bookkeeping, r is a total master condition over N and r forces that level δ of the derived tree $T_{\vec{x}}$ is contained in $\dot{E}$. □

Corollary 3.19. *Assuming that T is a free Suslin tree, the forcing poset $\mathbb{P}$ forces that T is an n-free Suslin tree all of whose derived trees with dimension $n+1$ are special. If CH holds, then $\mathbb{P}$ preserves all cardinals.*

4. Application II: Adding Subtrees of a Free Suslin Tree

In this section we give our second application of the abstract framework of Section 2: adding almost disjoint uncountable downwards closed subtrees of a free Suslin tree. In this example we will make use of most of the main ideas of the article, but in a form which is simpler and easier to understand than the automorphism forcing of Section 5. One of the reasons for this relative simplicity is that separation for a tuple in the subtree forcing depends on a property of the elements of the tuple considered one at a time, rather than on how the elements of the tuple relate to each other (compare Definitions 4.4 and 5.3). We recommend that the reader use this section as a warm-up for Section 5.

4.1. **Consistency and Separation for Subtree Functions.**

Definition 4.1 (Subtree Functions)**.** *Let $\beta < \omega_1$. A function $g : T \upharpoonright (\beta + 1) \to 2$ is called a* subtree function on $T \upharpoonright (\beta + 1)$ *if:*

1. *the value of g on the root of T equals 1;*
2. *if $g(a) = 1$ then for all $\gamma < \mathrm{ht}_T(a)$, $g(a \upharpoonright \gamma) = 1$;*
3. *for all $a \in T \upharpoonright (\beta + 1)$ and for any ξ with $\mathrm{ht}_T(a) < \xi \leq \beta$, there exist infinitely many $b \in T_\xi$ above a such that $g(b) = 1$.*

In the above, we refer to β as the top level of g.

If g is a subtree function on $T \upharpoonright (\beta+1)$ and $\alpha < \beta$, we will write $g \upharpoonright (\alpha+1)$ for $g \upharpoonright (T \upharpoonright (\alpha+1))$, which is easily seen to be a subtree function on $T \upharpoonright (\alpha + 1)$. If $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of subtree functions on $T \upharpoonright (\beta + 1)$, we will write $\mathcal{G} \upharpoonright (\alpha + 1)$ for the indexed family $\{g_\tau \upharpoonright (\alpha + 1) : \tau \in I\}$.

Definition 4.2 (Consistency)**.** *Let $\alpha < \beta < \omega_1$ and let $g : T \upharpoonright (\beta + 1) \to 2$ be a subtree function.*

1. *Let $X \subseteq T_\beta$ be finite with unique drop-downs to α. We say that $X \upharpoonright \alpha$ and X are g-*consistent *if for all $x \in X$, $g(x \upharpoonright \alpha) = 1$ iff $g(x) = 1$.*
2. *Let $\vec{a} = (a_0, \ldots, a_{n-1})$ be an injective tuple consisting of elements of T_β. We say that $\vec{a} \upharpoonright \alpha$ and $\vec{a}$ are g-*consistent *if for all $i < n$, $g(a_i \upharpoonright \alpha) = 1$ iff $g(a_i) = 1$.*

Note that in (1) above, the sets $X \upharpoonright \alpha$ and X are g-consistent if and only if for all $x \in X$, $g(x \upharpoonright \alpha) = 1$ implies that $g(x) = 1$. For the reverse implication follows from (2) of Definition 4.1 (Subtree Functions). A similar comment applies to (2).

The following lemma is easy to check.

Lemma 4.3 (Transitivity)**.** *Let $\alpha < \beta < \gamma < \omega_1$ and let $X \subseteq T_\gamma$ be a finite set with unique drop-downs to α. Let g be a subtree function on $T \upharpoonright (\gamma+1)$. If $X \upharpoonright \alpha$ and $X \upharpoonright \beta$ are $(g \upharpoonright (\beta+1))$-consistent and $X \upharpoonright \beta$ and X are g-consistent, then $X \upharpoonright \alpha$ and X are g-consistent.*

Definition 4.4 (Separation)**.** *Let $\alpha < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ an indexed family of subtree functions on $T \upharpoonright (\alpha+1)$ and $X \subseteq T_\alpha$. We say that $\mathcal{G}$ is* separated on X *if for all $x \in X$ there exists at most one $\tau \in I$ such that $g_\tau(x) = 1$. We say that $\mathcal{G}$ is* separated *if it is separated on T_α.*

Note that separation is defined for indexed families of subtree functions, rather than for sets of subtree functions. This is required in order for the subtree forcing of Subsection 4.3 to satisfy property (C) (Persistence) from Section 2.

Lemma 4.5 (Persistence)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ an indexed family of subtree functions on $T \upharpoonright (\beta+1)$. Let $X \subseteq T_\alpha$. If $\mathcal{G} \upharpoonright (\alpha+1)$ is separated on X, then $\mathcal{G}$ is separated on $\{y \in T_\beta : y \upharpoonright \alpha \in X\}$. In particular, if $\mathcal{G} \upharpoonright (\alpha+1)$ is separated, then $\mathcal{G}$ is separated.*

Proof. Consider $y \in T_\beta$ such that $y \upharpoonright \alpha \in X$. By (2) of Definition 4.1 (Subtree Functions), if $\tau \in I$ and $g_\tau(y) = 1$, then $g_\tau(y \upharpoonright \alpha) = 1$. Since $\mathcal{G} \upharpoonright (\alpha+1)$ is separated on X, there exists at most one such τ. □

Proposition 4.6 (Key Property)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $a_0, \dots, a_{n-1}$ are distinct elements of T_α and $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of subtree functions on $T \upharpoonright (\beta+1)$ such that $\mathcal{G} \upharpoonright (\alpha+1)$ is separated on $\{a_0, \dots, a_{n-1}\}$. Let $t \subseteq T_\beta$ be finite. Then there exist $b_0, \dots, b_{n-1}$ in $T_\beta \setminus t$ such that $a_i <_T b_i$ for all $i < n$ and for all $\tau \in I$, $\{a_0, \dots, a_{n-1}\}$ and $\{b_0, \dots, b_{n-1}\}$ are g_τ-consistent.*

Proof. Consider $i < n$ and we will choose b_i. If there does not exist any $\tau \in I$ such that $g_\tau(a_i) = 1$, then let b_i be an arbitrary element of $T_\beta \setminus t$ above a_i. This is possible since T is infinitely splitting. Otherwise by the separation assumption, there exists a unique $\tau \in I$ such that $g_\tau(a_i) = 1$. By (3) of Definition 4.1 (Subtree Functions), there are infinitely many $b \in T_\beta$ above a_i such that $g_\tau(b) = 1$. So we can pick some $b_i \in T_\beta \setminus t$ above a_i such that $g_\tau(b_i) = 1$. Now note that for any $\tau \in I$ and $i < n$, $g_\tau(a_i) = 1$ implies that $g_\tau(b_i) = 1$. □

4.2. **Constructing and Extending Subtree Functions.**

Proposition 4.7. *Assume the following:*

- *$\gamma < \beta < \omega_1$;*
- *$X \subseteq T_\beta$ is finite and has unique drop-downs to γ;*
- *$\{f_\tau : \tau \in I\}$ is a non-empty countable collection of subtree functions on $T \upharpoonright (\gamma+1)$;*
- *$A \subseteq I$ is finite.*

Then there exists a family $\{g_\tau : \tau \in I\}$ of subtree functions on $T \upharpoonright (\beta+1)$ satisfying:

(1) *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
(2) *for all $\tau \in A$, $X \upharpoonright \gamma$ and X are g_τ-consistent;*
(3) *if $\{f_\tau : \tau \in A\}$ is separated on $X \upharpoonright \gamma$, then $\{g_\tau : \tau \in I\}$ is separated.*

Proof. We begin by defining a family $\{f_\tau^+ : \tau \in I\}$ of subtree functions on $T \upharpoonright (\gamma+2)$. For each $\tau \in I$, let $f_\tau^+ \upharpoonright (\gamma+1) = f_\tau$. For every $\tau \in A$ and $x \in X \upharpoonright (\gamma+1)$, define $f_\tau^+(x) = 1$ if $f_\tau(x \upharpoonright \gamma) = 1$ and $f_\tau^+(x) = 0$ if $f_\tau(x \upharpoonright \gamma) = 0$. For every $\tau \in I \setminus A$ and $x \in X \upharpoonright (\gamma+1)$, define $f_\tau^+(x) = 0$.

For each $a \in T_\gamma$, fix a partition $\{Z_{a,\tau} : \tau \in I\}$ of the set of immediate successors of a minus the elements of $X \upharpoonright (\gamma+1)$ into infinite sets. This is possible since X is finite, I is countable, and T is infinitely splitting. Consider $b \in T_{\gamma+1} \setminus (X \restriction (\gamma+1))$, and let a be the immediate predecessor of b. Fix $\tau \in I$ such that $b \in Z_{a,\tau}$. Define $f_\tau^+(b) = 1$ if $f_\tau(a) = 1$ and $f_\tau^+(b) = 0$ if $f_\tau(a) = 0$. For each $\sigma \in I$ different from τ, define $f_\sigma^+(b) = 0$.

It is easy to check that each f_τ^+ is a subtree function on $T \restriction (\gamma+2)$ and for all $\tau \in A$, $X \restriction \gamma$ and $X \restriction (\gamma+1)$ are f_τ^+-consistent. Suppose that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$, and we will prove that $\{f_\tau^+ : \tau \in I\}$ is separated. Consider $b \in T_{\gamma+1}$ and let a be its immediate predecessor. If $b \in X \restriction (\gamma+1)$, then $a \in X \restriction \gamma$, so by separation there exists at most one $\tau \in A$ such that $f_\tau(a) = 1$. So by construction, there exists at most one $\tau \in A$ such that $f_\tau^+(b) = 1$, and for all $\sigma \in I \setminus A$, $f_\sigma^+(b) = 0$. If $b \in T_{\gamma+1} \setminus (X \restriction (\gamma+1))$, then for some $\tau \in I$, $b \in Z_{a,\tau}$. By definition, $f_\sigma^+(b) = 1$ implies $\sigma = \tau$. So there is at most one $\tau \in I$ such that $f_\tau^+(b) = 1$.

If $\beta = \gamma+1$, then we are done, so assume that $\gamma+1 < \beta$. Let $\tau \in I$. Define $g_\tau \restriction (\gamma+2) = f_\tau^+$. Consider $b \in T \restriction (\beta+1)$ such that $\mathrm{ht}_T(b) > \gamma+1$. Define $g_\tau(b) = 1$ if $f_\tau^+(b \restriction (\gamma+1)) = 1$ and $g_\tau(b) = 0$ if $f_\tau(b \restriction (\gamma+1)) = 0$. It is easy to check that each g_τ is a subtree function. If $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \alpha$, then $\{f_\tau^+ : \tau \in I\}$ is separated. By Lemma 4.5 (Persistence), $\{g_\tau : \tau \in I\}$ is separated. □

The following variation of Proposition 4.7 will be used in Subsection 4.5 for showing that the forcing poset for adding subtrees to a free Suslin tree does not add new cofinal branches of any ω_1-tree in the ground model.

Lemma 4.8. *Assume the following:*

- $\gamma < \omega_1$*;*
- *C and D are disjoint finite subsets of $T_{\gamma+1}$, each with unique drop-downs to γ, such that $C \restriction \gamma = D \restriction \gamma$;*
- *$\{f_\tau : \tau \in I\}$ is a non-empty countable collection of subtree functions on $T \restriction (\gamma+1)$;*
- *$A \subseteq I$ is finite and $\{f_\tau : \tau \in A\}$ is separated on $C \restriction \gamma$.*

Then there exists a family $\{g_\tau : \tau \in I\}$ of subtree functions on $T \restriction (\gamma+2)$ satisfying:

1. *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
2. *for all $\tau \in A$, $C \restriction \gamma$ and C are g_τ-consistent and $D \restriction \gamma$ and D are g_τ-consistent;*
3. *$\{g_\tau : \tau \in A\}$ is separated on $C \cup D$.*

Proof. For each $\tau \in I$, define $g_\tau \restriction (\gamma+1) = f_\tau$. For every $x \in C \cup D$ and $\tau \in A$, define $g_\tau(x) = 1$ if $f_\tau(x \restriction \gamma) = 1$ and $g_\tau(x) = 0$ if $f_\tau(x \restriction \gamma) = 0$. For every $x \in C \cup D$ and $\tau \in I \setminus A$, define $g_\tau(x) = 0$. For any $b \in T_{\gamma+1} \setminus (C \cup D)$ and $\tau \in I$, define $g_\tau(b) = 1$ if $f_\tau(b \restriction \gamma) = 1$ and $g_\tau(b) = 0$ if $f_\tau(b \restriction \gamma) = 0$.

It is easy to check that each g_τ is a subtree function and $f_\tau \subseteq g_\tau$. Clearly, for all $\tau \in A$, $C \restriction \gamma$ and C are g_τ-consistent and $D \restriction \gamma$ and D are g_τ-consistent. To show that $\{g_\tau : \tau \in A\}$ is separated on $C \cup D$, consider $x \in C \cup D$. Then $x \restriction \gamma \in C \restriction \gamma$, so there is at most one $\tau \in A$ such that $f_\tau(x \restriction \gamma) = 1$. Since $g_\tau(x) = 1$ implies $f_\tau(x \restriction \gamma) = 1$ for any $\tau \in I$, there is at most one $\tau \in A$ such that $g_\tau(x) = 1$. □

4.3. **The Forcing Poset for Adding Subtrees.**

Definition 4.9. *Let $\mathbb{Q}$ be the forcing poset whose conditions are all subtree functions on $T \restriction (\alpha+1)$, for some $\alpha < \omega_1$, ordered by $q \leq_{\mathbb{Q}} p$ if $p \subseteq q$. If $p \in \mathbb{Q}$ is a subtree function on $T \restriction (\alpha+1)$, then α is the* top level *of p.*

Definition 4.10. *Let $\mathbb{P}$ be the forcing poset whose conditions are all functions p satisfying:*

1. *the domain of p is a countable subset of κ;*
2. *there exists an ordinal $\alpha < \omega_1$, which we call the* top level *of p, such that for all $\tau \in \mathrm{dom}(p)$, $p(\tau)$ is a subtree function on $T \restriction (\alpha+1)$.*

Let $q \le p$ *if* $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ *and for all* $\tau \in \mathrm{dom}(p)$*,* $p(\tau) \subseteq q(\tau)$*.*

Definition 4.11 (Consistency)**.** *Let* $\alpha < \beta < \omega_1$ *and let* $q \in \mathbb{Q}$*.*

(1) *Let* $X \subseteq T_\beta$ *be finite with unique drop-downs to* α*. We say that* $X \restriction \alpha$ *and* X *are* q-consistent *if for all* $x \in X$*,* $q(x \restriction \alpha) = 1$ *iff* $q(x) = 1$*;*
(2) *Let* $\vec{a} = (a_0, \dots, a_{n-1})$ *be an injective tuple consisting of elements of* T_β*. We say that* $\vec{a} \restriction \alpha$ *and* $\vec{a}$ *are* q-consistent *if for all* $i < n$*,* $q(a_i \restriction \alpha) = 1$ *iff* $q(a_i) = 1$*.*

Definition 4.12 (Separation)**.** *Let* $\alpha < \omega_1$*. Suppose that* $p \in \mathbb{P}$ *has top level* α *and* $A \subseteq \mathrm{dom}(p)$*.*

(1) *Let* $X \subseteq T_\alpha$*. We say that* $\{p(\tau) : \tau \in A\}$ *is* separated on X *if for all* $x \in X$*, there exists at most one* $\tau \in A$ *such that* $p(\tau)(x) = 1$*. And* $\{p(\tau) : \tau \in A\}$ *is* separated *if* $\{p(\tau) : \tau \in A\}$ *is separated on* T_α*.*
(2) *Let* $\vec{x} = (x_0, \dots, x_{n-1})$ *be an injective tuple consisting of distinct elements of* α*. Then* $\{p(\tau) : \tau \in A\}$ *is* separated on $\vec{x}$ *if* $\{p(\tau) : \tau \in A\}$ *is separated on* $\{x_0, \dots, x_{n-1}\}$*.*

Observe that if $\{p(\tau) : \tau \in A\}$ is separated on X, then for any $B \subseteq A$ and $Y \subseteq X$, $\{p(\tau) : \tau \in B\}$ is separated on Y. We will use this fact implicitly going forward.

We have now defined for the subtree forcing the objects and properties described in Section 2, where we let $\mathbb{Q}_\tau = \mathbb{Q}$ for all $\tau < \kappa$. We now work towards verifying properties (A)-(E) of Section 2. (A) is clear. (B) (Transitivity) follows from Lemma 4.3. The next lemma implies (C) Persistence:

Lemma 4.13 (Persistence)**.** *Let* $\alpha < \beta < \omega_1$*. Suppose that* $p \in \mathbb{P}$ *has top level* α*,* $q \le p$ *has top level* β*,* $X \subseteq T_\beta$ *has unique drop-downs to* α*, and* $A \subseteq \mathrm{dom}(p)$*. If* $\{p(\tau) : \tau \in A\}$ *is separated on* $X \restriction \alpha$*, then* $\{q(\tau) : \tau \in A\}$ *is separated on* X*.*

Proof. Immediate from Lemma 4.5 (Persistence) letting $\mathcal{G} = \{q(\tau) : \tau \in A\}$. □

The next lemma implies (D) (Extension).

Lemma 4.14 (Extension)**.** *Let* $\alpha < \beta < \omega_1$ *and let* $X \subseteq T_\beta$ *be finite with unique drop-downs to* α*. Let* $p \in \mathbb{P}$ *have top level* α *and let* $A \subseteq \mathrm{dom}(p)$ *be finite. Then there exists some* $q \le p$ *with top level* β *and with the same domain as* p *such that for all* $\tau \in A$*,* $X \restriction \alpha$ *and* X *are* $q(\tau)$*-consistent.*

Proof. Immediate from Proposition 4.7 (ignoring conclusion (3)). □

Finally, (E) Key Property holds by the next proposition.

Proposition 4.15 (Key Property)**.** *Let* $\alpha < \beta < \omega_1$*. Suppose that* $a_0, \dots, a_{n-1}$ *are distinct elements of* T_α*,* $p \in \mathbb{P}$ *has top level* α*,* $A \subseteq \mathrm{dom}(p)$ *is finite, and* $\{p(\tau) : \tau \in A\}$ *is separated on* $\{a_0, \dots, a_{n-1}\}$*. Then for any* $q \le p$ *with top level* β *and any finite set* $t \subseteq T_\beta$*, there exist* $b_0, \dots, b_{n-1}$ *in* $T_\beta \setminus t$ *such that* $a_i <_T b_i$ *for all* $i < n$*, and for all* $\tau \in A$*,* $\{a_0, \dots, a_{n-1}\}$ *and* $\{b_0, \dots, b_{n-1}\}$ *are* $q(\tau)$*-consistent.*

Proof. Immediate from Proposition 4.6 letting $\mathcal{G} = \{q(\tau) : \tau \in A\}$. □

This completes the verification of properties (A)-(E) from Section 2 for the subtree forcing. We are now free to apply Proposition 2.2 (Consistent Extensions Into Dense Sets).

Definition 4.16 (Separated Conditions)**.** *A condition* $p \in \mathbb{P}$ *is* separated *if* $\{p(\tau) : \tau \in \mathrm{dom}(p)\}$ *is separated on* T_α*, where* α *is the top level of* p*.*

Lemma 4.17 (Separated Conditions are Dense)**.** *The set of separated conditions is dense in* $\mathbb{P}$*. In fact, let* $\alpha < \beta < \omega_1$ *and let* $X \subseteq T_\beta$ *be finite with unique drop-downs to* α*. Suppose that* $p \in \mathbb{P}$ *has top level* α*,* $A \subseteq \mathrm{dom}(p)$ *is finite, and* $\{p(\tau) : \tau \in A\}$ *is separated on* $X \restriction \alpha$*. Then there exists some* $q \le p$ *with top level* β *and having the same domain as* p *such that* q *is separated and for all* $\tau \in A$*,* $X \restriction \alpha$ *and* X *are* $q(\tau)$*-consistent.*

Proof. The second part follows immediately from Proposition 4.7. The first statement follows from the second statement letting $X = \emptyset$. □

Lemma 4.18 (Strong Persistence)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α, $X \subseteq T_\alpha$, $A \subseteq \mathrm{dom}(p)$, and $\{p(\tau) : \tau \in A\}$ is separated on X. Let $Y = \{y \in T_\beta : y \restriction \alpha \in X\}$. Then for any $q \leq p$ with top level β, $\{q(\tau) : \tau \in A\}$ is separated on Y. In particular, if $\{p(\tau) : \tau \in A\}$ is separated, then $\{q(\tau) : \tau \in A\}$ is separated. And if p is separated, then q is separated.*

Proof. Immediate by Lemma 4.5 (Persistence) letting $\mathcal{G} = \{q(\tau) : \tau \in A\}$. □

Lemma 4.19. *Let $B \subseteq \kappa$ be countable and let $\rho < \omega_1$. Let D be the set of conditions $r \in \mathbb{P}$ such that the top level γ of r is at least ρ, $B \subseteq \mathrm{dom}(r)$, and $\{r(\tau) : \tau \in B\}$ is separated. Then D is dense open.*

Proof. To show that D is dense, consider $p \in \mathbb{P}$ with top level α. Define p^* with domain equal to $\mathrm{dom}(p) \cup B$ so that $p^* \restriction \mathrm{dom}(p) = p$ and for all $\tau \in B \setminus \mathrm{dom}(p)$, $p^*(\tau)(x) = 1$ for all $x \in T \restriction (\alpha + 1)$. Then $p^* \in \mathbb{P}$ and $p^* \leq p$. Applying Lemma 4.17 (Separated Conditions are Dense), letting $X = \emptyset$, find $q \leq p^*$ with top level at least ρ such that q is separated. Then $q \in D$.

For openness, let $r \leq q$ where $q \in D$. We will show that $r \in D$. Let q have top level ξ and let r have top level γ. Since $q \in D$ and $r \leq q$, $B \subseteq \mathrm{dom}(r)$ and $\gamma \geq \rho$. As q is separated, $\{q(\tau) : \tau \in B\}$ is separated. By Lemma 4.18 (Strong Persistence), $\{r(\tau) : \tau \in B\}$ is separated. So $r \in D$. □

4.4. **Basic Properties of the Subtree Forcing.** In this subsection we will prove, assuming that T is a free Suslin tree, that the forcing poset $\mathbb{P}$ is totally proper and adds a sequence of length κ of uncountable downwards closed infinitely splitting normal subtrees of T. Also, assuming CH, $\mathbb{P}$ is ω_2-c.c.

The next proposition establishes the existence of total master conditions over countable elementary substructures. The additional part of the proposition concerning the tuples $\vec{c}$ and $\vec{d}$ will be used in the next subsection.

Proposition 4.20 (Existence of Total Master Conditions)**.** *Suppose that T is a free Suslin tree. Let λ be a large enough regular cardinal and assume that N is a countable elementary substructure of $H(\lambda)$ which contains as elements T, κ, $\mathbb{Q}$, and $\mathbb{P}$. Let $\delta = N \cap \omega_1$.*

Suppose that $p \in N \cap \mathbb{P}$ has top level α, $\vec{x}$ is an injective tuple consisting of elements of T_α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{x}$. Let $\vec{c}$ and $\vec{d}$ be tuples of height δ each above $\vec{x}$ such that $\vec{c} \restriction (\alpha + 1)$ and $\vec{d} \restriction (\alpha + 1)$ are disjoint.

Then there exists a total master condition $q \leq p$ over N with top level δ and with domain equal to $N \cap \kappa$ such that for all $\tau \in A$, $\vec{x}$ and $\vec{c}$ are $q(\tau)$-consistent and $\vec{x}$ and $\vec{d}$ are $q(\tau)$-consistent.

Proof. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \alpha$, and fix an enumeration $\langle D_n : n < \omega \rangle$ of all of the dense open subsets of $\mathbb{P}$ which lie in N. Fix a surjection $g : \omega \to (T \restriction \delta) \times (N \cap \kappa)$ such that every element of the codomain has an infinite preimage. Let C consist of the elements of $\vec{c}$ and let D consist of the elements of $\vec{d}$.

We will define the following objects by induction in ω-many steps:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;
- a non-decreasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ;
- a decreasing sequence $\langle p_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}$ such that $p_0 = p$, and for all $n < \omega$, the top level of p_n is δ_n;
- for each $n < \omega$ and $\tau \in A_n$, a function $h_{n,\tau} : X_n \to 2$.

In addition to the properties listed above, we will maintain the following inductive hypotheses for all $n < \omega$:

(1) X_n has unique drop-downs to δ_n;
(2) $A_n \subseteq \mathrm{dom}(p_n)$ and $\{p_n(\tau) : \tau \in A_n\}$ is separated on $X_n \restriction \delta_n$;
(3) for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent;
(4) for all $\tau \in A_n$ and $x \in X_n$,

$$h_{n,\tau}(x) = 1 \iff p_n(\tau)(x \restriction \delta_n) = 1.$$

Stage 0: Apply Lemma 4.8 to find $p_0 \leq p$ with top level $\alpha + 1$ and with the same domain as p satisfying:

- $\{p(\tau) : \tau \in A\}$ is separated on $(C \cup D) \restriction (\alpha + 1)$;
- for all $\tau \in A$, $\vec{x}$ and $\vec{c} \restriction (\alpha + 1)$ are $p_0(\tau)$-consistent and $\vec{x}$ and $\vec{d} \restriction (\alpha + 1)$ are $p_0(\tau)$-consistent.

Let $X_0 = C \cup D$, $A_0 = A$, and $\delta_0 = \alpha + 1$. For each $\tau \in A_0$, define $h_{0,\tau} : X_0 \to 2$ as described in inductive hypothesis (4).

Stage $n + 1$: Let $n < \omega$ and assume that we have completed stage n. In particular, we have defined X_n, A_n, δ_n, p_n, and $h_{n,\tau}$ for all $\tau \in A_n$ satisfying the required properties. Let $g(n) = (x, \sigma)$.

Fix $\rho < \delta$ larger than δ_n, γ_{n+1}, and $\mathrm{ht}_T(x)$. Let E be the set of conditions r in D_n such that the top level γ of r is at least ρ, $A_n \cup \{\sigma\} \subseteq \mathrm{dom}(r)$, and $\{r(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated. By Lemma 4.19, E is dense open in $\mathbb{P}$, and $E \in N$ by elementarity.

By Proposition 2.2 (Consistent Extensions Into Dense Sets), fix $p_{n+1} \leq p_n$ in $N \cap E$ with some top level δ_{n+1} such that for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent. So $\{p_{n+1}(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated.

We consider the two possibilities of whether or not $p_{n+1}(\sigma)(x) = 1$. If not, then let $X_{n+1} = X_n$ and $A_{n+1} = A_n \cup \{\sigma\}$. Define $h_{n+1,\tau} : X_{n+1} \to 2$ for all $\tau \in A_{n+1}$ as described in inductive hypothesis (4).

Now assume that $p_{n+1}(\sigma)(x) = 1$. Since $p_{n+1}(\sigma)$ is a subtree function, by (3) of Definition 4.1 (Subtree Functions) we can find some x^+ above x in $T_{\delta_{n+1}} \setminus (X_n \restriction \delta_{n+1})$ such that $p_{n+1}(\sigma)(x^+) = 1$. Now pick some $z_x \in T_\delta$ above x^+ and define $X_{n+1} = X_n \cup \{z_x\}$. Note that $z_x \restriction \delta_{n+1} = x^+$ which is not in $X_n \restriction \delta_{n+1}$, so X_{n+1} has unique drop-downs to δ_{n+1}. Define $h_{n+1,\tau}$ for all $\tau \in A_{n+1}$ as described in inductive hypothesis (4). Observe that $h_{n+1,\sigma}(z_x) = 1$. This completes stage $n + 1$. It is routine to check that the inductive hypotheses hold.

This completes the construction. We claim that for all $n < \omega$ and $\tau \in A_n$, $h_{n+1,\tau} \restriction X_n = h_{n,\tau}$. Let $z \in X_n$. Then $h_{n+1,\tau}(z) = 1$ iff $p_{n+1}(\tau)(z \restriction \delta_{n+1}) = 1$ (by inductive hypothesis (4) for $n + 1$) iff $p_n(\tau)(z \restriction \delta_n) = 1$ (by inductive hypothesis (3) for n) iff $h_{n,\tau}(z) = 1$ (by inductive hypothesis (4) for n).

We define a condition q with domain $N \cap \kappa$ as follows. Consider $\tau \in N \cap \kappa$. By the previous paragraph,

$$h^*_\tau = \bigcup\{h_{n,\tau} : n < \omega,\ \tau \in A_n\}$$

is a function from $\bigcup_n X_n$ into 2. Define $h_\tau : T_\delta \to 2$ by letting $h_\tau(z) = h^*_\tau(z)$ for all $z \in \mathrm{dom}(h^*_\tau)$ and $h_\tau(z) = 0$ for all $z \in T_\delta \setminus \mathrm{dom}(h^*_\tau)$. Now define

$$q(\tau) = \bigcup\{p_n(\tau) : n < \omega,\ \tau \in \mathrm{dom}(p_n)\} \cup h_\tau.$$

We claim that for all $\tau \in N \cap \kappa$, $q(\tau)$ is a subtree function on $T \restriction (\delta+1)$. We verify properties (1)-(3) of Definition 4.1 (Subtree Functions). (1) is immediate. (2) follows from inductive hypothesis (4). (3) follows easily from our bookkeeping and the successor case of the construction. It follows that $q \in \mathbb{P}$ and $q \leq p_n$ for all $n < \omega$. So $q \leq p$ and q is a total master condition over N with domain $N \cap \kappa$ and top level δ.

Consider $\tau \in A$ and $z \in C \cup D = X_0$. Then $q(\tau)(z) = 1$ iff $h^*_\tau(z) = 1$ iff $h^0_\tau(z) = 1$ iff $p_0(\tau)(z \upharpoonright (\alpha+1)) = 1$ (by inductive hypothesis (4)) iff $p_0(\tau)(z \upharpoonright \alpha) = 1$ (by the choice of p_0 in stage 0) iff $q(\tau)(z \upharpoonright \alpha) = 1$. So $\vec{x}$ and $\vec{c}$ are $q(\tau)$-consistent and $\vec{x}$ and $\vec{d}$ are $q(\tau)$-consistent. □

The following is now immediate from Proposition 4.20 (Existence of Total Master Conditions) letting $\vec{x}$, $\vec{c}$, and $\vec{d}$ be the empty tuples.

Corollary 4.21. *Assuming that T is a free Suslin tree, the forcing poset $\mathbb{P}$ is totally proper.*

Proposition 4.22. *Assuming* CH*, the forcing poset $\mathbb{P}$ is ω_2-c.c.*

This proposition follows by a standard application of the Δ-system lemma, assuming CH, to an ω_2-sized collection of countable sets.

Proposition 4.23. *Assuming that T is a free Suslin tree, the forcing poset $\mathbb{P}$ adds an almost disjoint sequence of length κ of uncountable downwards closed infinitely splitting normal subtrees of T.*

Proof. For each $\tau < \kappa$ let $\dot{f}_\tau$ be a $\mathbb{P}$-name for the set $\bigcup\{p(\tau) : p \in \dot{G},\ \tau \in \mathrm{dom}(p)\}$ and let $\dot{U}_\tau$ be a $\mathbb{P}$-name for the set $\{x \in T : \dot{f}_\tau(x) = 1\}$. From the definition of a subtree function and Lemma 4.19, it is easy to check each $\dot{U}_\tau$ is forced to be an uncountable downwards closed infinitely splitting normal subtree of T. To see that these subtrees are almost disjoint, consider a condition p and $\tau_0 < \tau_1 < \kappa$. By Lemma 4.19, we can find $q \leq p$ with some top level γ such that $\tau_0, \tau_1 \in \mathrm{dom}(q)$ and $\{q(\tau) : \tau \in \{\tau_0, \tau_1\}\}$ is separated. So for all $x \in T_\gamma$, it is not the case that both $q(\tau_0)(x) = 1$ and $q(\tau_1)(x) = 1$. So q forces that $\dot{U}_{\tau_0} \cap \dot{U}_{\tau_1} \subseteq T \upharpoonright \gamma$. By genericity, $\mathbb{P}$ forces that $\dot{U}_{\tau_0}$ and $\dot{U}_{\tau_1}$ are almost disjoint. □

4.5. **The Subtree Forcing Adds No New Cofinal Branches.** The goal of this subsection is to prove the following theorem.

Theorem 4.24. *Suppose that T is a free Suslin tree. Let U be an ω_1-tree. Then $\mathbb{P}$ forces that every cofinal branch of U in $V^{\mathbb{P}}$ is in V. In particular, $\mathbb{P}$ forces that T is an Aronszajn tree.*

For the remainder of the section, fix an ω_1-tree U and a $\mathbb{P}$-name $\dot{b}$ for a branch of U. Without loss of generality assume that for each $\gamma < \omega_1$, U_γ consists of ordinals in the interval $[\omega \cdot \gamma, \omega \cdot (\gamma+1))$. Fix a large enough regular cardinal λ. A set N is said to be *suitable* if it is a countable elementary substructure of $(H(\lambda), \in)$ which contains as members the objects T, κ, $\mathbb{Q}$, $\mathbb{P}$, U, and $\dot{b}$.

Proposition 4.25. *Suppose that $\bar{p} \in \mathbb{P}$ has top level β and $\bar{p}$ forces that $\dot{b}$ is a cofinal branch of U which is not in V. Assume that $\vec{x}$ is an injective tuple consisting of elements of T_β, $A \subseteq \mathrm{dom}(\bar{p})$ is finite, and $\{\bar{p}(\tau) : \tau \in A\}$ is separated on $\vec{x}$. Define $\mathcal{X}$ as the set of all tuples $\vec{b}$ in the derived tree $T_{\vec{x}}$ for which there exist conditions $q_0, q_1 \leq p$ with top level equal to the height γ of $\vec{b}$ such that:*

1. *for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;*
2. *there exists some $\zeta < \gamma$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.*

Then $\mathcal{X}$ is dense open in $T_{\vec{x}}$.

Proof. To prove that $\mathcal{X}$ is open, assume that $\vec{b} \in \mathcal{X}$ has height γ and $\vec{c} > \vec{b}$ has height ξ. Fix $q_0, q_1 \leq p$ with top level γ which witness that $\vec{b} \in \mathcal{X}$. Apply Lemma 4.14 (Extension) to find r_0 and r_1 below q_0 and q_1 respectively with top level ξ such that for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{c}$ are $r_j(\tau)$-consistent. Then r_0 and r_1 witness that $\vec{c} \in \mathcal{X}$.

Now we prove that $\mathcal{X}$ is dense. Suppose for a contradiction that $\vec{b} \in T_{\vec{x}}$ and for all $\vec{c} \geq \vec{b}$, $\vec{c} \notin \mathcal{X}$. Let α be the height of $\vec{b}$. Apply Lemma 4.14 (Extension) to find $p \leq \bar{p}$ with top level α such that for all $\tau \in A$, $\vec{x}$ and $\vec{b}$ are $p(\tau)$-consistent. Since $\{\bar{p}(\tau) : \tau \in A\}$ is separated on $\vec{x}$, $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$ by Lemma 4.13 (Persistence). Fix a $\in$-increasing and continuous chain $\langle N_\gamma : \gamma < \omega_1 \rangle$ of suitable sets such that p is in N_0. Let $\delta_\gamma = N_\gamma \cap \omega_1$ for all $\gamma < \omega_1$.

We define a function F which takes as inputs any tuple $\vec{a}$ satisfying that for some $\gamma < \omega_1$:

(a) $\vec{a}$ has height δ_γ;
(b) $\vec{b} < \vec{a}$;
(c) there exists some $r \leq p$ with top level δ_γ such that r decides $\dot{b} \cap \delta_\gamma$ and for all $\tau \in A$, $\vec{b}$ and $\vec{a}$ are $r(\tau)$-consistent.

For any such tuple $\vec{a}$, define $F(\vec{a})$ to be equal to b^*, where for some r as in (c), $r \Vdash_{\mathbb{P}} \dot{b} \cap \delta_\gamma = b^*$.

Claim 1: F is well-defined. Proof: Let $\vec{a}$ and γ be as above, and consider two conditions r_0 and r_1 as described in (c). For each $j < 2$ fix b_j such that $r_j \Vdash_{\mathbb{P}} \dot{b} \cap \delta_\gamma = b_j$. If $b_0 \neq b_1$, then there is some $\zeta < \delta_\gamma$ such that $r_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $r_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$. But then r_0 and r_1 witness that $\vec{a} \in \mathcal{X}$, which is a contradiction. This completes the proof of claim 1.

Claim 2: For all $\gamma < \omega_1$, there exists some $\vec{a}$ with height δ_γ which is in the domain of F. Proof: By Proposition 4.20 (Existence of Total Master Conditions), fix a total master condition $r \leq p$ over N_γ with top level δ_γ. By Proposition 4.15 (Key Property), there exists some tuple $\vec{a}$ above $\vec{b}$ with height δ_γ such that for all $\tau \in A$, $\vec{b}$ and $\vec{a}$ are $r(\tau)$-consistent. Since r is a total master condition over N_γ, r decides $\dot{b} \cap \delta_\gamma$. So r witnesses that $\vec{a}$ is in the domain of F. This completes the proof of claim 2.

Claim 3: If $\vec{c}$ and $\vec{d}$ are both in the domain of F and have the same height δ_γ, then $F(\vec{c}) = F(\vec{d})$. Proof: Let r be a condition which witnesses that $\vec{d}$ is in the domain of F. So r has top level δ_γ, r decides $\dot{b} \cap \delta_\gamma$, and for all $\tau \in A$, $\vec{b}$ and $\vec{d}$ are $r(\tau)$-consistent. Define r^* to be the condition with the same domain as r such that for all $\tau \in \operatorname{dom}(r^*)$, $r^*(\tau) = r(\tau) \restriction (\alpha + 1)$. Then $r^* \leq p$ has top level $\alpha + 1$. By Proposition 4.15 (Key Property), find a tuple $\vec{e}^*$ above $\vec{b}$ with height $\alpha + 1$ which is disjoint from $\vec{c} \restriction (\alpha + 1)$ such that for all $\tau \in A$, $\vec{b}$ and $\vec{e}^*$ are $r^*(\tau)$-consistent. By Lemma 4.13 (Persistence), $\{r^*(\tau) : \tau \in A\}$ is separated on $\vec{e}^*$. Since $r \leq r^*$, we can apply Proposition 4.15 (Key Property) again to find a tuple $\vec{e}$ above $\vec{e}^*$ of height δ_γ such that for all $\tau \in A$, $\vec{e}^*$ and $\vec{e}$ are $r(\tau)$-consistent. Then for all $\tau \in A$, $\vec{b}$ and $\vec{e}$ are $r(\tau)$-consistent, and $\vec{e}$ is in the domain of F as witnessed by r.

Since $\vec{c} \restriction (\alpha + 1)$ and $\vec{e} \restriction (\alpha + 1) = \vec{e}^*$ are disjoint, by Proposition 4.20 (Existence of Total Master Conditions) we can fix a total master condition $s \leq p$ over N_γ with top level δ_γ such that for all $\tau \in A$, $\vec{b}$ and $\vec{c}$ are $s(\tau)$-consistent and $\vec{b}$ and $\vec{e}$ are $s(\tau)$-consistent. Since s is a total master condition, we can fix some b^* such that $s \Vdash_{\mathbb{P}} \dot{b} \cap \delta_\gamma = b^*$. Then $F(\vec{c}) = F(\vec{e}) = b^*$ as witnessed by s, and $F(\vec{e}) = F(\vec{d})$ as witnessed by r. So $F(\vec{c}) = F(\vec{d})$. This completes the proof of claim 3.

Based on claims 2 and 3, for each $\gamma < \omega_1$ we can define $F(\gamma)$ to be the unique value of $F(\vec{a})$ for some (any) $\vec{a}$ above $\vec{b}$ with height δ_γ in the domain of F. Then each $F(\gamma)$ is a cofinal branch of $U \restriction \delta_\gamma$.

Claim 4: For all $\gamma < \xi < \omega_1$, $F(\gamma) = F(\xi) \cap \delta_\gamma$. Proof: Since $N_\gamma \in N_\xi$, by elementarity we can fix a tuple $\vec{a}$ with height δ_γ and some $r \in N_\xi$ satisfying properties (a)-(c) in the definition of $\vec{a}$ being in the domain of F. Then $r \Vdash_{\mathbb{P}} F(\gamma) = \dot{b} \cap \delta_\gamma$. By Lemma 4.13 (Persistence), $\{r(\tau) : \tau \in A\}$ is separated on $\vec{b}$. By Proposition 4.20 (Existence of Total Master Conditions), letting $\vec{x}$, $\vec{c}$, and $\vec{d}$ be the empty tuples, fix $s \leq r$ with top level δ_ξ which is a total master condition over N_ξ. By

Proposition 4.15 (Key Property), fix $\vec{c}$ above $\vec{a}$ with height δ_ξ such that for all $\tau \in A$, $\vec{a}$ and $\vec{c}$ are $s(\tau)$-consistent. Then $s \Vdash_{\mathbb{P}} F(\xi) = \dot{b} \cap \delta_\xi$. Since $s \le r$, $s \Vdash_{\mathbb{P}} F(\gamma) = \dot{b} \cap \delta_\gamma = F(\xi) \restriction \delta_\gamma$. So indeed $F(\gamma) = F(\xi) \cap \delta_\gamma$. This completes the proof of claim 4.

Define $c = \bigcup\{F(\gamma) : \gamma < \omega_1\}$. By claim 4, c is a cofinal branch of U and for all $\gamma < \omega_1$, $c \cap \delta_\gamma = F(\gamma)$.

Claim 5: p forces that $c = \dot{b}$. Proof: Since c is a chain, it suffices to show that $p \Vdash_{\mathbb{P}} \dot{b} \subseteq c$. If not, then for some $q \le p$ and $\zeta < \omega_1$, $q \Vdash_{\mathbb{P}} \zeta \in \dot{b} \setminus c$. Then $\zeta \notin c$. Fix $\gamma < \omega_1$ such that $\zeta < \delta_\gamma$. Then $\zeta \notin F(\gamma)$, so $q \Vdash_{\mathbb{P}} \zeta \in \dot{b} \setminus F(\gamma)$. Note that $F(\gamma)$ is definable in $N_{\gamma+1}$, so by elementarity we may assume that $q \in N_{\gamma+1}$. Apply Proposition 4.20 (Existence of Total Master Conditions) to find $r \le q$ which is a total master condition over $N_{\gamma+1}$ with top level $\delta_{\gamma+1}$. Since $r \le q$, $r \Vdash_{\mathbb{P}} \zeta \in \dot{b}$. By the Key Property, r decides $\dot{b} \cap \delta_{\gamma+1}$ as the set $F(\gamma+1)$. So $\zeta \in F(\gamma+1) \cap \delta_\gamma$, and by claim 4, $F(\gamma+1) \cap \delta_\gamma = F(\gamma)$. Hence, $\zeta \in F(\gamma)$, which is a contradiction. This completes the proof of claim 5.

Now c is in the ground model, so p forces that $\dot{b}$ is in the ground model, which contradicts that $p \le p^*$. □

Lemma 4.26. *Suppose that T is a free Suslin tree. Let N be suitable and let $\delta = N \cap \omega_1$. Suppose that $p_0, \ldots, p_{l-1}$ are conditions in $N \cap \mathbb{P}$ all of which have top level β and each of which forces that $\dot{b}$ is a cofinal branch of U which is not in V. Assume that $X \subseteq T_\delta$ is finite and has unique drop-downs to β, $A \subseteq \bigcap_{k<l} \operatorname{dom}(p_k)$ is finite, and for all $k < l$, $\{p_k(\tau) : \tau \in A\}$ is separated on $X \restriction \beta$. Then there exists $\gamma < \delta$ and for all $k < l$ there exist conditions $q_{k,0}, q_{k,1} \le p_k$ in N satisfying that for all $j < 2$:*

(1) *$q_{k,j}$ has top level γ;*
(2) *for all $\tau \in A$, $X \restriction \beta$ and $X \restriction \gamma$ are $q_{k,j}(\tau)$-consistent;*
(3) *there exists some $\zeta < \gamma$ such that $q_{k,0} \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_{k,1} \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.*

Proof. The proof is by induction on l. Let N and δ be as above and let $X \subseteq T_\delta$ be finite.

Base case: Suppose that $p \in N \cap \mathbb{P}$ has top level β and forces that $\dot{b}$ is a cofinal branch of U which is not in V. Assume that X has unique drop-downs to β, $A \subseteq \operatorname{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $X \restriction \beta$. Fix an injective tuple $\vec{a}$ which lists the elements of X. Let $\vec{x} = \vec{a} \restriction \beta$.

Define $\mathcal{X}$ as the set of all tuples $\vec{b}$ in the derived tree $T_{\vec{x}}$ for which there exist $q_0, q_1 \le p$ with top level equal to the height ρ of $\vec{b}$ such that:

- for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;
- there exists some $\zeta < \rho$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.

By Proposition 4.25, $\mathcal{X}$ is dense open in $T_{\vec{x}}$. Also, $\mathcal{X} \in N$ by elementarity. Since T is a free Suslin tree, $T_{\vec{x}}$ is Suslin. So fix some $\gamma < \delta$ greater than β such that every member of $T_{\vec{x}}$ with height at least γ is in $\mathcal{X}$. In particular, $\vec{a} \restriction \gamma \in \mathcal{X}$. Fix $q_0, q_1 \le p$ which witness that $\vec{a} \restriction \gamma \in \mathcal{X}$. Then γ, q_0, and q_1 satisfy conclusions (1)-(3).

Inductive step: Let $l < \omega$ be positive. Assume that the statement of the lemma is true for l, and we will prove that it is true for $l+1$. Suppose that $p_0, \ldots, p_l$ are conditions in $N \cap \mathbb{P}$ all of which have top level β and each of which forces that $\dot{b}$ is a cofinal branch of U which is not in V. Assume that X has unique drop-downs to β. Let $A \subseteq \bigcap_{k \le l} \operatorname{dom}(p_k)$ be finite and suppose that for all $k \le l$, $\{p_k(\tau) : \tau \in A\}$ is separated on $X \restriction \beta$.

By the inductive hypothesis, we can fix $\gamma < \delta$ and conditions $q_{k,0}, q_{k,1} \le p_k$ in N for all $k < l$ satisfying conclusions (1)-(3). Apply Lemma 4.14 (Extension) to find some $q \le p_l$ with top level γ and the same domain as p_l such that for all $\tau \in A$, $X \restriction \beta$ and $X \restriction \gamma$ are $q(\tau)$-consistent. Note that by Lemma 4.13 (Persistence), $\{q(\tau) : \tau \in A\}$ is separated on $X \restriction \gamma$. Fix an injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which enumerates X. Let $\vec{x} = \vec{a} \restriction \gamma$.

Define $\mathcal{X}$ as the set of all tuples $\vec{b}$ in the derived tree $T_{\vec{x}}$ for which there exist $q_0, q_1 \leq q$ with top level equal to the height ρ of $\vec{b}$ such that:

- for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;
- there exists some $\zeta < \rho$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.

By Proposition 4.25, $\mathcal{X}$ is dense open in $T_{\vec{x}}$. Also, $\mathcal{X} \in N$ by elementarity. Since T is a free Suslin tree, $T_{\vec{x}}$ is Suslin. So fix some $\xi < \delta$ greater than γ such that every member of $T_{\vec{x}}$ with height at least ξ is in $\mathcal{X}$. In particular, $\vec{a} \restriction \xi \in \mathcal{X}$.

Fix $\bar{q}_0, \bar{q}_1 \leq q$ which witness that $\vec{a} \restriction \xi \in \mathcal{X}$. Now apply Lemma 4.14 (Extension) in N to find, for each $k < l$ and $j < 2$, a condition $\bar{q}_{k,j} \leq q_{k,j}$ in N with top level ξ such that for all $\tau \in A$, $X \restriction \gamma$ and $X \restriction \xi$ are $\bar{q}_{k,j}(\tau)$-consistent. Now the ordinal ξ and the conditions $\bar{q}_{k,j}$ for $k \leq l$ and $j < 2$ are as required. □

The following lemma now completes the proof of Theorem 4.24.

Lemma 4.27. *Assuming that T is a free Suslin tree, the forcing poset $\mathbb{P}$ forces that if $\dot{b}$ is a cofinal branch of U, then $\dot{b}$ is in V.*

Proof. Suppose for a contradiction that there exists a condition p which forces that $\dot{b}$ is a cofinal branch of U which is not in V. We will prove that U has an uncountable level, which contradicts that U is an ω_1-tree. Let α be the top level of p.

Fix a suitable model N such that $p \in N$ and let $\delta = N \cap \omega_1$. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \alpha$, and fix an enumeration $\langle D_n : n < \omega \rangle$ of all dense open subsets of $\mathbb{P}$ which lie in N. Fix a surjection $g : \omega \to (T \restriction \delta) \times (N \cap \kappa)$ in which each element of the codomain has an infinite preimage.

We will define by induction the following objects in ω-many steps:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;
- an increasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ;
- for all $s \in {}^{<\omega}2$, a condition $r^s \in N \cap \mathbb{P}$ and an ordinal $\zeta_s < \delta$;
- for all $s \in {}^{<\omega}2$ and $\tau \in A_{|s|}$, a function $h^s_\tau : X_{|s|} \to 2$.

We will maintain the following inductive hypotheses for all $n < \omega$ and $s \in {}^n2$:

(1) X_n has unique drop-downs to δ_n;
(2) δ_n is the top level of r^s;
(3) $r^s \leq p$, and if $m > n$, $t \in {}^m2$, and $s \subseteq t$, then $r^t \leq r^s$;
(4) $A_n \subseteq \mathrm{dom}(r^s)$;
(5) $\zeta_s < \delta_{n+1}$;
(6) $r^{s^\frown 0} \Vdash_{\mathbb{P}} \zeta_s \in \dot{b}$ and $r^{s^\frown 1} \Vdash_{\mathbb{P}} \zeta_s \notin \dot{b}$;
(7) for all $\tau \in A_n$ and $j < 2$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $r^{s^\frown j}(\tau)$-consistent;
(8) $r^{s^\frown 0}$ and $r^{s^\frown 1}$ are in D_n;
(9) $\{r^s(\tau) : \tau \in A_n\}$ is separated on $X_n \restriction \delta_n$;
(10) for all $\tau \in A_n$ and $x \in X_n$,

$$h^s_\tau(x) = 1 \iff r^s(\tau)(x \restriction \delta_n) = 1.$$

<u>Stage 0</u>: Let $X_0 = \emptyset$, $A_0 = \emptyset$, $\delta_0 = \alpha$, and $r^\emptyset = p$.

Stage $n+1$: Let $n < \omega$ and assume that we have completed stage n. In particular, we have defined X_n, A_n, δ_n, r^s and h^s_τ for all $s \in {}^n2$ and $\tau \in A_n$ which satisfy the required properties. Let $g(n) = (x, \sigma)$.

Fix $\rho < \delta$ larger than δ_n, γ_{n+1}, and $\mathrm{ht}_T(x)$. Let D be the set of conditions r in D_n such that the top level γ of r is at least ρ, $A_n \cup \{\sigma\} \subseteq \mathrm{dom}(r)$, and $\{r(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated. By Lemma 4.19, D is dense open in $\mathbb{P}$, and $D \in N$ by elementarity.

Applying Proposition 2.2 (Consistent Extension Into Dense Sets), for each $s \in {}^n2$ fix $\bar{r}^s \leq r^s$ in $N \cap D$ with some top level γ_s such that for all $\tau \in A_n$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright \gamma_s$ are $\bar{r}^s(\tau)$-consistent. Then $\{\bar{r}^s(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated. Now fix $\gamma < \delta$ larger than each γ_s, and apply Lemma 4.14 (Extension) to find for each $s \in {}^n2$ a condition $\hat{r}^s \leq \bar{r}^s$ with top level γ such that for all $\tau \in A_n$, $X_n \upharpoonright \gamma_s$ and $X_n \upharpoonright \gamma$ are $\hat{r}^s(\tau)$-consistent. By Lemma 4.18 (Strong Persistence), for each $s \in {}^n2$, $\{\hat{r}^s(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated.

Let B be the set of $s \in {}^n2$ such that $\hat{r}^s(\sigma)(x) = 1$. For each $s \in B$, choose some x^+_s above x in $T_\gamma \setminus (X_n \upharpoonright \gamma)$ such that $\hat{r}^s(\sigma)(x^+_s) = 1$, and such that whenever $s \neq t$ are in B then $x^+_s \neq x^+_t$. This is possible by (3) of Definition 4.1 (Subtree Functions). Now for each $s \in B$ choose some $x^*_s \in T_\delta$ above x^+_s. Define $X_{n+1} = X_n \cup \{x^*_s : s \in B\}$. Observe that X_{n+1} has unique drop-downs to γ. Define $A_{n+1} = A_n \cup \{\sigma\}$. So for each $s \in {}^n2$, $\{\hat{r}^s(\tau) : \tau \in A_{n+1}\}$ is separated on $X_{n+1} \upharpoonright \gamma$.

Now apply Lemma 4.26 to find some $\delta_{n+1} < \delta$, and for each $s \in {}^n2$ find conditions $r^{s^\frown 0}, r^{s^\frown 1} \leq \hat{r}^s$ with top level δ_{n+1} and $\zeta_s < \delta_{n+1}$ such that $r^{s^\frown 0} \Vdash_{\mathbb{P}} \zeta_s \in \dot{b}$, $r^{s^\frown 1} \Vdash_{\mathbb{P}} \zeta_s \notin \dot{b}$, and for all $\tau \in A_{n+1}$, $X_{n+1} \upharpoonright \gamma$ and $X_{n+1} \upharpoonright \delta_{n+1}$ are $r^{s^\frown 0}(\tau)$-consistent and $r^{s^\frown 1}(\tau)$-consistent. For each $s \in {}^n2$, $j < 2$, and $\tau \in A_{n+1}$, define a function $h^{s^\frown j}_\tau : X_{n+1} \to 2$ by letting, for all $x \in X_{n+1}$, $h^{s^\frown j}_\tau(x) = 1$ iff $r^{s^\frown j}(\tau)(x \upharpoonright \delta_{n+1}) = 1$. Observe that for all $s \in B$ and $j < 2$, since $X_{n+1} \upharpoonright \gamma$ and $X_{n+1} \upharpoonright \delta_{n+1}$ are $r^{s^\frown 1}(\sigma)$-consistent, $h^{s^\frown j}_\tau(x^*_s) = 1$. This completes stage $n+1$. It is easy to check that the required properties are satisfied.

This completes the construction. Consider a function $f \in {}^\omega 2$. As in the proof of Proposition 4.20 (Existence of Total Master Conditions), we can define a condition r_f with domain equal to $N \cap \kappa$ such that for all $n < \omega$, $r_f \leq r_{f \upharpoonright n}$ and for all $\tau \in N \cap \kappa$, $X_n \upharpoonright \delta_n$ and X_n are $r_f(\tau)$-consistent. In particular, r_f is a total master condition over N.

For each $f \in {}^\omega 2$, let b_f be such that $r_f \Vdash_{\mathbb{P}} \dot{b} \cap \delta = b_f$. Suppose that $f \neq g$. Let n be least such that $f(n) \neq g(n)$, and assume without of generality that $f(n) = 0$ and $g(n) = 1$. Let $s = f \upharpoonright n = g \upharpoonright n$. Then $r_f \leq r^{s^\frown 0}$ and $r_g \leq r^{s^\frown 1}$. So $r_f \Vdash_{\mathbb{P}} \zeta_s \in \dot{b} \cap \delta$ and $r_g \Vdash_{\mathbb{P}} \zeta_s \notin \dot{b} \cap \delta$. Hence, $b_f \neq b_g$.

For each $f \in {}^\omega 2$, b_f is a cofinal branch of $U \upharpoonright \delta$ and r_f forces that $b_f = \dot{b} \cap \delta$. Hence, r_f forces that $\dot{b}(\delta)$ is an upper bound of b_f. Since having an upper bound in U is absolute between V and $V^{\mathbb{P}}$, b_f does in fact have an upper bound in U_δ, which we will denote by x_f. By construction, if $f \neq g$ then $b_f \neq b_g$, so $x_f \neq x_g$. Hence, U_δ is uncountable, which contradicts that U is an ω_1-tree. □

Corollary 4.28. *Assuming that T is a free Suslin tree, the forcing poset $\mathbb{P}$ forces that T is an Aronszajn tree for which there exists an almost disjoint sequence of length κ of uncountable downwards closed infinitely splitting normal subtrees.*

The forcing poset $\mathbb{P}$ of this section satisfies a stronger property than not adding cofinal branches of ω_1-trees in the ground model. Let $\theta < \kappa$ and define $\mathbb{P}_\theta$ as the suborder of $\mathbb{P}$ consisting of all conditions whose domain is a subset of θ. Then $\mathbb{P}_\theta$ is a regular suborder of $\mathbb{P}$. It turns out that for any ω_1-tree U in the intermediate extension $V^{\mathbb{P}_\theta}$, every cofinal branch of U in $V^{\mathbb{P}}$ is already in $V^{\mathbb{P}_\theta}$. The proof is a simpler variation of the proof of the same fact about the automorphism forcing which is given in Subsections 5.6-5.9. We omit the proof since we do not need it, although we note that this result can be used to give a more direct construction of a model in which there exists a non-saturated Aronszajn tree but no Kurepa tree.

5. Application III: Adding Automorphisms of a Free Suslin Tree

We have now arrived at the most substantial part of the article. Our goal is to develop a forcing poset which adds almost disjoint automorphisms of a free Suslin tree. This section mirrors the structure of Section 4 but with some additional complications. The notion of separation for the automorphism forcing is more complex than separation for the subtree forcing, which in particular makes the construction of total master conditions and related objects more difficult. In order to prove the main results of the article, we will need to prove that quotient forcings of the automorphism forcing do not add new cofinal branches of ω_1-trees appearing in intermediate extensions. However, since the free Suslin tree T is no longer free in an intermediate extension, the arguments we have been giving throughout the article using freeness do not apply there. To overcome this difficulty, we introduce and prove the existence of so-called *nice conditions* for regular suborders.

5.1. **Consistency and Separation for Automorphisms.** If $\beta < \omega_1$, g is an automorphism of $T \upharpoonright (\beta + 1)$, and $\alpha < \beta$, we will write $g \upharpoonright (\alpha + 1)$ for $g \upharpoonright (T \upharpoonright (\alpha + 1))$, which is obviously an automorphism of $T \upharpoonright (\alpha + 1)$. If $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of automorphisms of $T \upharpoonright (\beta + 1)$, we will write $\mathcal{G} \upharpoonright (\alpha + 1)$ for the indexed family $\{g_\tau \upharpoonright (\alpha + 1) : \tau \in I\}$.

Definition 5.1 (Consistency)**.** *Let $\alpha < \beta < \omega_1$ and let g be an automorphism of $T \upharpoonright (\beta + 1)$.*

1. *Let $X \subseteq T_\beta$ be finite with unique drop-downs to α. We say that $X \upharpoonright \alpha$ and X are g-*consistent *if for all $x, y \in X$, $g(x \upharpoonright \alpha) = y \upharpoonright \alpha$ iff $g(x) = y$.*
2. *Let $\vec{a} = (a_0, \ldots, a_{n-1})$ be an injective tuple consisting of elements of T_β. We say that $\vec{a} \upharpoonright \alpha$ and $\vec{a}$ are g-*consistent *if for all $i, j < n$, $g(a_i \upharpoonright \alpha) = a_j \upharpoonright \alpha$ iff $g(a_i) = a_j$.*

Note that in (1) above, the sets $X \upharpoonright \alpha$ and X are g-consistent if and only if for all $x, y \in X$, $g(x \upharpoonright \alpha) = y \upharpoonright \alpha$ implies $g(x) = y$. For the reverse implication follows from the fact that g is strictly increasing. A similar comment applies to (2).

The following lemma is easy to check.

Lemma 5.2 (Transitivity)**.** *Let $\alpha < \beta < \gamma < \omega_1$ and let $X \subseteq T_\gamma$ be finite with unique drop-downs to α. Let g be an automorphism of $T \upharpoonright (\gamma + 1)$. If $X \upharpoonright \alpha$ and $X \upharpoonright \beta$ are $g \upharpoonright (\beta + 1)$-consistent and $X \upharpoonright \beta$ and X are g-consistent, then $X \upharpoonright \alpha$ and X are g-consistent.*

Definition 5.3 (Separation)**.** *Let $\alpha < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of automorphisms of $T \upharpoonright (\alpha + 1)$ and $\vec{a} = (a_0, \ldots, a_{n-1})$ consists of distinct elements of T_α. We say that $\mathcal{G}$ is* separated on $\vec{a}$ *if for all $k < n$:*

1. *for all $\tau \in I$, $g_\tau(a_k) \neq a_k$;*
2. *there exists at most one triple (i, m, τ), where $i < k$, $m \in \{-1, 1\}$, and $\tau \in I$, such that $g_\tau^m(a_k) = a_i$.*

We will sometimes refer to an equation of the form $g_\tau^m(a_k) = a_i$ as in (2) above as a *relation*. So separation means that each member of the tuple has at most one relation with previous members of the tuple, and no relation with itself. In contrast to the subtree forcing from Section 4, the way in which a tuple is ordered is essential to whether or not separation holds.

Lemma 5.4 (Persistence)**.** *Let $\alpha < \beta < \omega_1$ and let $\mathcal{G} = \{g_\tau : \tau \in I\}$ be an indexed family of automorphisms of $T \upharpoonright (\beta + 1)$. Let $\vec{b} = (b_0, \ldots, b_{n-1})$ consist of distinct elements of T_β. If the indexed family $\mathcal{G} \upharpoonright (\alpha + 1)$ is separated on $\vec{b} \upharpoonright \alpha$, then $\mathcal{G}$ is separated on $\vec{b}$.*

Proof. For all $\tau \in I$, the fact that g_τ is strictly increasing implies that for all $i, j < n$, if $g_\tau(b_i) = b_j$ then $g_\tau(b_i \upharpoonright \alpha) = b_j \upharpoonright \alpha$. So any violation of separation of $\mathcal{G}$ on $\vec{b}$ would imply a violation of separation of $\mathcal{G} \upharpoonright (\alpha+1)$ on $\vec{b} \upharpoonright \alpha$. □

Lemma 5.5. *Let $\alpha < \omega_1$. Let $\vec{a} = (a_0, \ldots, a_{n-1})$ consist of distinct elements of T_α, and let $\mathcal{G} = \{g_\tau : \tau \in A\}$ be a finite indexed family of automorphisms of $T \upharpoonright (\alpha+1)$. Then for all $\bar{n} < n$, there exists a sequence*

$$\langle i_0, (i_1, m_1, \tau_1), \ldots, (i_{l-1}, m_{l-1}, \tau_{l-1})\rangle,$$

for some $l \leq \bar{n} + 1$, such that:

(1) *$\bar{n} = i_0 > i_1 > \cdots > i_{l-1} \geq 0$;*
(2) *for all $0 < k < l$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $g_{\tau_k}^{m_k}(a_{i_{k-1}}) = a_{i_k}$;*
(3) *there does not exist a triple (i, m, τ) such that $i < i_{l-1}$, $m \in \{-1, 1\}$, $\tau \in A$, and $g_\tau^m(a_{i_{l-1}}) = a_i$.*

Moreover, if $\mathcal{G}$ is separated on $\vec{a}$, then the above sequence is unique.

Proof. We construct the sequence by induction. Let $i_0 = \bar{n}$. Now let $k \geq 0$ and assume that we have defined $\langle i_0, (i_1, m_1, \tau_1), \ldots, (i_k, m_k, \tau_k)\rangle$ as described in (1) and (2). If there does not exist a triple (i, m, τ) such that $i < i_k$, $m \in \{-1, 1\}$, $\tau \in A$, and $g_\tau^m(a_{i_k}) = a_i$, then let $l = k + 1$ and we are done. Otherwise fix such a triple (i, m, τ) (which is unique in the case that $\mathcal{G}$ is separated on $\vec{a}$), and let $i_{k+1} = i$, $m_{k+1} = m$, and $\tau_{k+1} = \tau$. This completes the construction. Note that (1) implies that $l \leq \bar{n} + 1$. □

While separation in the context of automorphisms depends on the way in which a tuple is ordered, the following notion of separation for sets is useful when we do not need to be explicit about what that order is.

Definition 5.6 (Separation for Sets)**.** *Let $\alpha < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of automorphisms of $T \upharpoonright (\alpha+1)$ and X is a finite subset of T_α. We say that $\mathcal{G}$ is* separated on *X if there exists some injective tuple $\vec{a}$ which lists the elements of X such that $\mathcal{G}$ is separated on $\vec{a}$.*

Lemma 5.7 (Persistence for Sets)**.** *Let $\alpha < \beta < \omega_1$ and let $\mathcal{G} = \{g_\tau : \tau \in I\}$ be an indexed family of automorphisms of $T \upharpoonright (\beta+1)$. Let $X \subseteq T_\beta$ be a finite set with unique drop-downs to α. If the indexed family $\mathcal{G} \upharpoonright (\alpha+1)$ is separated on $X \upharpoonright \alpha$, then $\mathcal{G}$ is separated on X.*

Proof. Let $\vec{a}$ be an injective tuple which lists the elements of X so that $\mathcal{G} \upharpoonright (\alpha+1)$ is separated on $\vec{a} \upharpoonright \alpha$. Now apply Lemma 5.4 (Persistence). □

The proof of the following lemma is easy.

Lemma 5.8. *Let $\alpha < \omega_1$ and let X be a finite subset of T_α. Suppose that $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are indexed families of automorphisms of $T \upharpoonright (\alpha+1)$, $h : I \to J$ is a bijection, and for all $i \in I$, $f_i = g_{h(i)}$. If $\{g_j : j \in J\}$ is separated on X, then $\{f_i : i \in I\}$ is separated on X.*

Lemma 5.9. *Let $\alpha < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of automorphisms of $T \upharpoonright (\alpha+1)$ and X is a finite subset of T_α. If $\mathcal{G}$ is separated on X, then for any $J \subseteq I$ and $Y \subseteq X$, $\{g_\tau : \tau \in J\}$ is separated on Y.*

Proof. Let $\vec{a}$ be an injective tuple which lists the elements of X such that $\mathcal{G}$ is separated on $\vec{a}$. Let $\vec{b}$ be an injective tuple which lists the elements of Y in the same order in which they appear in $\vec{a}$. Now any counter-example to the indexed family $\{g_\tau : \tau \in J\}$ being separated on $\vec{b}$ would yield a counter-example to $\mathcal{G}$ being separated on $\vec{a}$. □

Definition 5.10. *Let $\alpha < \omega_1$. Suppose that $\mathcal{G} = \{g_\tau : \tau \in I\}$ is an indexed family of automorphisms of $T \upharpoonright (\alpha + 1)$. We say that $\mathcal{G}$ is* separated *if for any finite set $X \subseteq T_\alpha$, $\{g_\tau : \tau \in I\}$ is separated on X.*

Additional remarks: Although we will not use it, there is a graph-theoretic characterization of when an indexed family $\mathcal{G} = \{g_\tau : \tau \in I\}$ of automorphisms of $T \upharpoonright (\alpha + 1)$ is separated on a finite set $X \subseteq T_\alpha$. Namely, equip the set X with a set of directed edges labeled by indices in I, where a is connected to b with a directed edge labeled with index τ if $g_\tau(a) = b$. Then $\mathcal{G}$ is separated on X if and only if:

(1) no element of X has an edge to itself;
(2) there exists at most one edge between any two distinct elements of X;
(3) there does not exist a *loop*, by which we mean a finite sequence $\langle x_0, \ldots, x_{m-1} \rangle$ of elements of X satisfying that $m \geq 4$, $\langle x_0, \ldots, x_{m-2} \rangle$ is injective, $x_0 = x_{m-1}$, and there exists an edge between x_i and x_{i+1} (in either direction) for each $i < m - 1$.

The forward direction of the equivalence is easy. For the reverse direction, we build a tuple $(a_0, \ldots, a_{n-1})$ listing the elements of X and witnessing that $\mathcal{G}$ is separated on X roughly as follows. This tuple will split into consecutive segments, where there is no edge between members of distinct segments, and for each member b of a segment other than the first element a of the segment, there exists a path from b down to a, by which we mean a finite sequence $\langle c_0, \ldots, c_{m-1} \rangle$ such that $m \geq 2$, $c_0 = b$, $c_{m-1} = a$, and for all $i < m - 1$, c_{i+1} appears earlier in the segment than c_i and c_i and c_{i+1} are connected by an edge (in either direction). For the first segment, let a_0 be arbitrary. Assuming that $(a_0, \ldots, a_k)$ has been defined and is part of the first segment, let a_{k+1} be any member of $X \setminus \{a_0, \ldots, a_k\}$ which is connected by an edge (in either direction) to some member of $\{a_0, \ldots, a_k\}$. If there does not exist such an element, then we move on to the second segment, and so forth, using the same instructions as above. Of course, when we run out of elements of X, we are done.

To see that this tuple witnesses separation, note that by construction there are no relations between members of distinct segments, (1) implies that there are no fixed points, and (2) implies that each element in the tuple has at most one relation with any element appearing earlier in the tuple. Finally, if a member of the tuple has a relation with two distinct elements appearing earlier in the same segment, then there are distinct paths of that member down to the first element of the segment; taking the first shared element of both paths, we get a loop which contradicts (3).

5.2. **The Key Property for Automorphisms.**

Proposition 5.11 (Key Property)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $a_0, \ldots, a_{n-1}$ are distinct elements of T_α and $\mathcal{G} = \{g_\tau : \tau \in A\}$ is a finite indexed set of automorphisms of $T \upharpoonright (\beta + 1)$ such that $\mathcal{G} \upharpoonright (\alpha + 1)$ is separated on $(a_0, \ldots, a_{n-1})$. Let $t \subseteq T_\beta$ be finite. Then there exist $b_0, \ldots, b_{n-1}$ in $T_\beta \setminus t$ such that $a_i <_T b_i$ for all $i < n$, and for all $\tau \in A$, $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ are g_τ-consistent.*

Proof. Let t^* be the set of all $y \in T_\beta$ such that either $y \in t$, or $y = g_{\tau_{l-1}}^{m_{l-1}}(\cdots (g_{\tau_1}^{m_1}(x))$, for some $x \in t$, $l \leq n + 1$, $\tau_1, \ldots, \tau_{l-1} \in A$, and $m_1, \ldots, m_{l-1} \in \{-1, 1\}$. Note that t^* is finite.

By induction on $k < n$, we will choose $b_k \in T_\beta$ above a_k, maintaining that for all $k < n$:

(a) for all $\tau \in A$, $(a_0, \ldots, a_k)$ and $(b_0, \ldots, b_k)$ are g_τ-consistent;
(b) if there does not exist a triple (j, m, τ), where $j < k$, $m \in \{-1, 1\}$, and $\tau \in A$, such that $g_\tau^m(a_k) = a_j$, then $b_k \notin t^*$.

For the base case, let b_0 be an arbitrary member of $T_\beta \setminus t^*$ above a_0, which is possible since T is infinitely splitting. Consider any $\tau \in A$. Since $\mathcal{G}$ is separated on $(a_0, \ldots, a_{n-1})$, $g_\tau(a_0) \neq a_0$, which implies that $g_\tau(b_0) \neq b_0$. So (a_0) and (b_0) are g_τ-consistent.

Now let $0 < k < n$ be given, and assume that we have chosen b_i for all $i < k$ so that the tuple $(b_0, \dots, b_{k-1})$ satisfies the inductive hypotheses.

Case 1: There does not exist a triple (j, m, τ), where $j < k$, $m \in \{-1, 1\}$, and $\tau \in A$, such that $g_\tau^m(a_k) = a_j$. In this case, let b_k be an arbitrary member of $T_\beta \setminus t^*$ above a_k, which is possible since T is infinitely splitting. Since there are no relations between a_k and members of the tuple $(a_0, \dots, a_{k-1})$, the inductive hypothesis together with the argument we gave for the base case easily imply that for all $\tau \in A$, $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are g_τ-consistent. So inductive hypothesis (a) holds, and (b) is immediate from the choice of b_k.

Case 2: There exists a triple (j, m, σ), where $j < k$, $m \in \{-1, 1\}$, and $\sigma \in A$, such that $g_\sigma^m(a_k) = a_j$. Hence, $a_k = g_\sigma^{1-m}(a_j)$. Since $\mathcal{G} \restriction (\alpha + 1)$ is separated on $(a_0, \dots, a_{n-1})$, this triple is unique. Let $b_k = g_\sigma^{1-m}(b_j)$. By the uniqueness of the triple (j, m, σ), the inductive hypotheses, and the argument we gave in the base case, it easily follows that for all $\tau \in A$, $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are g_τ-consistent. So inductive hypothesis (a) holds, and (b) holds vacuously.

It remains to show that for all $\bar{n} < n$, $b_{\bar{n}} \notin t$. Suppose for a contradiction that there exists some $\bar{n} < n$ such that $b_{\bar{n}} \in t$. Applying Lemma 5.5, fix a sequence

$$\langle i_0, (i_1, m_1, \tau_1), \dots, (i_{l-1}, m_{l-1}, \tau_{l-1}) \rangle,$$

for some $l \leq \bar{n} + 1$, such that:

(1) $\bar{n} = i_0 > i_1 > \cdots > i_{l-1} \geq 0$;
(2) for all $0 < k < l$, $m_k \in \{-1, 1\}$, $\tau_k \in A$, and $g_{\tau_k}^{m_k}(a_{i_{k-1}}) = a_{i_k}$;
(3) there does not exist a triple (i, m, τ) such that $i < i_{l-1}$, $m \in \{-1, 1\}$, $\tau \in I$, and $g_\tau^m(a_{i_{l-1}}) = a_i$.

By (3) and inductive hypothesis (b), $b_{i_{l-1}} \notin t^*$. By (2) and Case 2, we have that

$$b_{i_{l-1}} = g_{\tau_{l-1}}^{m_{l-1}}(\cdots(g_{\tau_1}^{m_1}(b_{i_0}))).$$

So $b_{i_{l-1}} \in t^*$, which is a contradiction. □

Proposition 5.12 (1-Key Property). *Let $\alpha < \beta < \omega_1$. Suppose that $a_0, \dots, a_{n-1}$ are distinct elements of T_α and $\mathcal{G} = \{g_\tau : \tau \in A\}$ is a finite indexed set of automorphisms of $T \restriction (\beta + 1)$ such that $\mathcal{G} \restriction (\alpha + 1)$ is separated on $(a_0, \dots, a_{n-1})$. Let $\bar{n} < n$. Fix $b \in T_\beta$ such that $a_{\bar{n}} <_T b$. Then there exist $b_0, \dots, b_{n-1}$ in T_β such that $a_i <_T b_i$ for all $i < n$, $b_{\bar{n}} = b$, and for all $\tau \in A$, $(a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$ are g_τ-consistent.*

Proof. Apply Lemma 5.5 to find some $l \leq \bar{n} + 1$ and a sequence

$$\langle i_0, (i_1, m_1, \tau_1), \dots, (i_{l-1}, m_{l-1}, \tau_{l-1}) \rangle,$$

satisfying (1)-(3) of that lemma. In particular, $i_0 = \bar{n}$. Define $(c_0, \dots, c_{l-1})$ inductively by letting $c_0 = b$, and for all $0 < k < l$, $c_k = g_{\tau_k}^{m_k}(c_{k-1})$. Using the fact that $a_{\bar{n}} <_T b$ and (2) of Lemma 5.5, it is easy to prove by induction that for all $k < l$, $a_{i_k} <_T c_k$.

By induction on $i < n$ we will choose b_i in T_β above a_i, maintaining that for all $k < n$:

(a) for all $\tau \in A$, $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are g_τ-consistent;
(b) for all $m < l$, if $i_m \leq k$ then $b_{i_m} = c_m$.

Assuming that we are able to define $(b_0, \dots, b_{n-1})$ with these properties, then for all $\tau \in A$, $(a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$ are g_τ-consistent, and $b_{\bar{n}} = b_{i_0} = c_0 = b$, which completes the proof.

For the base case, if $0 \in \{i_0, \dots, i_{l-1}\}$, then clearly $0 = i_{l-1}$, so in this case we let $b_0 = b_{i_{l-1}} = c_{l-1}$. Otherwise, let b_0 be an arbitrary element of T_β above a_0. Consider any $\tau \in A$. Since $\mathcal{G} \restriction (\alpha + 1)$ is separated on $(a_0, \dots, a_{n-1})$, $g_\tau(a_0) \neq a_0$, and hence $g_\tau(b_0) \neq b_0$. So (a_0) and (b_0) are g_τ-consistent. Clearly, the inductive hypotheses are maintained.

Now let $0 < k < n$ and assume that we have chosen b_i for all $i < k$ so that $(b_0, \ldots, b_{k-1})$ satisfies the inductive hypotheses.

Case 1: There does not exist a triple (j, m, τ) such that $j < k$, $m \in \{-1, 1\}$, $\tau \in A$, and $g_\tau^m(a_k) = a_j$. If $k \in \{i_0, \ldots, i_{l-1}\}$, then clearly $k = i_{l-1}$, and we let $b_k = b_{i_{l-1}} = c_{l-1}$. So inductive hypothesis (b) holds. Otherwise, choose b_k above a_k arbitrarily. For all $\tau \in A$, $g_\tau(a_k) \neq a_k$, which implies that $g_\tau(b_k) \neq b_k$. So the inductive hypothesis together with the fact that there are no relations between a_k and members of $(a_0, \ldots, a_{k-1})$ easily imply inductive hypothesis (a).

Case 2: There exists a triple (j, m, σ) such that $j < k$, $m \in \{-1, 1\}$, $\sigma \in A$, and $g_\sigma^m(a_k) = a_j$. Then $a_k = g_\sigma^{1-m}(a_j)$. Define $b_k = g_\sigma^{1-m}(b_j)$. By the inductive hypothesis, the uniqueness of the triple (j, m, σ), and the fact that $g_\tau(b_k) \neq b_k$ for all $\tau \in A$, it easily follows that for all $\tau \in A$, $(a_0, \ldots, a_k)$ and $(b_0, \ldots, b_k)$ are g_τ-consistent. In the case that $k \in \{i_0, \ldots, i_{l-1}\}$, by the uniqueness of the triple (j, m, σ) and the assumption of Case 2 it must be the case that $k = i_{q-1}$ for some q such that $0 < q \leq l - 1$, $j = i_q$, $m = m_q$, and $\tau = \tau_q$. By the induction hypothesis, $b_{i_q} = c_q$, and by definition of $b_{i_{q-1}}$ and c_{q-1}, $b_{i_{q-1}} = g_{\tau_q}^{1-m_q}(b_{i_q}) = g_{\tau_q}^{1-m_q}(c_q) = c_{q-1}$. □

We cannot improve Proposition 5.12 to get the Strong 1-Key Property of Definition 2.5. For example, using the configuration described in Proposition 5.12, if $g_\tau(a_{\bar{n}}) = a_k$ for some $\tau \in A$ and $k < \bar{n}$, then letting $t = \{g_\tau(b)\}$, we cannot find $b_0, \ldots, b_{n-1}$ as described in Proposition 5.12 which are all not in t.

5.3. **Constructing and Extending Automorphisms.**

Lemma 5.13. *Let $\gamma < \omega_1$ and let $\{f_\tau : \tau \in I\}$ be a countable family of automorphisms of $T \restriction (\gamma + 1)$. Then there exists a family $\{g_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\gamma + 2)$ such that:*

(1) *for all $\tau \in I$, $f_\tau \subseteq g_\tau$;*
(2) *for all distinct τ_0 and τ_1 in I and for all $x \in T_{\gamma+1}$, $g_{\tau_0}(x) \neq g_{\tau_1}(x)$.*

Proof. Fix a bijection $h : \omega \to T_{\gamma+1} \times I$. Let $g_\tau \restriction (\gamma + 1) = f_\tau$ for all $\tau \in I$. We will define the values of each g_τ on $T_{\gamma+1}$ in ω-many stages, where at any given stage we will have defined only finitely many values of finitely many g_τ's. We also define a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of $T_{\gamma+1}$.

At stage 0, we do nothing. Let $X_0 = \emptyset$. Now let $n < \omega$ and suppose that stage n is complete. In particular, the finite set $X_n \subseteq T_{\gamma+1}$ has been defined. Let $h(n) = (z, \sigma)$. Stage $n + 1$ will consist of two steps. In the first step, if $g_\sigma(z)$ is already defined, then move on to step 2. Otherwise, define $g_\sigma(z)$ to be some element of $T_{\gamma+1}$ above $f_\sigma(z \restriction \gamma)$ which is not in X_n. In the second step, if $g_\sigma^{-1}(z)$ is already defined, then we are done. If not, then define $g_\sigma^{-1}(z)$ to be some element of $T_{\gamma+1}$ above $f_\sigma^{-1}(z \restriction \gamma)$ which is not in X_n. Now let $X_{n+1} = X_n \cup \{z, g_\sigma(z), g_\sigma^{-1}(z)\}$.

This completes the construction. It is easy to check that for all $\tau \in I$, g_τ is a strictly increasing function from $T \restriction (\gamma + 2)$ onto $T \restriction (\gamma + 2)$. Suppose for a contradiction that for some $z \in T_{\gamma+1}$, $g_{\tau_0}(z) = g_{\tau_1}(z)$ for distinct τ_0 and τ_1 in I. Assume that $g_{\tau_0}(z)$ was defined at stage n and $g_{\tau_1}(z)$ was defined at stage m, where $n < m$. At stage n, either $h(n) = (\tau_0, z)$ and we defined $g_{\tau_0}(z)$, or for some y_0, $h(n) = (\tau_0, y_0)$ and we defined $z = g_{\tau_0}^{-1}(y_0)$. In either case, both z and $g_{\tau_0}(z)$ are in X_n. At stage m, either $h(n) = (\tau_1, z)$ and we defined $g_{\tau_1}(z)$ which is not in X_n, or for some y_1, $h(n) = (\tau_1, y_1)$ and we defined $z = g_{\tau_1}^{-1}(y_1)$ which is not in X_n. In the first case, $g_{\tau_1}(z)$ cannot equal $g_{\tau_0}(z)$ since the latter element is in X_n, and the second case is impossible since $z \in X_n$. So we have a contradiction. A similar argument shows that each g_τ is injective, and hence is an automorphism of $T \restriction (\gamma + 2)$. □

Lemma 5.14. *Assume the following:*

- *$\gamma < \omega_1$;*
- *$X \subseteq T_{\gamma+1}$ is finite and has unique drop-downs to γ;*
- *$\{f_\tau : \tau \in I\}$ is a countable collection of automorphisms of $T \restriction (\gamma + 1)$;*
- *$A \subseteq I$ is finite.*

Then there exists a family $\{g_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\gamma + 2)$ satisfying:

(1) *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
(2) *for all $\tau \in A$, $X \restriction \gamma$ and X are g_τ-consistent;*
(3) *if $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$, then $\{g_\tau : \tau \in I\}$ is separated.*

Proof. Fix a bijection $h : \omega \to T_{\gamma+1} \times I$. For each $\tau \in I$, define $g_\tau \restriction (\gamma + 1) = f_\tau$.

We will define the values of the functions g_τ on $T_{\gamma+1}$ in ω-many stages. The following describes the construction:

- at any given stage n, we will have defined only finitely many values of the functions g_τ for finitely many $\tau \in I$;
- we will define a sequence $\langle a_k : k < \omega\rangle$ which enumerates $T_{\gamma+1}$, where at any stage n we will have defined $\langle a_k : k < l_n\rangle$ for some $l_n < \omega$, and let $X_n = \{a_k : k < l_n\}$.

We will maintain the following inductive hypotheses:

(i) for all n, if the value $g_\tau^m(a) = b$ is defined at stage n, where $\tau \in I$ and $m \in \{-1, 1\}$, then a and b are in X_n, $f_\tau^m(a \restriction \gamma) = b \restriction \gamma$, and if $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$, then $a \neq b$;
(ii) for all $n_0 < n_1$, if a and b are in X_{n_0} and $g_\tau^m(a) = b$ has been defined by the end of stage n_1, where $\tau \in I$ and $m \in \{-1, 1\}$, then $g_\tau^m(a) = b$ has been defined by the end of stage n_0;
(iii) in the case that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$, then for all n and $k < l_n$ there exists at most one triple (j, m, τ), where $j < k$, $m \in \{-1, 1\}$, and $\tau \in I$, such that $g_\tau^m(a_k)$ has been defined by stage n and $g_\tau^m(a_k) = a_j$.

<u>Stage 0</u>: For each $\tau \in A$ and $x, y \in X$, define $g_\tau(x) = y$ iff $f_\tau(x \restriction \gamma) = y \restriction \gamma$. Let $A_0 = A$ and let $l_0 = |X|$. In the case that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$, fix an injective enumeration $\vec{a} = (a_0, \ldots, a_{l_0-1})$ of X such that $\{f_\tau : \tau \in A\}$ is separated on $\vec{a} \restriction \gamma$. Otherwise, let $\vec{a} = (a_0, \ldots, a_{l_0-1})$ be an arbitrary injective enumeration of X. It is easy to check that the required properties hold.

<u>Stage $n + 1$</u>: Let $n < \omega$ and suppose that stage n is complete. Let $h(n) = (z, \sigma)$. Stage $n + 1$ consists of two steps. In the first step, if $g_\sigma(z)$ is already defined, then move on to step two. Otherwise, define $g_\sigma(z)$ to be some member of $T_{\gamma+1}$ above $f_\sigma(z \restriction \gamma)$ which is not in $X_n \cup \{z\}$. This is possible since T is infinitely splitting. In the second step, if $g_\sigma^{-1}(z)$ is already defined, then we are done. Otherwise, define $g_\sigma^{-1}(z)$ to be some member of $T_{\gamma+1}$ above $g_\sigma^{-1}(z \restriction \gamma)$ which is not in $X_n \cup \{z, g_\sigma(z)\}$. Again, this is possible since T is infinitely splitting. Now define $\langle a_k : k < l_{n+1}\rangle$ by adding at the end of the sequence $\langle a_k : k < l_n\rangle$ the elements among z, $g_\sigma(z)$, and $g_\sigma^{-1}(z)$ which are not already in X_n, in the order just listed.

Let us check that inductive hypotheses (i)-(iii) hold. (i) is clear. For (ii), the only new equations of the form $g_\tau^m(a) = b$ which were introduced at stage $n + 1$, where $\tau \in I$, $m \in \{-1, 1\}$, and $a, b \in T_{\gamma+1}$, is when $\tau = \sigma$ and at least one of a or b is in $X_{n+1} \setminus X_n$. So (ii) easily follows from the inductive hypothesis.

Now we prove (iii). Assume that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$. Consider first the case when z is not in X_n. Then by inductive hypothesis (i), neither $g_\sigma(z)$ nor $g_\sigma^{-1}(z)$ were defined at any earlier stage. So by definition, $g_\sigma(z)$ and $g_\sigma^{-1}(z)$ are not in X_n. Hence, the last three elements of $\langle a_k : k < l_{n+1}\rangle$ are z, $g_\sigma(z)$, and $g_\sigma^{-1}(z)$. The relations introduced between these three elements at stage $n+1$ yield no counter-example to (iii), and z, $g_\sigma(z)$, and $g_\sigma^{-1}(z)$ have no relations to any elements of $\langle a_k : k < l_n \rangle$. (iii) follows from these observations and the inductive hypothesis.

Next, consider the case when z is in X_n. Then z already appears on the sequence $\langle a_k : k < l_n \rangle$. At stage $n+1$, no new relations are introduced between elements of $\langle a_k : k < l_n \rangle$. Each new element in $X_{n+1} \setminus X_n$ has exactly one relation between any other member of X_{n+1}, namely z. (iii) follows from these observations and the inductive hypothesis.

This completes the construction. It is straightforward to check that each g_τ is an automorphism of $T \restriction (\gamma+2)$. By what we did at stage 0, for all $\tau \in A$, $X \restriction \gamma$ and X are g_τ-consistent. Now assume that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \gamma$. To show that $\{g_\tau : \tau \in I\}$ is separated, let $Y \subseteq T_{\gamma+1}$ be finite. Fix a large enough n so that $Y \subseteq X_n$. Then by Lemma 5.9, it suffices to show that $\{g_\tau : \tau \in I\}$ is separated on X_n, as witnessed by the tuple $(a_0, \ldots, a_{l_n - 1})$. Suppose that $k < l_n$ and the triple (j, m, τ) satisfies that $j < k$, $m \in \{-1, 1\}$, $\tau \in I$, and $g_\tau^m(a_k) = a_j$. By inductive hypothesis (ii), the relation $g_\tau^m(a_k) = a_j$ was introduced by the end of stage n. By inductive hypothesis (iii), there is at most one such triple. □

Proposition 5.15. *Assume the following:*

- $\alpha < \delta < \omega_1$;
- $X \subseteq T_\delta$ *is finite and has unique drop-downs to* α;
- $\{f_\tau : \tau \in I\}$ *is a countable collection of automorphisms of* $T \restriction (\alpha+1)$;
- $A \subseteq I$ *is finite.*

Then there exists a collection $\{g_\tau : \tau \in I\}$ *of automorphisms of* $T \restriction (\delta+1)$ *such that:*

(1) $f_\tau \subseteq g_\tau$ *for all* $\tau \in I$;
(2) *for all* $\tau \in A$, $X \restriction \alpha$ *and* X *are* g_τ*-consistent;*
(3) *if* $\{f_\tau : \tau \in A\}$ *is separated on* $X \restriction \alpha$, *then* $\{g_\tau : \tau \in I\}$ *is separated.*

Proof. The proof is by induction on δ, where the base case and the successor case follow easily from Lemma 5.14 and the inductive hypothesis. Assume that δ is a limit ordinal and the statement holds for all β with $\alpha < \beta < \delta$. Fix a surjection $h : \omega \to T_\delta \times I$ such that each member of the codomain has an infinite preimage. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \alpha$.

We will define by induction in ω-many stages the following objects satisfying the listed properties:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of I with union equal to I;
- a non-decreasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ;
- for each $n < \omega$ a collection $\{f_\tau^n : \tau \in I\}$ of automorphisms of $T \restriction (\delta_n + 1)$, where $f_\tau^n \subseteq f_\tau^m$ for all $m > n$;
- partial injective functions h_τ^n from X_n to X_n for all $n < \omega$ and $\tau \in A_n$.

The following inductive hypotheses will be maintained for each $n < \omega$:

(a) X_n has unique drop-downs to δ_n;
(b) if $n > 0$, then for all $\tau \in A_{n-1}$, $X_{n-1} \restriction \delta_{n-1}$ and $X_{n-1} \restriction \delta_n$ are f_τ^n-consistent;
(c) if $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \alpha$, then $\{f_\tau^n : \tau \in I\}$ is separated;
(d) for all $\tau \in A_n$ and $x, y \in X_n$,

$$h_\tau^n(x) = y \iff f_\tau^n(x \restriction \delta_n) = y \restriction \delta_n.$$

Stage 0: Let $X_0 = X$, $A_0 = A$, and $\delta_0 = \alpha + 1$. Apply the inductive hypothesis to find a collection $\{f_\tau^0 : \tau \in I\}$ of automorphisms of $T \restriction (\alpha+2)$ satisfying:

- $f_\tau \subseteq f_\tau^0$ for all $\tau \in I$;

- for all $\tau \in A$, $X \restriction \alpha$ and $X \restriction (\alpha+1)$ are f^0_τ-consistent;
- if $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \alpha$, then $\{f^0_\tau : \tau \in I\}$ is separated.

For all $x, y \in X$ and $\tau \in A$, define $h^0_\tau(x) = y$ iff $f^0_\tau(x \restriction \delta_0) = y \restriction \delta_0$.

Stage $n > 0$: Let $0 < n < \omega$ and assume that we have completed stage $n-1$. In particular, we have defined X_{n-1}, A_{n-1}, δ_{n-1}, $\{f^{n-1}_\tau : \tau \in I\}$, and $\{h^{n-1}_\tau : \tau \in I\}$ satisfying the required properties. Let $h(n-1) = (z, \sigma)$.

If $z \in X_{n-1}$, $\sigma \in A_{n-1}$, $z \in \mathrm{dom}(h^{n-1}_\sigma)$, and $z \in \mathrm{ran}(h^{n-1}_\sigma)$, then there is nothing for us to do at stage n. So let $X_n = X_{n-1}$, $A_n = A_{n-1}$, $\delta_n = \delta_{n-1}$, $f^n_\tau = f^{n-1}_\tau$ and $h^n_\tau = h^{n-1}_\tau$ for all $\tau \in I$. The required properties are immediate. If not, then exactly one of the following cases holds.

Case 1: Either $z \notin X_{n-1}$ or $\sigma \notin A_{n-1}$. Fix $\delta_n < \delta$ larger than δ_{n-1} and γ_n such that $X_{n-1} \cup \{z\}$ has unique drop-downs to δ_n. Apply the inductive hypothesis to find a collection $\{f^n_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\delta_n + 1)$ satisfying:

(1) $f^{n-1}_\tau \subseteq f^n_\tau$ for all $\tau \in I$;
(2) for all $\tau \in A_{n-1}$, $X_{n-1} \restriction \delta_{n-1}$ and $X_{n-1} \restriction \delta_n$ are f^n_τ-consistent;
(3) if $\{f^{n-1}_\tau : \tau \in A_{n-1}\}$ is separated on $X_{n-1} \restriction \delta_{n-1}$, then $\{f^n_\tau : \tau \in I\}$ is separated.

Define $X_n = X_{n-1} \cup \{z\}$ and $A_n = A_{n-1} \cup \{\sigma\}$.

Case 2: $z \in X_{n-1}$, $\sigma \in A_{n-1}$, and $z \notin \mathrm{dom}(h^{n-1}_\sigma)$. Define $\delta_n = \delta_{n-1}$, $A_n = A_{n-1}$, and $f^n_\tau = f^{n-1}_\tau$ for all $\tau \in I$. We claim that $f^n_\sigma(z \restriction \delta_n)$ is not in $X_{n-1} \restriction \delta_n$. For otherwise let x be the unique element of X_{n-1} such that $x \restriction \delta_n = f^n_\sigma(z \restriction \delta_n)$, which exists by unique drop-downs. By inductive hypothesis (d) applied to $n-1$, $h^{n-1}_\sigma(z) = x$, which is a contradiction. Choose some $c \in T_\delta$ which is above $f^n_\sigma(z \restriction \delta_n)$. Define $X_n = X_{n-1} \cup \{c\}$, and note that by the claim just proved, X_n has unique drop-downs to δ_n.

Case 3: $z \in X_{n-1}$, $\sigma \in A_{n-1}$, $z \in \mathrm{dom}(h^{n-1}_\sigma)$, and $z \notin \mathrm{ran}(h^{n-1}_\sigma)$. Define $\delta_n = \delta_{n-1}$, $A_n = A_{n-1}$, and $f^n_\tau = f^{n-1}_\tau$ for all $\tau \in I$. By the same argument as in Case 2, $(f^n_\sigma)^{-1}(z \restriction \delta_n)$ is not in $X_{n-1} \restriction \delta_{n-1}$. Let d be some element of T_δ which is above $(f^n_\sigma)^{-1}(z \restriction \delta_n)$. Define $X_n = X_{n-1} \cup \{d\}$, and note that X_n has unique drop-downs to δ_n.

Now in any case, define for all $\tau \in A_n$ and $x, y \in X_n$,

$$h^n_\tau(x) = y \iff f^n_\tau(x \restriction \delta_n) = y \restriction \delta_n.$$

Note that in Case 2, $h^n_\sigma(z) = c$, so $z \in \mathrm{dom}(h^n_\sigma)$, and in Case 3, $h^n_\sigma(d) = z$, so $z \in \mathrm{ran}(h^n_\sigma)$.

Inductive hypotheses (a), (b), and (d) are clear. Let us verify inductive hypothesis (c). Suppose that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \alpha$. By (c) for $n-1$, $\{f^{n-1}_\tau : \tau \in I\}$ is separated. In Cases 2 and 3 we are done. For Case 1, we clearly have that $\{f^{n-1}_\tau : \tau \in A_{n-1}\}$ is separated on $X_{n-1} \restriction \delta_{n-1}$. So by statement (3) of Case 1, $\{f^n_\tau : \tau \in I\}$ is separated.

This completes the construction. We claim that for all $0 < n < \omega$ and $\tau \in A_{n-1}$, $h^n_\tau \cap X^2_{n-1} = h^{n-1}_\tau$. Let $x, y \in X_{n-1}$. Then $h^n_\tau(x) = y$ iff $f^n_\tau(x \restriction \delta_n) = y \restriction \delta_n$ (by (d) for n) iff $f^{n-1}_\tau(x \restriction \delta_{n-1}) = y \restriction \delta_{n-1}$ (by (b) for n) iff $h^{n-1}_\tau(x) = y$ (by (d) for $n-1$).

Fix $\tau \in I$. By our bookkeeping, it is clear that $h_\tau = \bigcup\{h^n_\tau : n < \omega,\ \tau \in A_n\}$ is a bijection from T_δ to T_δ. And by (d), it is straightforward to show that $g_\tau = \bigcup\{f^n_\tau : n < \omega\} \cup h_\tau$ is an automorphism of $T \restriction (\delta+1)$ satisfying that $f_\tau \subseteq g_\tau$. Let $\tau \in A = A_0$. For any $x, y \in X = X_0$, by (d) we have that $g_\tau(x) = y$ iff $h^0_\tau(x) = y$ iff $f_\tau(x \restriction \alpha) = y \restriction \alpha$ iff $g_\tau(x \restriction \alpha) = y \restriction \alpha$. Hence, $X \restriction \alpha$ and X are g_τ-consistent.

Now assume that $\{f_\tau : \tau \in A\}$ is separated on $X \restriction \alpha$. To prove that $\{g_\tau : \tau \in I\}$ is separated, let $Y \subseteq T_\delta$ be finite. Fix n large enough so that $Y \subseteq X_n$. Then by inductive hypothesis (a), Y has unique drop-downs to δ_n. By inductive hypothesis (c), $\{f^n_\tau : \tau \in I\}$ is separated on $X_n \restriction \delta_n$. By Lemma 5.7 (Persistence for Sets), it follows that $\{g_\tau : \tau \in B\}$ is separated on Y. □

5.4. **The Forcing Poset for Adding Automorphisms.**

Definition 5.16. *Let $\mathbb{Q}$ be the forcing poset whose conditions are all automorphisms $f : T \restriction (\alpha+1) \to T \restriction (\alpha+1)$, for some $\alpha < \omega_1$, ordered by $g \leq f$ if $f \subseteq g$. If $f \in \mathbb{Q}$ is an automorphism of $T \restriction (\alpha+1)$, then α is the* top level of f.

Definition 5.17. *Let $\mathbb{P}$ be the forcing poset whose conditions are all functions p satisfying:*

(1) *the domain of p is a countable subset of κ;*
(2) *there exists an ordinal $\alpha < \omega_1$, which we call the* top level of p, *such that for all $\tau \in \operatorname{dom}(p)$, $p(\tau)$ is an automorphism of $T \restriction (\alpha+1)$.*

Let $q \leq p$ if $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$ and for all $\tau \in \operatorname{dom}(p)$, $p(\tau) \subseteq q(\tau)$.

Definition 5.18 (Consistency)**.** *Let $\alpha < \beta < \omega_1$ and let $q \in \mathbb{Q}$.*

(1) *Let $X \subseteq T_\beta$ be finite with unique drop-downs to α. We say that $X \restriction \alpha$ and X are q-*consistent *if for all $x, y \in X$, $q(x \restriction \alpha) = y \restriction \alpha$ iff $q(x) = y$.*
(2) *Let $\vec{a} = (a_0, \ldots, a_{n-1})$ consist of distinct elements of T_β. We say that $\vec{a} \restriction \alpha$ and $\vec{a}$ are q-*consistent *if for all $i, j < n$, $q(a_i \restriction \alpha) = a_j \restriction \alpha$ iff $q(a_i) = a_j$.*

Definition 5.19 (Separation)**.** *Let $\alpha < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α, $A \subseteq \operatorname{dom}(p)$, and $\vec{a} = (a_0, \ldots, a_{n-1})$ is an injective tuple whose members are in T_α. We say that $\{p(\tau) : \tau \in A\}$ is* separated on $\vec{a}$ *if for all $k < n$:*

(1) *for all $\tau \in A$, $p(\tau)(a_k) \neq a_k$;*
(2) *there exists at most one triple (j, m, τ), where $j < k$, $m \in \{-1, 1\}$, and $\tau \in A$, such that $p(\tau)^m(a_k) = a_j$.*

Definition 5.20 (Separation for Sets)**.** *Let $\alpha < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α, $A \subseteq \operatorname{dom}(p)$, and $X \subseteq T_\alpha$ is finite. We say that $\{p(\tau) : \tau \in A\}$ is* separated on X *if there exists some injective tuple $\vec{a}$ which lists the elements of X such that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$.*

We have now defined for the automorphism forcing the objects and properties described in Section 2, where we let $\mathbb{Q}_\tau = \mathbb{Q}$ for all $\tau < \kappa$. We now work towards verifying properties (A)-(E) of Section 2. (A) is clear, and (B) (Transitivity) follows from Lemma 5.2 (Transitivity). For (C) (Persistence):

Lemma 5.21 (Persistence)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α, $A \subseteq \operatorname{dom}(p)$, $\vec{a}$ is an injective tuple whose members are in T_α, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a}$. Then for any $q \leq p$ with top level β and any tuple $\vec{b}$ above $\vec{a}$ whose members are in T_β, $\{q(\tau) : \tau \in B\}$ is separated on $\vec{b}$.*

Proof. Immediate by Lemma 5.4 (Persistence). ☐

Lemma 5.22 (Persistence for Sets)**.** *Let $\alpha < \beta < \omega_1$. Assume that p is a condition with top level α, $q \leq p$, q has top level β, $X \subseteq T_\beta$ is finite and has unique drop-downs to α, and $A \subseteq \operatorname{dom}(p)$ is finite. If $\{p(\tau) : \tau \in A\}$ is separated on $X \restriction \alpha$, then $\{q(\tau) : \tau \in A\}$ is separated on X.*

Proof. Let $\vec{a}$ be an injective tuple which lists the elements of X in such a way that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a} \restriction \alpha$. Then $\{q(\tau) : \tau \in A\}$ is separated on $\vec{a}$ by Lemma 5.21 (Persistence). ☐

Definition 5.23. *A condition $p \in \mathbb{P}$ with top level $\alpha < \omega_1$ is* separated *if for any finite set $X \subseteq T_\alpha$, $\{p(\tau) : \tau \in \operatorname{dom}(p)\}$ is separated on X.*

The next lemma implies (D) (Extension).

Lemma 5.24 (Extension)**.** *Let $\alpha < \beta < \omega_1$ and let $X \subseteq T_\beta$ be finite with unique drop-downs to α. Suppose that $p \in \mathbb{P}$ has top level α and $A \subseteq \mathrm{dom}(p)$ is finite. Then there exists some $q \leq p$ with top level β and with the same domain as p such that for all $\tau \in A$, $X \restriction \alpha$ and X are $q(\tau)$-consistent. Moreover, if $\{p(\tau) : \tau \in A\}$ is separated on $X \restriction \alpha$, then we can find such a condition q which is separated.*

Proof. Immediate from Proposition 5.15. ☐

Finally, (E) (Key Property) follows from the next proposition.

Proposition 5.25 (Key Property)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $a_0, \ldots, a_{n-1}$ are distinct elements of T_α, $p \in \mathbb{P}$ has top level α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $(a_0, \ldots, a_{n-1})$. Then for any $q \leq p$ with top level β and any finite set $t \subseteq T_\beta$, there exist $b_0, \ldots, b_{n-1}$ in $T_\beta \setminus t$ such that $a_i <_T b_i$ for all $i < n$, and for all $\tau \in A$, $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ are $q(\tau)$-consistent.*

Proof. Immediate from Proposition 5.11 letting $\mathcal{G} = \{q(\tau) : \tau \in A\}$. ☐

Proposition 5.26 (1-Key Property)**.** *Let $\alpha < \beta < \omega_1$. Suppose that $a_0, \ldots, a_{n-1}$ are distinct elements of T_α, $p \in \mathbb{P}$ has top level α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $(a_0, \ldots, a_{n-1})$. Fix $\bar{n} < n$ and $b \in T_\beta$ with $a_{\bar{n}} <_T b$. Then for any $q \leq p$ with top level β, there exist $b_0, \ldots, b_{n-1}$ in T_β such that $a_i <_T b_i$ for all $i < n$, $b_{\bar{n}} = b$, and for all $\tau \in A$, $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ are $q(\tau)$-consistent.*

Proof. Immediate from Proposition 5.12 letting $\mathcal{G} = \{q(\tau) : \tau \in A\}$. ☐

Lemma 5.27. *For any $\tau < \kappa$ and $\rho < \omega_1$, the set of conditions $q \in \mathbb{P}$ such that $\tau \in \mathrm{dom}(q)$ and the top level of q is at least ρ is dense open.*

Proof. Given any condition $p \in \mathbb{P}$, we can easily add τ to the domain of p, for example, by attaching to it the identity automorphism. The second part of the statement follows from Lemma 5.24 (Extension). ☐

Lemma 5.28. *For any $\tau_0 < \tau_1 < \kappa$, the set of conditions $q \in \mathbb{P}$ with some top level α satisfying that $\tau_0, \tau_1 \in \mathrm{dom}(q)$ and for all $x \in T_\alpha$, $q(\tau_0)(x) \neq q(\tau_1)(x)$, is dense open.*

Proof. The set of such conditions is dense by Lemmas 5.13 and 5.27. It is easy to check that if $\alpha < \beta$, f and g are automorphisms of $T \restriction (\beta + 1)$, $x \in T_\alpha$, and $f(x) \neq g(x)$, then for any $y \in T_\beta$ above x, $f(y) \neq f(y)$. This fact easily implies that the set of such conditions in open. ☐

Lemma 5.29 (Separated Conditions are Dense)**.** *The set of separated conditions is dense in $\mathbb{P}$. In fact, suppose that p is a condition with top level $\gamma < \omega_1$, $X \subseteq T_{\gamma+1}$ is finite with unique drop-downs to γ, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $X \restriction \gamma$. Then there exists $q \leq p$ with top level $\gamma + 1$ such that q is separated on $T_{\gamma+1}$ and for all $\tau \in A$, $X \restriction \gamma$ and X are $q(\tau)$-consistent.*

Proof. Immediate from Lemma 5.14. ☐

Lemma 5.30. *A condition $p \in \mathbb{P}$ with top level α is separated if and only if for any finite set $A \subseteq \mathrm{dom}(p)$ and finite set $X \subseteq T_\alpha$, $\{p(\tau) : \tau \in A\}$ is separated on X.*

Proof. The forward direction of the if and only if is immediate. For the converse, assume the second statement. It easily follows that for any $\tau \in A$, $p(\tau)$ has no fixed-points in T_α. Let $X \subseteq T_\alpha$ be finite with size $n < \omega$ and suppose for a contradiction that $\{p(\tau) : \tau \in \mathrm{dom}(p)\}$ is not separated on X. For any injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which lists the elements of X, we can find some $k < n$ and distinct triples (j_0, m_0, τ_0) and (j_1, m_1, τ_1) such that for $i < 2$, $j_i < k$, $m_i \in \{-1, 1\}$, $\tau_i \in \mathrm{dom}(p)$, and $p(\tau_i)^{m_i}(a_k) = a_{j_i}$. There are only finitely many such enumerations, so we can find a finite set $A \subseteq \mathrm{dom}(p)$ such that for any such enumeration, the fixed triples (j_0, m_0, τ_0) and (j_1, m_1, τ_1) described above satisfy that τ_0 and τ_1 are in A. Now it is easy to check that $\{p(\tau) : \tau \in A\}$ is not separated on X, which is a contradiction. □

Lemma 5.31 (Generalized Key Property)**.** *Let $\alpha < \xi < \omega_1$. Suppose that $p \in \mathbb{P}$ has top level α, $\vec{b}$ is a finite tuple with height α, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$. Assume that $r_0, \ldots, r_{n-1} \leq p$ are conditions with top level ξ, $\{h_\tau : \tau \in B\}$ is a finite family of automorphisms of $T \upharpoonright (\xi + 1)$, and $t \subseteq T_\xi$ is finite. Then there exist tuples $\vec{a}^0, \ldots, \vec{a}^{n-1}$ above $\vec{b}$ with height ξ such that:*

1. *for all $\tau \in A$ and $j < n$, $\vec{b}$ and $\vec{a}^j$ are $r_j(\tau)$-consistent;*
2. *$\vec{a}^0 \upharpoonright (\alpha + 1), \ldots, \vec{a}^{n-1} \upharpoonright (\alpha + 1)$ and $t \upharpoonright (\alpha + 1)$ are pairwise disjoint;*
3. *for all $x \in t$, $\tau \in B$, and $m \in \{-1, 1\}$, $h_\tau^m(x)$ is not in any of the tuples $\vec{a}^0, \ldots, \vec{a}^{n-1}$;*
4. *for all $\tau \in B$, $m \in \{-1, 1\}$, and distinct $i, j < n$, if x is in the tuple $\vec{a}^i$ then $h_\tau^m(x)$ is not in the tuple $\vec{a}^j$.*

Proof. The proof is by induction on $n > 0$. Let $n > 0$ be given, and assume that the statement holds for $n - 1$ (in the case that $n > 1$). Assume that $r_0, \ldots, r_{n-1} \leq p$ are conditions with top level ξ, $\{h_\tau : \tau \in B\}$ is a finite family of automorphisms of $T \upharpoonright (\xi + 1)$, and $t \subseteq T_\xi$ is finite. Applying the inductive hypothesis in the case that $n > 1$, fix $\vec{a}^0, \ldots, \vec{a}^{n-2}$ above $\vec{b}$ with height ξ satisfying (1)-(4).

Define r_{n-1}^* to be the condition with the same domain as r_{n-1} so that for all $\tau \in \mathrm{dom}(r_{n-1})$, $r_{n-1}^*(\tau) = r_{n-1}(\tau) \upharpoonright (\alpha + 1)$. Then $r_{n-1} \leq r_{n-1}^* \leq p$ and r_{n-1}^* has top level $\alpha + 1$. Let Z be the finite set of elements of $T_{\alpha+1}$ which are in one of $\vec{a}^0 \upharpoonright (\alpha + 1), \ldots, \vec{a}^{n-1} \upharpoonright (\alpha + 1)$ or $t \upharpoonright (\alpha + 1)$, or else of the form $h_\tau^m(x) \upharpoonright (\alpha + 1)$, where $\tau \in B$, $m \in \{-1, 1\}$, and x is in one of $\vec{a}^0, \ldots, \vec{a}^{n-1}$ or in t. By Proposition 5.25 (Key Property), find a tuple $\vec{b}^{n-1} > \vec{b}$ with height $\alpha + 1$ such that for all $\tau \in A$, $\vec{b}$ and $\vec{b}^{n-1}$ are $r_{n-1}^*(\tau)$-consistent, and $\vec{b}^{n-1}$ is disjoint from Z. By Lemma 5.21 (Persistence), $\{r_{n-1}^*(\tau) : \tau \in A\}$ is separated on $\vec{b}^{n-1}$. Apply Proposition 5.25 (Key Property) again to find a tuple $\vec{a}^n$ with height ξ above $\vec{b}^{n-1}$ such that for all $\tau \in A$, $\vec{b}^{n-1}$ and $\vec{a}^{n-1}$ are $r_{n-1}(\tau)$-consistent. □

5.5. **Basic Properties of the Automorphism Forcing.** In this subsection we will prove, assuming that T is a free Suslin tree, that the forcing poset $\mathbb{P}$ is totally proper, preserves the fact that T is Suslin, and adds a sequence of length κ of almost disjoint automorphisms of T. Also, assuming CH, $\mathbb{P}$ is ω_2-c.c. In particular, if T is a free Suslin tree, CH holds, and $\kappa \geq \omega_2$, then $\mathbb{P}$ forces that T is an almost Kurepa Suslin tree.

Many of the proofs in the rest of the article will involve constructing total master conditions over countable elementary substructures. The next two lemmas provide tools for such constructions.

Lemma 5.32 (Constructing Total Master Conditions)**.** *Let λ be a large enough regular cardinal and assume that N is a countable elementary substructure of $H(\lambda)$ which contains as elements T, κ, $\mathbb{Q}$, and $\mathbb{P}$. Let $\delta = N \cap \omega_1$.*

Assume the following:

1. *$\langle \delta_n : n < \omega \rangle$ is a non-decreasing sequence of ordinals cofinal in δ;*
2. *$\langle p_n : n < \omega \rangle$ is a decreasing sequence of conditions in $N \cap \mathbb{P}$, where each p_n has top level δ_n;*
3. *$\langle A_n : n < \omega \rangle$ is a subset-increasing sequence of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;*

(4) *$\langle X_n : n < \omega \rangle$ is a subset-increasing sequence of finite subsets of T_δ with union equal to T_δ, where each X_n has unique drop-downs to δ_n;*
(5) *$\{h_{n,\tau} : n < \omega,\ \tau \in A_n\}$ is a family of functions, where each $h_{n,\tau}$ is an injective partial function from X_n to X_n;*
(6) *for all $z \in T_\delta$ and $\tau \in N \cap \kappa$, there exists some $n < \omega$ such that z is in the domain and in the range of $h_{n,\tau}$;*
(7) *for all $n < \omega$ and $\tau \in A_n$:*
   (a) *$\tau \in \mathrm{dom}(p_n)$;*
   (b) *$X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent;*
   (c) *for all $x, y \in X_n$, $h_{n,\tau}(x) = y$ iff $p_n(\tau)(x \restriction \delta_n) = y \restriction \delta_n$.*

Let q be the function with domain $N \cap \kappa$ such that for all $\tau \in N \cap \kappa$,

$$q(\tau) = \bigcup\{p_n(\tau) : n < \omega,\ \tau \in A_n\} \cup \bigcup\{h_{n,\tau} : n < \omega,\ \tau \in A_n\}.$$

Then $q \in \mathbb{P}$, q has top level δ, and $q \leq p_n$ for all $n < \omega$. For all $n < \omega$ and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and X_n are $q(\tau)$-consistent. Moreover, if for all dense open sets $D \in N$ there exists some n such that $p_n \in D$, then q is a total master condition over N.

Proof. We claim that for all $n < \omega$ and $\tau \in A_n$, $h_{n+1,\tau} \cap X_n^2 = h_{n,\tau}$. Let $x, y \in X_n$. By (2) and (7), $h_{n,\tau}(x) = y$ iff $p_n(\tau)(x \restriction \delta_n) = y \restriction \delta_n$ iff $p_{n+1}(\tau)(x \restriction \delta_{n+1}) = y \restriction \delta_{n+1}$ iff $h_{n+1,\tau}(x) = y$. It follows from this fact together with (4), (5), and (6) that for each $\tau \in N \cap \kappa$, $\bigcup\{h_{n,\tau} : n < \omega,\ \tau \in A_n\}$ is a bijection from T_δ onto T_δ.

Let $\tau \in N \cap \kappa$ and we will show that $q(\tau)$ is strictly increasing. It suffices to show that if $x \in T_\delta$ and $\gamma < \delta$, then $q(\tau)(x)$ is above $q(\tau)(x \restriction \gamma)$. Fix $n < \omega$ large enough so that $x \in X_n$, $\tau \in A_n$, and $\delta_n > \gamma$. By (7), $q(\tau)(x) = h_{n,\tau}(x)$ is above $p_n(\tau)(x \restriction \delta_n)$. Since p_n is strictly increasing, $p_n(\tau)(x \restriction \delta_n)$ is above $p_n(\tau)(x \restriction \gamma) = q(\tau)(x \restriction \gamma)$.

So $q \in \mathbb{P}$ and $q \leq p_n$ for all $n < \omega$. The other statements are easy to verify. □

Lemma 5.33 (Augmentation)**.** *Let λ be a large enough regular cardinal and assume that N is a countable elementary substructure of $H(\lambda)$ which contains as elements T, κ, $\mathbb{Q}$, and $\mathbb{P}$. Let $\delta = N \cap \omega_1$. Suppose the following:*

- *$p \in N \cap \mathbb{P}$ has top level γ;*
- *$B \subseteq \mathrm{dom}(p)$ is finite;*
- *$z \in T_\delta$ and $\sigma \in \mathrm{dom}(p)$;*
- *$X \subseteq T_\delta$ is finite and $X \cup \{z\}$ has unique drop-downs to γ;*
- *$\{p(\tau) : \tau \in B\}$ is separated on $X \restriction \gamma$.*

Then there exists some $q \leq p$ in $N \cap \mathbb{P}$ with top level $\gamma + 1$ and there exists a finite set $Y \subseteq T_\delta$ satisfying:

(1) *for all $\tau \in B$, $X \restriction \gamma$ and $X \restriction (\gamma + 1)$ are $q(\tau)$-consistent;*
(2) *$X \cup \{z\} \subseteq Y$ and Y has unique drop-downs to $\gamma + 1$;*
(3) *$\{q(\tau) : \tau \in B \cup \{\sigma\}\}$ is separated on $Y \restriction (\gamma + 1)$;*
(4) *let h_σ^+ be the partial injective function from Y to Y defined by letting, for all $x, y \in Y$, $h_\sigma^+(x) = y$ iff $q(\sigma)(x \restriction (\gamma + 1)) = y \restriction (\gamma + 1)$; then z is in the domain and range of h_σ^+.*

Proof. Apply Lemma 5.29 (Separated Conditions are Dense) and elementarity to find a separated condition $q \leq p$ in $N \cap \mathbb{P}$ with top level $\gamma + 1$ satisfying that for all $\tau \in B$, $X \restriction \gamma$ and $X \restriction (\gamma + 1)$ are $q(\tau)$-consistent. Define

$$W = \{\, z \restriction (\gamma + 1),\ q(\sigma)(z \restriction (\gamma + 1)),\ q(\sigma)^{-1}(z \restriction (\gamma + 1)) \,\} \setminus (X \restriction (\gamma + 1)).$$

Note that by unique drop-downs of $X \cup \{z\}$, if $z \notin X$ then $z \upharpoonright (\gamma+1)$ is not in $X \upharpoonright (\gamma+1)$. So if z is not in X, then $z \upharpoonright (\gamma+1)$ is in W. Choose a set $Y \subseteq T_\delta$ consisting of the elements of X together with exactly one element of T_δ above each member of W, and such that if $z \notin X$ then the element of Y above $z \upharpoonright (\gamma+1)$ is z. Note that Y has unique drop-downs to $\gamma+1$ and $Y \upharpoonright (\gamma+1) = (X \upharpoonright (\gamma+1)) \cup W$.

Conclusions (1), (2), and (3) are clear. (4): Note that $q(\sigma)(z \upharpoonright (\gamma+1))$ is either in $X \upharpoonright (\gamma+1)$ or in W. As $Y \upharpoonright (\gamma+1) = (X \upharpoonright (\gamma+1)) \cup W$, in either case we can fix $c \in Y$ such that $c \upharpoonright (\gamma+1) = q(\sigma)(z \upharpoonright (\gamma+1))$. Then by the definition of h_σ^+, $q(\sigma)(z \upharpoonright (\gamma+1)) = c \upharpoonright (\gamma+1)$ implies that $h_\sigma^+(z) = c$, so $z \in \mathrm{dom}(h_\sigma^+)$. The proof that z is in the range of h_σ^+ is similar. □

Theorem 5.34. *Suppose that T is a free Suslin tree. Then the forcing poset $\mathbb{P}$ is totally proper and preserves the fact that T is Suslin.*

Proof. Let λ be a large enough regular cardinal. Let N be a countable elementary substructure of $H(\lambda)$ containing as members T, κ, $\mathbb{Q}$, and $\mathbb{P}$. Let $\delta = N \cap \omega_1$. Suppose that $p \in N \cap \mathbb{P}$ and $\dot{E} \in N$ is a $\mathbb{P}$-name for a dense open subset of T. We will prove that there exists a total master condition $q \le p$ over N such that $q \Vdash T_\delta \subseteq \dot{E}$. The theorem easily follows.

So let N, δ, p, and $\dot{E}$ be given. Let α be the top level of p. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \alpha$ and an enumeration $\langle D_n : n < \omega \rangle$ of all of the dense open subsets of $\mathbb{P}$ which lie in N. Fix a surjection $g : \omega \to 3 \times T_\delta \times (N \cap \kappa)$ such that every element of the codomain has an infinite preimage.

We will define the following objects by induction in ω-many stages:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;
- a non-decreasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ;
- a decreasing sequence $\langle p_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}$ such that $p_0 = p$ and for all n, δ_n is the top level of p_n;
- for each $n < \omega$ and $\sigma \in N \cap \kappa$, an injective partial function $h_{n,\sigma}$ from X_n to X_n.

In addition to the properties listed above, we will maintain the following inductive hypotheses for all $n < \omega$:

(1) X_n has unique drop-downs to δ_n;
(2) $A_n \subseteq \mathrm{dom}(p_n)$ and $\{p_n(\tau) : \tau \in A_n\}$ is separated on $X_n \upharpoonright \delta_n$;
(3) for all $\tau \in A_n$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent;
(4) for all $\tau \in A_n$ and $x, y \in X_n$,
$$h_{n,\tau}(x) = y \iff p_n(\tau)(x \upharpoonright \delta_n) = y \upharpoonright \delta_n.$$

Stage 0: Let $X_0 = \emptyset$, $A_0 = \emptyset$, $\delta_0 = \alpha$, and $p_0 = p$.

Stage $n+1$: Let $n < \omega$ and assume that we have completed stage n. In particular, we have defined X_n, A_n, δ_n, p_n, and $h_{n,\tau}$ for all $\tau \in A_n$ satisfying the required properties. Let $g(n) = (n_0, z, \sigma)$.

Case a: $n_0 = 0$. Apply Proposition 2.2 (Consistent Extensions Into Dense Sets) and Lemma 5.27 to find a condition $p_{n+1} \le p_n$ in $N \cap \bigcap_{k<n} D_k$ with $\sigma \in \mathrm{dom}(p_{n+1})$ and top level some ordinal δ_{n+1} greater than γ_{n+1} such that for all $\tau \in A_n$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent. Define $X_{n+1} = X_n$ and $A_{n+1} = A_n$. Define for all $\tau \in A_{n+1}$ a partial function $h_{n+1,\tau}$ as described in inductive hypothesis (4). It is routine to check that the required properties are satisfied.

Case b: $n_0 = 1$. If $z \notin X_n$, then let $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n$, $p_{n+1} = p_n$, and $h_{n+1,\tau} = h_{n,\tau}$ for all $\tau \in A_n$. Suppose that $z \in X_n$. Fix an injective tuple $\vec{a} = (a_0, \dots, a_{l-1})$ which lists the elements of X_n, and fix $j < l$ such that $z = a_j$. We apply Lemma 5.26 (1-Key

Property) and Proposition 2.4 (Consistent Extensions for Sealing), where the derived tree in the statement of Proposition 2.4 is just T itself. So fix $p_{n+1} \leq p$ in $N \cap \mathbb{P}$ with some top level δ_{n+1} such that $p_{n+1} \Vdash_{\mathbb{P}} a_j \in \dot{E}$ and for all $\tau \in A_n$, $\vec{a} \restriction \delta_n$ and $\vec{a} \restriction \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent. Then $p_{n+1} \Vdash_{\mathbb{P}} z \in \dot{E}$ and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $p_{n+1}(\tau)$-consistent. Let $X_{n+1} = X_n$ and $A_{n+1} = A_n$. Define for all $\tau \in A_{n+1}$ a partial function $h_{n+1,\tau}$ as described in inductive hypothesis (4). The required properties are clearly satisfied.

Case c: $n_0 = 2$. If $\sigma \notin \mathrm{dom}(p_n)$, then let $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n$, $p_{n+1} = p_n$, and $h_{n+1,\tau} = h_{n,\tau}$ for all $\tau \in A_n$. Now suppose that $\sigma \in \mathrm{dom}(p_n)$. Fix $\gamma < \delta$ larger than δ_n and γ_{n+1} such that $X_n \cup \{z\}$ has unique drop-downs to γ. Apply Lemma 5.24 (Extension) to find $p_n' \leq p_n$ with top level γ such that for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \gamma$ are $p_n'(\tau)$-consistent.

Apply Lemma 5.33 (Augmentation) to find $p_{n+1} \leq p_n'$ with top level $\gamma + 1$ and a finite set $Y \subseteq T_\delta$ satisfying:

(1) for all $\tau \in A_n$, $X_n \restriction \gamma$ and $X_n \restriction (\gamma + 1)$ are $p_{n+1}(\tau)$-consistent;
(2) $X_n \cup \{z\} \subseteq Y$ and Y has unique drop-downs to $\gamma + 1$;
(3) $\{p_{n+1}(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated on $Y \restriction (\gamma + 1)$;
(4) let $h^+_{n,\sigma}$ be the partial injective function from Y to Y defined by letting, for all $x, y \in Y$, $h^+_\sigma(x) = y$ iff $p_{n+1}(\sigma)(x \restriction (\gamma + 1)) = y \restriction (\gamma + 1)$; then z is in the domain and range of $h^+_{n,\sigma}$.

Let $X_{n+1} = Y$, $A_{n+1} = A_n \cup \{\sigma\}$, and $\delta_{n+1} = \gamma + 1$. For each $\tau \in A_{n+1}$, define a partial injective function $h_{n+1,\tau}$ from X_{n+1} to X_{n+1} as described in inductive hypothesis (4). Note that $h_{n+1,\sigma} = h^+_{n,\sigma}$, so z is in the domain and range of $h_{n+1,\sigma}$. The inductive hypotheses are clearly satisfied.

This completes the construction. By our bookkeeping, it is routine to check that the assumptions of Lemma 5.32 (Constructing Total Master Conditions) hold. Fix a total master condition q over N such that $q \leq p_n$ for all n. Reviewing Case (b) and our bookkeeping, it is easy to show q forces that $T_\delta \subseteq \dot{E}$. □

Lemma 5.35. *Let λ be a large enough regular cardinal and let N be a countable elementary substructure of $H(\lambda)$ which contains as members T, κ, $\mathbb{Q}$, and $\mathbb{P}$. If q is a total master condition over N such that* $\mathrm{dom}(q) = N \cap \kappa$, *then q is separated.*

Proof. Let $\delta = N \cap \omega_1$. By Lemma 5.30, to show that q is separated it suffices to show that whenever $A \subseteq \mathrm{dom}(q)$ and $X \subseteq T_\delta$ are finite sets, then $\{q(\tau) : \tau \in A\}$ is separated on X. Fix $\xi < \delta$ large enough so that X has unique drop-downs to ξ. Let D be the set of conditions s with top level at least ξ such that $A \subseteq \mathrm{dom}(s)$ and s is separated. By Lemmas 5.27 and 5.29 (Separated Conditions are Dense), D is dense and $D \in N$ by elementarity. Since q is a total master condition, we can find some $s \in N \cap D$ such that $q \leq s$. Let ρ be the top level of s. As $\rho \geq \xi$, X has unique drop-downs to ρ. Since s is separated, $\{s(\tau) : \tau \in A\}$ is separated on $X \restriction \rho$. By Lemma 5.22 (Persistence for Sets), $\{q(\tau) : \tau \in A\}$ is separated on X. □

Proposition 5.36. *Assuming CH, the forcing poset $\mathbb{P}$ is ω_2-c.c.*

This follows by a standard application of the Δ-system lemma, assuming CH, to an ω_2-sized collection of countable sets.

Let us say that two automorphisms of T are *almost disjoint* if they agree on only countably many elements of T, or in other words, their graphs have countable intersection.

Proposition 5.37. *Suppose that T is a free Suslin tree. Then $\mathbb{P}$ forces that there exists an almost disjoint sequence of length κ of automorphisms of T.*

Proof. Let G be a generic filter on $\mathbb{P}$. For each $\tau < \kappa$, let $f_\tau := \bigcup\{p(\tau) : p \in G,\ \tau \in \mathrm{dom}(p)\}$. By Lemma 5.27 and a density argument, it is easy to check that each f_τ is an automorphism of T. Consider $\tau_0 < \tau_1 < \kappa$. By Lemma 5.28, we can find $\alpha = \alpha_{\tau_0,\tau_1} < \omega_1$ such that for all $x \in T_\alpha$, $f_{\tau_0}(x) \neq f_{\tau_1}(x)$. But if $x <_T y$ and $f_{\tau_0}(x) \neq f_{\tau_1}(x)$, then $f_{\tau_0}(y) \neq f_{\tau_1}(y)$. So $\{x \in T : f_{\tau_0}(x) = f_{\tau_1}(x)\} \subseteq T \restriction \alpha_{\tau_0,\tau_1}$. □

Proposition 5.38. *Suppose that T is a free Suslin tree and $\kappa \geq \omega_2$. Then $\mathbb{P}$ forces that T is an almost Kurepa Suslin tree.*

Proof. Let G be a generic filter on $\mathbb{P}$. By Theorem 5.34, T is Suslin in $V[G]$. In $V[G]$, let $\{f_\tau : \tau < \kappa\}$ be an almost disjoint family of automorphisms of T. Force with T over $V[G]$ to get a generic branch b of T. In $V[G][b]$, define $b_\tau = f_\tau[b]$ for all $\tau < \kappa$. For any $\tau_0 < \tau_1 < \kappa$, since f_{τ_0} and f_{τ_1} are almost disjoint, it is easy to conclude that $b_{\tau_0} = f_{\tau_0}[b]$ and $b_{\tau_1} = f_{\tau_1}[b]$ have countable intersection. So $\langle b_\tau : \tau < \kappa\rangle$ is a sequence of κ-many distinct cofinal branches of T. Hence, T is a Kurepa tree in $V[G][b]$. □

5.6. **More About Constructing and Extending Automorphisms.** We now turn towards proving that the automorphism forcing does not add new cofinal branches of ω_1-trees appearing in certain intermediate extensions, a task which will occupy us for the remainder of the section. In this subsection we prove three technical lemmas about extending countable families of automorphisms one level higher in order to achieve some desirable properties. These lemmas anticipate configurations which will appear in proofs occurring later in the section. Specifically, Lemmas 5.39 and 5.40 are used in the proof of Lemma 5.53, and Lemma 5.41 is used in the proof of Lemma 5.47. The proofs of Lemmas 5.39 and 5.40 are fairly simple and almost identical. Lemma 5.41 is essentially an expansion of Lemma 5.14 to a more elaborate context.

Lemma 5.39. *Let $\gamma < \omega_1$ and let $\{f_\tau : \tau \in I\}$ be a countable family of automorphisms of $T \restriction (\gamma + 1)$. Let $A \subseteq B \subseteq I$ be finite. Let $Y, Z \subseteq T_{\gamma+1}$ be finite sets each with unique drop-downs to γ such that $Y \cap Z = \emptyset$. Then there exists a family $\{g_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\gamma + 2)$ such that:*

(1) *for all $\tau \in I$, $f_\tau \subseteq g_\tau$;*
(2) *for all $\tau \in B$, $Y \restriction \gamma$ and Y are g_τ-consistent;*
(3) *for all $\tau \in A$, $Z \restriction \gamma$ and Z are g_τ-consistent;*
(4) *for all $\tau \in B \setminus A$ and for all $x \in Z$, $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $Y \cup Z$.*

Proof. Fix a bijection $h : \omega \to T_{\gamma+1} \times I$. Let $g_\tau \restriction (\gamma + 1) = f_\tau$ for all $\tau \in I$. We define the values of the functions g_τ on $T_{\gamma+1}$ in ω-many stages, where at any given stage we will have defined only finitely many values for finitely many g_τ's. We also define a subset-increasing sequence $\langle X_n : n < \omega\rangle$ of finite subsets of $T_{\gamma+1}$.

At stage 0, for all $\tau \in B$ and $x, y \in Y$, let $g_\tau(x) = y$ iff $f_\tau(x \restriction \gamma) = y \restriction \gamma$. And for all $\tau \in A$ and $x, y \in Z$, let $g_\tau(x) = y$ iff $f_\tau(x \restriction \gamma) = y \restriction \gamma$. Let $X_0 = Y \cup Z$.

Now let $n < \omega$ and assume that we have completed stage n. In particular, we have defined the finite set $X_n \subseteq T_{\gamma+1}$. Consider $h(n) = (z, \sigma)$. Stage $n + 1$ will consist of two steps. For the first step, if $g_\sigma(z)$ is already defined, then move on to step 2. Otherwise, define $g_\sigma(z)$ to be some element of $T_{\gamma+1}$ above $f_\sigma(z \restriction \gamma)$ which is not in $X_n \cup \{z\}$. This is possible since T is infinitely splitting. For the second step, if $g_\sigma^{-1}(z)$ is already defined, then we are done. Otherwise, define $g_\sigma^{-1}(z)$ to be some element of $T_{\gamma+1}$ which is above $f_\sigma^{-1}(z \restriction \gamma)$ and is not in $X_n \cup \{z, g_\sigma(z)\}$. Again, this is possible since T is infinitely splitting. Define $X_{n+1} = X_n \cup \{z, g_\sigma(z), g_\sigma^{-1}(z)\}$.

This completes the construction. It is routine to check that this works, using what we did at stage 0 to show (2) and (3), and using the sets X_n to show injectivity and (4). □

Lemma 5.40. *Let $\gamma < \omega_1$ and let $\{f_\tau : \tau \in I\}$ be a family of automorphisms of $T \restriction (\gamma+1)$. Let $A \subseteq B \subseteq I$ be finite sets. Let $b_0, \dots, b_{n-1}$ be distinct elements of T_γ, and let $c_0, \dots, c_{n-1}, d_0, \dots, d_{n-1}$ be distinct elements of $T_{\gamma+1}$ such that for all $k < n$, $b_k <_T c_k$ and $b_k <_T d_k$. Define $C = \{c_k : k < n\}$ and $D = \{d_k : k < n\}$. Let $Y \subseteq T_{\gamma+1}$ be finite and assume that $(C \cup D) \cap Y = \emptyset$.*

Then there exists a family $\{g_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\gamma + 2)$ such that:

(1) *for all $\tau \in I$, $f_\tau \subseteq g_\tau$;*
(2) *for all $\tau \in A$, $(b_0, \dots, b_{n-1})$ and $(c_0, \dots, c_{n-1})$ are g_τ-consistent and $(b_0, \dots, b_{n-1})$ and $(d_0, \dots, d_{n-1})$ are g_τ-consistent;*
(3) *for all $x \in C$: if $\tau \in A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $Y \cup D$, and if $\tau \in B \setminus A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $C \cup D \cup Y$;*
(4) *for all $x \in D$: if $\tau \in A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $Y \cup C$, and if $\tau \in B \setminus A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $C \cup D \cup Y$;*
(5) *for all $x \in Y$ and $\tau \in B$, $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $C \cup D \cup Y$.*

Proof. Fix a bijection $h : \omega \to T_{\gamma+1} \times I$. Let $g_\tau \restriction (\gamma + 1) = f_\tau$ for all $\tau \in I$. We define the values of the functions g_τ on $T_{\gamma+1}$ in ω-many stages, where at any given stage we will have defined only finitely many values for finitely many g_τ's. We also define a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of $T_{\gamma+1}$.

At stage 0, for all $\tau \in A$ and $i, j < n$, define $g_\tau(c_i) = c_j$ iff $f_\tau(b_i) = b_j$ and $g_\tau(d_i) = d_j$ iff $f_\tau(b_i) = b_j$. Let $X_0 = Y \cup C \cup D$.

Now let $n < \omega$ and assume that we have completed stage n. In particular, the finite set $X_n \subseteq T_{\gamma+1}$ has been defined. Consider $h(n) = (z, \sigma)$. Stage $n + 1$ will consist of two steps. For the first step, if $g_\sigma(z)$ is already defined, then move on to step 2. Otherwise define $g_\sigma(z)$ to be some element of $T_{\gamma+1}$ above $f_\sigma(z \restriction \gamma)$ which is not in $X_n \cup \{z\}$. This is possible since T is infinitely splitting. For the second step, if $g_\sigma^{-1}(z)$ is already defined, then we are done. Otherwise define $g_\sigma^{-1}(z)$ to be some element of $T_{\gamma+1}$ which is above $f_\sigma^{-1}(z)$ and not in $X_n \cup \{z, g_\sigma(z)\}$. Again, this is possible since T is infinitely splitting. Define $X_{n+1} = X_n \cup \{z, g_\sigma(z), g_\sigma^{-1}(z)\}$.

This completes the construction. It is routine to check that this works. □

Lemma 5.41. *Assume the following:*

- *$\gamma < \omega_1$ and $n < \omega$;*
- *$X \subseteq T_{\gamma+1}$ is finite and has unique drop-downs to γ;*
- *$\{f_\tau : \tau \in I\}$ is a countable collection of automorphisms of $T \restriction (\gamma + 1)$;*
- *$\{I_0\} \cup \{J_k : k < n\}$ is a partition of I;*
- *$\{A\} \cup \{A_k : k < n\}$ is a family of finite sets, where $A \subseteq I_0$ and $A_k \subseteq J_k$ for each $k < n$;*
- *for all $k < n$, $\{f_\tau : \tau \in A \cup A_k\}$ is separated on $X \restriction \gamma$.*

Then there exists a family $\{g_\tau : \tau \in I\}$ of automorphisms of $T \restriction (\gamma + 2)$ satisfying:

(1) *$f_\tau \subseteq g_\tau$ for all $\tau \in I$;*
(2) *for all $\tau \in A \cup \bigcup_{k<n} A_k$, $X \restriction \gamma$ and X are g_τ-consistent;*
(3) *for all $k < n$, $\{g_\tau : \tau \in I_0 \cup J_k\}$ is separated.*

Proof. Fix a bijection $h : \omega \to T_{\gamma+1} \times I$. For each $\tau \in I$, define $g_\tau \restriction (\gamma + 1) = f_\tau$.

We will define the values of the functions g_τ on $T_{\gamma+1}$ in ω-many stages. In addition, for each $k < n$ we will define an injective sequence $\langle a^k_l : l < \omega \rangle$ which enumerates $T_{\gamma+1}$. At any given stage $p < \omega$, we will have defined a set X_p of some finite size l_p, and also defined, for each $k < n$, an injective enumeration $\langle a^k_l : l < l_p \rangle$ of X_p which will be an initial segment of the sequence $\langle a^k_l : l < \omega \rangle$.

We will maintain the following inductive hypotheses:

(i) for all $p < \omega$, if the value $g^m_\tau(a) = b$ is defined at stage p, where $\tau \in I$ and $m \in \{-1, 1\}$, then a and b are in X_p, $a \neq b$, and $f^m_\tau(a \restriction \gamma) = b \restriction \gamma$;

(ii) for all $p_0 < p_1 < \omega$, if a and b are in X_{p_0} and $g^m_\tau(a) = b$ has been defined by the end of stage p_1, where $\tau \in I$ and $m \in \{-1, 1\}$, then $g^m_\tau(a) = b$ has been defined by the end of stage p_0;

(iii) for all $k < n$, $p < \omega$, and $l < l_p$, there exists at most one triple (j, m, τ), where $j < l$, $m \in \{-1, 1\}$, and $\tau \in I_0 \cup J_k$, such that $g^m_\tau(a^k_l)$ has been defined by the end of stage p and $g^m_\tau(a^k_l) = a^k_j$.

Stage 0: For each $\tau \in A \cup \bigcup_{k<n} A_k$ and $x, y \in X$, define $g_\tau(x) = y$ iff $f_\tau(x \restriction \gamma) = y \restriction \gamma$. Define $X_0 = X$ and $l_0 = |X|$. For each $k < n$, since $\{f_\tau : \tau \in A \cup A_k\}$ is separated on $X \restriction \gamma$, we can fix an injective sequence $\langle a^k_0, \ldots, a^k_{l_0-1} \rangle$ which lists the elements of X so that $\{f_\tau : \tau \in A \cup A_k\}$ is separated on $\vec{a}^k \restriction \gamma$, where $\vec{a}^k = (a^k_0, \ldots, a^k_{l_0-1})$.

Let us check that the inductive hypotheses hold. (i) Suppose $g^m_\tau(a) = b$ is defined at stage 0, where $\tau \in I$ and $m \in \{-1, 1\}$. By flipping a and b if necessary, we may assume without loss of generality that $m = 1$. Then by construction, a and b are in $X = X_0$, $\tau \in A \cup \bigcup_{k<n} A_k$, and $f_\tau(a \restriction \gamma) = b \restriction \gamma$. Fix $k < n$ such that $\tau \in A \cup A_k$. Since $\{f_\tau : \tau \in A \cup A_k\}$ is separated on $X \restriction \gamma$, $a \restriction \gamma \neq b \restriction \gamma$, and hence $a \neq b$.

(ii) is vacuously true. (iii) Fix $k < n$ and $l < l_0$. Suppose that there exists a triple (j, m, τ), where $j < l$, $m \in \{-1, 1\}$, and $\tau \in I_0 \cup J_k$, such that $g^m_\tau(a^k_l) = a^k_j$ has been defined by the end of stage 0. Then by what we did at stage 0, $\tau \in A \cup A_k$. Since $\{f_\tau : \tau \in A \cup A_k\}$ is separated on $\vec{a}^k \restriction \gamma$, the triple (j, m, τ) must be unique.

Stage $p + 1$: Let $p < \omega$ and suppose that stage p is complete. In particular, we have defined X_p and for each $k < n$ we have defined an injective sequence $\langle a^k_l : l < l_p \rangle$ which lists the elements of X_p satisfying the required properties. Let $h(p) = (z, \sigma)$.

We will define $g_\sigma(z)$ and $g^{-1}_\sigma(z)$ in two steps, where at each step we use the fact that T is infinitely splitting. In the first step, if $g_\sigma(z)$ is already defined, then move on to step two. Otherwise, define $g_\sigma(z)$ to be some member of $T_{\gamma+1}$ which is above $f_\sigma(z \restriction \gamma)$ and is not in $X_p \cup \{z\}$. In the second step, if $g^{-1}_\sigma(z)$ is already defined, then we are done. Otherwise, define $g^{-1}_\sigma(z)$ to be some member of $T_{\gamma+1}$ which is above $g^{-1}_\sigma(z \restriction \gamma)$ and is not in $X_p \cup \{z, g_\sigma(z)\}$.

Let $X_{p+1} = X_p \cup \{z, g_\sigma(z), g^{-1}_\sigma(z)\}$ and $l_{p+1} = |X_{p+1}|$. For each $k < n$, define $\langle a^k_l : l < l_{p+1} \rangle$ by adding at the end of the sequence $\langle a^k_l : l < l_p \rangle$ those elements among z, $g_\sigma(z)$, and $g^{-1}_\sigma(z)$ which are not already in X_p, in the order just listed.

Let us check that inductive hypotheses (i)-(iii) hold for $p + 1$. (i) is clear. For (ii), the only new equations of the form $g^m_\tau(a) = b$ which were introduced at stage $p + 1$, where $\tau \in I$, $m \in \{-1, 1\}$, and $a, b \in T_{\gamma+1}$, is when at least one of a or b is in $X_{n+1} \setminus X_n$. So (ii) easily follows from the inductive hypothesis.

Now we prove (iii). Fix $k < n$. Consider first the case when z is not in X_p. Then by inductive hypothesis (i), neither $g_\sigma(z)$ nor $g^{-1}_\sigma(z)$ were defined at any stage earlier than $p + 1$. So by how we defined $g_\sigma(z)$ and $g^{-1}_\sigma(z)$ at stage $p + 1$, $g_\sigma(z)$ and $g^{-1}_\sigma(z)$ are not in X_p. Hence, the last three elements of $\langle a^k_l : l < l_{p+1} \rangle$ are z, $g_\sigma(z)$, and $g^{-1}_\sigma(z)$. The relations introduced between these three elements at stage $p+1$ yield no counter-example to (iii), and z, $g_\sigma(z)$, and $g^{-1}_\sigma(z)$ have no relations to any elements of $\langle a^k_l : l < l_p \rangle$. So (iii) follows by the inductive hypothesis.

Next, consider the case when z is in X_p. Then z already appears on the sequence $\langle a^k_l : l < l_p \rangle$. At stage $p + 1$, no new relations are introduced between elements of $\langle a^k_l : l < l_p \rangle$. Each new element in $X_{p+1} \setminus X_p$ has exactly one relation with elements of X_{p+1}, namely with z. So (iii) follows by the inductive hypothesis.

This completes the construction. It is routine to check that each g_τ is an automorphism of $T \upharpoonright (\gamma+2)$ extending f_τ. By what we did at stage 0, for all $\tau \in A \cup \bigcup_n A_n$, $X \upharpoonright \gamma$ and X are g_τ-consistent.

Now consider $k < n$, and we will show that $\{g_\tau : \tau \in I_0 \cup J_k\}$ is separated. By inductive hypothesis (i), g_τ has no fixed points in $T_{\gamma+1}$ for any $\tau \in I_0 \cup J_k$. Let $Y \subseteq T_{\gamma+1}$ be finite. Fix a large enough $p < \omega$ so that $Y \subseteq X_p$. Then by Lemma 5.9, it suffices to show that $\{g_\tau : \tau \in I_0 \cup J_k\}$ is separated on X_p as witnessed by the tuple $(a_0^k, \ldots, a_{l_p-1}^k)$. Suppose that $l < l_p$ and (j, m, τ) satisfies that $j < l$, $m \in \{-1, 1\}$, $\tau \in I_0 \cup J_k$, and $g_\tau^m(a_l^k) = a_j^k$. By inductive hypothesis (ii), the relation $g_\tau^m(a_l^k) = a_j^k$ was introduced by the end of stage p. By inductive hypothesis (iii), there is at most one such triple. ☐

5.7. **Regular Suborders and Generalized Properties.** In this subsection we generalize many of the main properties of $\mathbb{P}$ to the context of regular suborders of $\mathbb{P}$.

Definition 5.42. *For any set $X \subseteq \kappa$, let $\mathbb{P}_X$ denote the suborder of $\mathbb{P}$ consisting of all $p \in \mathbb{P}$ such that* $\mathrm{dom}(p) \subseteq X$.

The proof of the following is routine.

Proposition 5.43. *For any set $X \subseteq \kappa$, $\mathbb{P}_X$ is a regular suborder of $\mathbb{P}$.*

Of particular interest for us will be $\mathbb{P}_\theta$, where $\theta < \kappa$. The goal for the remainder of Section 5 is to show that whenever $\theta < \kappa$ and $\dot{U}$ is a $\mathbb{P}_\theta$-name for an ω_1-tree, then $\mathbb{P}$ forces that any cofinal branch of $\dot{U}$ in $V^{\mathbb{P}}$ is in $V^{\mathbb{P}_\theta}$. When $\theta < \kappa$ is fixed, we will write $\leq_\theta$ for the ordering on $\mathbb{P}_\theta$, mainly to emphasize that the conditions we are relating are in $\mathbb{P}_\theta$. On the other hand, when we write $\leq$ we mean the ordering on $\mathbb{P}$.

The remaining results of this subsection can be described as follows. Many previously discussed properties of $\mathbb{P}$ were of the form that some condition can be extended to a higher level satisfying some additional information. Now we will have finitely many conditions, all with the same restriction to $\mathbb{P}_\theta$, and we will simultaneously extend those conditions so that the extended conditions also have the same restriction to $\mathbb{P}_\theta$. We refer to this type of result as *generalized* versions of the earlier results.

Lemma 5.44 (Simple Generalized Extension)**.** *Let $\theta < \kappa$. Assume the following:*

- *$\gamma \leq \xi < \omega_1$;*
- *$p \in \mathbb{P}$ has top level γ, $w \in \mathbb{P}_\theta$ has top level ξ, and $w \leq_\theta p \upharpoonright \theta$;*
- *$B \subseteq \mathrm{dom}(p)$;*
- *X is a finite subset of T_ξ with unique drop-downs to γ;*
- *for all $\tau \in B \cap \theta$, $X \upharpoonright \gamma$ and X are $w(\tau)$-consistent.*

Then there exists a condition $q \leq p$ with top level ξ and domain equal to $\mathrm{dom}(w) \cup \mathrm{dom}(p)$ *such that $q \upharpoonright \theta = w$ and for all $\tau \in B$, $X \upharpoonright \gamma$ and X are $q(\tau)$-consistent.*

Proof. Apply Lemma 5.24 (Extension) to the condition $p \upharpoonright [\theta, \kappa)$ to find a condition $s \leq p \upharpoonright [\theta, \kappa)$ with top level ξ and with the same domain as $p \upharpoonright [\theta, \kappa)$ such that for all $\tau \in B \setminus \theta$, $X \upharpoonright \gamma$ and X are $s(\tau)$-consistent. Now let $q = w \cup s$. ☐

Lemma 5.45 (Generalized Extension)**.** *Let $\theta < \kappa$. Assume the following:*

- *$\gamma \leq \xi < \omega_1$;*
- *$\{p_0, \ldots, p_{n-1}\}$ is a finite set of conditions in $\mathbb{P}$ all with top level γ;*
- *$v \in \mathbb{P}_\theta$ and for all $k < n$, $p_k \upharpoonright \theta = v$;*
- *$B \subseteq \bigcap_{k<n} \mathrm{dom}(p_k)$;*

- *X is a finite subset of T_ξ with unique drop-downs to γ.*

Then there exists a set of conditions $\{\hat{p}_0, \dots, \hat{p}_{n-1}\}$ all with top level ξ and there exists some $\hat{v} \in \mathbb{P}_\theta$ such that for all $k < n$:

(1) *$\hat{p}_k \leq p_k$;*
(2) *$\hat{p}_k \upharpoonright \theta = \hat{v}$;*
(3) *for all $\tau \in B$, $X \upharpoonright \gamma$ and X are $\hat{p}_k(\tau)$-consistent.*

Moreover, if $w \leq_\theta v$ is a fixed condition with top level ξ such that for all $\tau \in B \cap \theta$, $X \upharpoonright \gamma$ and X are $w(\tau)$-consistent, then we can also arrange that $\hat{v} = w$.

Proof. We apply Lemma 5.24 (Extension) several times. In the case of the moreover clause, let $\hat{v} = w$. Otherwise, apply Lemma 5.24 (Extension) to find $\hat{v} \leq v$ with top level ξ and the same domain as v such that for all $\tau \in B \cap \theta$, $X \upharpoonright \gamma$ and X are $\hat{v}(\tau)$-consistent. For each $k < n$, apply Lemma 5.24 (Extension) to find $s_k \leq p_k \upharpoonright [\theta, \kappa)$ with top level ξ and the same domain as $p_k \upharpoonright [\theta, \kappa)$ such that for all $\tau \in B \setminus \theta$, $X \upharpoonright \gamma$ and X are $p_k(\tau)$-consistent. Now let $\hat{p}_k = \hat{v} \cup s_k$ for all $k < n$. □

Lemma 5.46 (Generalized Consistent Extensions Into Dense Sets)**.** *Suppose that T is a free Suslin tree. Let $\theta < \kappa$. Let λ be a large enough regular cardinal and let N be a countable elementary substructure of $H(\lambda)$ which contains as members T, κ, $\mathbb{Q}$, $\mathbb{P}$, and θ. Let $\delta = N \cap \omega_1$. Assume that:*

- *$D \in N$ is a dense open subset of $\mathbb{P}$;*
- *$\{p_0, \dots, p_{n-1}\}$ is a finite set of conditions in $N \cap \mathbb{P}$ all with top level ξ;*
- *$v \in N \cap \mathbb{P}_\theta$ and for all $k < n$, $p_k \upharpoonright \theta = v$;*
- *$X \subseteq T_\delta$ is finite and has unique drop-downs to ξ;*
- *$B \subseteq \bigcap_{k<n} \mathrm{dom}(p_k)$ is finite;*
- *for each $k < n$, $\{p_k(\tau) : \tau \in B\}$ is separated on $X \upharpoonright \xi$.*

Then there exist $q_0, \dots, q_{n-1}$ in $N \cap D$, $w \in N \cap \mathbb{P}_\theta$, and $\gamma < \delta$ satisfying that for all $k < n$,

(1) *$q_k \leq p_k$, q_k has top level γ, and $q_k \upharpoonright \theta = w$;*
(2) *for all $\tau \in B$, $X \upharpoonright \xi$ and $X \upharpoonright \gamma$ are $q_k(\tau)$-consistent.*

Proof. The proof is by induction on n. For the base case $n = 1$, the statement follows immediately from Proposition 2.2 (Consistent Extensions Into Dense Sets). Now assume that $n \geq 1$ and the statement holds for n. Consider D, $\{p_0, \dots, p_n\}$, ξ, v, X, and B as above. By the inductive hypothesis applied to the set $\{p_0, \dots, p_{n-1}\}$, fix $q_0, \dots, q_{n-1}$ in $N \cap D$, $w \in N \cap \mathbb{P}_\theta$, and $\gamma < \delta$ satisfying properties (1) and (2). In particular, (2) implies that for all $\tau \in B \cap \theta$, $X \upharpoonright \xi$ and $X \upharpoonright \gamma$ are $w(\tau)$-consistent.

Apply Lemma 5.44 (Simple Generalized Extension) inside N to find $q_n \leq p_n$ with top level γ such that $q_n \upharpoonright \theta = w$ and for all $\tau \in B$, $X \upharpoonright \xi$ and $X \upharpoonright \gamma$ are $q_n(\tau)$-consistent. Since $\{p_n(\tau) : \tau \in B\}$ is separated on $X \upharpoonright \xi$ and $q_n \leq p_n$, by Lemma 5.22 (Persistence for Sets), $\{q_n(\tau) : \tau \in B\}$ is separated on $X \upharpoonright \gamma$. So we can apply Proposition 2.2 (Consistent Extensions Into Dense Sets) to find $\bar{q}_n \leq q_n$ in $N \cap D$ with some top level $\rho < \delta$ and some $\bar{w} \in N \cap \mathbb{P}_\theta$ such that $\bar{q}_n \upharpoonright \theta = \bar{w}$ and for all $\tau \in B$, $X \upharpoonright \gamma$ and $X \upharpoonright \rho$ are $\bar{q}_n(\tau)$-consistent. Now apply Lemma 5.45 (Generalized Extension) inside N to find a family $\{\bar{q}_k : k < n\}$ of conditions in N such that for all $k < n$, $\bar{q}_k \leq q_k$, $\bar{q}_k \upharpoonright \theta = \bar{w}$, and for all $\tau \in B$, $X \upharpoonright \gamma$ and $X \upharpoonright \rho$ are $\bar{q}_k(\tau)$-consistent. Since D is open, each $\bar{q}_k$ is in D. So $\bar{q}_0, \dots, \bar{q}_n$ and $\bar{w}$ are as required. □

Lemma 5.47 (Generalized Separated Conditions are Dense)**.** *Let $\theta < \kappa$. Assume the following:*

- *$\{p_0, \dots, p_{n-1}\}$ is a finite set of conditions in $\mathbb{P}$ all with top level γ;*
- *$v \in \mathbb{P}_\theta$ and for all $k < n$, $p_k \upharpoonright \theta = v$;*
- *X is a finite subset of $T_{\gamma+1}$ with unique drop-downs to γ;*

- $B \subseteq \bigcap_{k<n} \mathrm{dom}(p_k)$ *is finite and for all* $k < n$, $\{p_k(\tau) : \tau \in B\}$ *is separated on* $X \restriction \gamma$.

Then there exist $q_0, \ldots, q_{n-1}$ *in* $\mathbb{P}$ *and* $w \in \mathbb{P}_\theta$, *all with top level* $\gamma+1$, *such that* $w \leq_\theta v$ *and for all* $k < n$, $q_k \leq p_k$, $q_k \restriction \theta = w$, q_k *is separated, and for all* $\tau \in B$, $X \restriction \gamma$ *and* X *are* $q_k(\tau)$*-consistent.*

Proof. Let $I_0 = \mathrm{dom}(v)$ and $A = B \cap \theta$. For each $k < n$, let $J_k = \{k\} \times (\mathrm{dom}(p_k) \setminus \theta)$ and $A_k = \{k\} \times (B \setminus \theta)$. Let $I = I_0 \cup \bigcup_{k<n} J_k$. For all $\tau \in I_0$, let $f_\tau = v(\tau)$, and for all $k < n$ and $\tau \in \mathrm{dom}(p_k) \setminus \theta$, let $f_{(k,\tau)} = p_k(\tau)$.

We would like to apply Lemma 5.41 to the above objects. The first five assumptions of this lemma clearly hold. For the last assumption, we need to show that for all $k < n$, $\{f_i : i \in A \cup A_k\}$ is separated on $X \restriction \gamma$. Define $h : A \cup A_k \to B$ by $h(\tau) = \tau$ for all $\tau \in A$, and $h((k,\tau)) = \tau$ for all $(k,\tau) \in A_k$. Then h is a bijection and $f_i = p_k(h(i))$ for all $i \in A$. Since $\{p_k(\tau) : \tau \in B\}$ is separated on $X \restriction \gamma$, by Lemma 5.8 so is $\{f_i : i \in A \cup A_k\}$.

Applying Lemma 5.41 fix a family $\{g_i : i \in I\}$ of automorphisms of $T \restriction (\gamma+2)$ satisfying:

(1) $f_i \subseteq g_i$ for all $i \in I$;
(2) for all $i \in A \cup \bigcup_{k<n} A_k$, $X \restriction \gamma$ and X are g_i-consistent;
(3) for all $k < n$, $\{g_i : i \in I_0 \cup J_k\}$ is separated.

Define w with the same domain as v so that for all $\tau \in \mathrm{dom}(v)$, $w(\tau) = g_\tau$. For each $k < n$, define q_k with the same domain as p_k so that $q_k \restriction \theta = w$ and for all $\tau \in \mathrm{dom}(p_k) \setminus \theta$, $q_k(\tau) = g_{(k,\tau)}$. It is easy to check that each q_k is a condition below p_k with top level $\gamma+1$.

Let $k < n$. We claim that for all $\tau \in B$, $X \restriction \gamma$ and X are $q_k(\tau)$-consistent. If $\tau \in B \cap \theta$, then $q_k(\tau) = w(\tau) = g_\tau$ and $\tau \in A$, and by (2), $X \restriction \gamma$ and X are g_τ-consistent. Suppose that $\tau \in B \setminus \theta$. Then $(k,\tau) \in A_k$ and $q_k(\tau) = g_{(k,\tau)}$. By (2), $X \restriction \gamma$ and X are $g_{(k,\tau)}$-consistent.

Finally, we claim that q_k is separated. So let $Y \subseteq T_{\gamma+1}$ be finite, and we will show that $\{q_k(\tau) : \tau \in \mathrm{dom}(q_k)\}$ is separated on Y. Define a function $h : \mathrm{dom}(q_k) \to I_0 \cup J_k$ by letting $h(\tau) = \tau$ if $\tau < \theta$, and $h(\tau) = (k,\tau)$ if $\tau \geq \theta$. Then h is a bijection, for all $\tau \in \mathrm{dom}(q_k)$, $q_k(\tau) = g_{h(\tau)}$, and by (3), $\{g_i : i \in I_0 \cup J_k\}$ is separated on Y. So by Lemma 5.8, $\{q_k(\tau) : \tau \in \mathrm{dom}(q_k)\}$ is separated on Y. □

Lemma 5.48 (Generalized Augmentation)**.** *Let* λ *be a large enough regular cardinal and assume that* N *is a countable elementary substructure of* $H(\lambda)$ *which contains as elements* T, κ, $\mathbb{Q}$, *and* $\mathbb{P}$. *Let* $\delta = N \cap \omega_1$. *Suppose the following:*

- $\{p_0, \ldots, p_{n-1}\}$ *is a finite set of conditions in* $N \cap \mathbb{P}$ *with top level* γ;
- $v \in \mathbb{P}_\theta$ *and for all* $k < n$, $p_k \restriction \theta = v$;
- $B \subseteq \bigcap_{k<n} \mathrm{dom}(p_k)$ *is finite;*
- $z \in T_\delta$ *and* $\sigma \in \bigcap_{k<n} \mathrm{dom}(p_k)$;
- $X \subseteq T_\delta$ *is finite and* $X \cup \{z\}$ *has unique drop-downs to* γ;
- *for all* $k < n$, $\{p_k(\tau) : \tau \in B\}$ *is separated on* $X \restriction \gamma$;

Then there exist $q_0, \ldots, q_{n-1}$ *in* $N \cap \mathbb{P}$ *and* $w \in N \cap \mathbb{P}_\theta$, *all with top level* $\gamma+1$, *and a finite set* $Y \subseteq T_\delta$ *such that* $X \cup \{z\} \subseteq Y$ *and* Y *has unique drop-downs to* $\gamma+1$, *satisfying that for all* $k < n$:

(1) $q_k \leq p_k$ *and* $q_k \restriction \theta = w$;
(2) *for all* $\tau \in B$, $X \restriction \gamma$ *and* $X \restriction (\gamma+1)$ *are* $q_k(\tau)$*-consistent;*
(3) $\{q_k(\tau) : \tau \in B \cup \{\sigma\}\}$ *is separated on* $Y \restriction (\gamma+1)$;
(4) *let* $h^+_{k,\sigma}$ *be the partial injective function from* Y *to* Y *defined by letting, for all* $x, y \in Y$, $h^+_{k,\sigma}(x) = y$ *iff* $q_k(\sigma)(x \restriction (\gamma+1)) = y \restriction (\gamma+1)$; *then* z *is in the domain and range of* $h^+_{k,\sigma}$.

Proof. Apply Lemma 5.47 (Generalized Separated Conditions are Dense) in N to find conditions $q_0, \ldots, q_{n-1}$ in $N \cap \mathbb{P}$ and $w \leq_\theta v$ in $N \cap \mathbb{P}_\theta$, all with top level $\gamma+1$, such that for all $k < n$,

$q_k \leq p_k$, $q_k \upharpoonright \theta = w$, q_k is separated, and for all $\tau \in B$, $X \upharpoonright \gamma$ and $X \upharpoonright (\gamma+1)$ are $q_k(\tau)$-consistent.

Define

$$W = ((\{ z \upharpoonright (\gamma+1) \}) \cup \{ q_k(\sigma)^m(z \upharpoonright (\gamma+1)) : k < n, m \in \{-1,1\} \}) \setminus (X \upharpoonright (\gamma+1)).$$

Note that by unique drop-downs of $X \cup \{z\}$, if $z \notin X$ then $z \upharpoonright (\gamma+1)$ is not in $X \upharpoonright (\gamma+1)$. So if z is not in X, then $z \upharpoonright (\gamma+1)$ is in W.

Let Y consist of the elements of X together with exactly one element of T_δ above each member of W, and such that if $z \notin X$ then the element of Y above $z \upharpoonright (\gamma+1)$ is z. Note that Y has unique drop-downs to $\gamma+1$ and $Y \upharpoonright (\gamma+1) = (X \upharpoonright (\gamma+1)) \cup W$. Now define $h^+_{k,\sigma}$ for all $k < n$ as described in (4).

Conclusions (1), (2), and (3) are clear. (4): Let $k < n$. Note that $q_k(\sigma)(z \upharpoonright (\gamma+1))$ is either in $X \upharpoonright (\gamma+1)$ or in W. As $Y \upharpoonright (\gamma+1) = (X \upharpoonright (\gamma+1)) \cup W$, in either case we can find $c \in Y$ such that $c \upharpoonright (\gamma+1) = q_k(\sigma)(z \upharpoonright (\gamma+1))$. Then by the definition of $h^+_{k,\sigma}$, $q(\sigma)(z \upharpoonright (\gamma+1)) = c \upharpoonright (\gamma+1)$ implies that $h^+_{k,\sigma}(z) = c$, so $z \in \mathrm{dom}(h^+_{k,\sigma})$. The proof that z is in the range of $h^+_{k,\sigma}$ is similar. □

5.8. **Existence of Nice Conditions.** Our goal for the rest of the section is to prove the following theorem:

Theorem 5.49 (No New Cofinal Branches)**.** *Suppose that T is a free Suslin tree and* CH *holds. Let $\theta < \kappa$ and suppose that $\dot{U}$ is a $\mathbb{P}_\theta$-name for an ω_1-tree. Then $\mathbb{P}$ forces that every branch of $\dot{U}$ in $V^{\mathbb{P}}$ is in $V^{\mathbb{P}_\theta}$.*

Lemma 5.50. *In Theorem 5.49, it suffices to prove the statement under the assumption that $\kappa < \omega_2$.*

Proof. Assume that the result holds when $\kappa < \omega_2$, and now let $\theta < \kappa$ be arbitrary. Suppose that $\dot{U}$ is a $\mathbb{P}_\theta$-name for an ω_1-tree and $\dot{b}$ is a $\mathbb{P}_\kappa$-name for a branch of $\dot{U}$. Without loss of generality, we may assume that $\dot{U}$ is forced to be a subset of ω_1, and $\dot{U}$ and $\dot{b}$ are nice names.

Since $\mathbb{P}$ is ω_2-c.c. by Lemma 5.36, conditions have countable domain, and $\dot{U}$ is a nice name, we can find a set $X \subseteq \theta$ of size at most ω_1 such that $\dot{U}$ is a $\mathbb{P}_X$-name. Similarly, we can find a set $Y \subseteq \kappa$ of size at most ω_1 such that $X \subseteq Y$ and $\dot{b}$ is a $\mathbb{P}_Y$-name. Then $\dot{U}$ is a $\mathbb{P}_{Y\cap\theta}$-name.

Consider a generic filter G on $\mathbb{P}$ and let $H = G \cap \mathbb{P}_\theta$. Let $U = \dot{U}^H$ and $b = \dot{b}^G$. We will show that $b \in V[H]$. By the choice of Y and the names, $U \in V[H \cap \mathbb{P}_{Y\cap\theta}]$ and $b \in V[G \cap \mathbb{P}_Y]$. Let κ_0 be the order type of Y and let θ_0 be the order type of $Y \cap \theta$. By standard arguments, there exists an isomorphism $\varphi : \mathbb{P}_Y \to \mathbb{P}_{\kappa_0}$ such that $\varphi \upharpoonright \mathbb{P}_{Y\cap\theta}$ is an isomorphism of $\mathbb{P}_{Y\cap\theta}$ onto $\mathbb{P}_{\theta_0}$. Let $\bar{G} = \varphi[G \cap \mathbb{P}_Y]$ and $\bar{H} = \varphi[H \cap \mathbb{P}_{Y\cap\theta}]$. Then $\bar{G}$ is a generic filter on $\mathbb{P}_{\kappa_0}$, $\bar{H} = \bar{G} \cap \mathbb{P}_{\theta_0}$ is a generic filter on $\mathbb{P}_{\theta_0}$, $U \in V[\bar{H}]$, and $b \in V[\bar{G}]$. Since Y has cardinality at most ω_1, $\theta_0 < \kappa_0 < \omega_2$. So $b \in V[\bar{H}]$. But $V[\bar{H}] \subseteq V[H]$. □

For the remainder of this section assume that $\theta < \kappa < \omega_2$, $\dot{U}$ is a $\mathbb{P}_\theta$-name for an ω_1-tree, and $\dot{b}$ is a $\mathbb{P}$-name for a branch of $\dot{U}$. We will prove that $\mathbb{P}$ forces that $\dot{b}$ is in $V^{\mathbb{P}_\theta}$. Without loss of generality assume that the underlying set of $\dot{U}$ is forced to equal ω_1, and in fact that for any $\gamma < \omega_1$, the elements of $\dot{U}_\gamma$ are ordinals in the interval $[\omega \cdot \gamma, \omega \cdot (\gamma+1))$.

Fix a large enough regular cardinal λ and a well-ordering $\trianglelefteq$ of $H(\lambda)$. Define a set N to be *suitable* if it is a countable elementary substructure of $(H(\lambda), \in, \trianglelefteq)$ which contains as members the objects T, κ, $\mathbb{Q}$, $\mathbb{P}$, θ, $\dot{U}$, and $\dot{b}$.

Definition 5.51 (Nice Conditions)**.** *Let N be suitable, $\delta = N \cap \omega_1$, and $\alpha < \delta$. Suppose that $p \in N \cap \mathbb{P}$ has top level α and $A \subseteq \mathrm{dom}(p)$ is finite. Let $\vec{b}$ be a tuple consisting of distinct elements of T_α and assume that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$. A condition $v \in \mathbb{P}_\theta$ is said to be N*-nice *for* p, $\vec{b}$, and A *if the following statements hold:*

(1) *$v \leq p \upharpoonright \theta$, v has top level δ, and v decides $\dot{U} \upharpoonright \delta$.*
(2) *For all $q \in N \cap \mathbb{P}$ such that $q \leq p$ and $v \leq q \upharpoonright \theta$, there exists some $r \leq q$ with top level δ such that $r \upharpoonright \theta = v$, $N \cap \kappa \subseteq \mathrm{dom}(r)$, r is separated, and r decides $\dot{b} \cap \delta$.*
(3) *Suppose that $\vec{a}^0$ and $\vec{a}^1$ are distinct tuples above $\vec{b}$ with height δ, and $q_0, q_1 \leq p$ have top level δ and satisfy: $q_0 \upharpoonright \theta = q_1 \upharpoonright \theta = v$, $N \cap \kappa \subseteq \mathrm{dom}(q_0) \cap \mathrm{dom}(q_1)$, q_0 and q_1 are separated, q_0 and q_1 decide $\dot{b} \cap \delta$, and for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{a}^j$ are $q_j(\tau)$-consistent. Then there exist q_0^* and q_1^* satisfying the same properties listed above for q_0 and q_1, there exists $r \leq p$ with top level δ, and there exist disjoint tuples $\vec{e}^0$ and $\vec{e}^1$ above $\vec{b}$ with height δ satisfying: $r \upharpoonright \theta = v$, $N \cap \kappa \subseteq \mathrm{dom}(r)$, r is separated, r decides $\dot{b} \cap \delta$, and for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{e}^j$ are $q_j^*(\tau)$-consistent and $r(\tau)$-consistent.*

Lemma 5.52. *Let M and N be suitable, $N \subseteq M$, $\delta = N \cap \omega_1 = M \cap \omega_1$, $\alpha < \delta$, $p \in N \cap \mathbb{P}$ has top level α, and $A \subseteq \mathrm{dom}(p)$ is finite. Let $\vec{b}$ be a tuple consisting of distinct elements of T_α and assume that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$. If v is M-nice for p, $\vec{b}$, and A, then v is N-nice for p, $\vec{b}$, and A.*

Proof. We verify that v satisfies properties (1)-(3) of Definition 5.51 (Nice Conditions) for N. (1) is immediate and (2) follows easily from the fact that $N \subseteq M$. (3) is easy to check once we show that $N \cap \kappa = M \cap \kappa$, which is where we use the assumption that $\kappa < \omega_2$. Let $g : \omega_1 \to \kappa$ be the $\trianglelefteq$-minimum function in $H(\lambda)$ which is a surjection of ω_1 onto κ. As $\kappa \in N \cap M$, it follows that $g \in N \cap M$ by elementarity. Also, by elementarity, $N \cap \kappa = g[N \cap \omega_1] = g[M \cap \omega_1] = M \cap \kappa$. □

We need the following technical lemma in order to prove the existence of nice conditions.

Lemma 5.53. *Assume the following:*

(1) *$\alpha < \rho < \omega_1$;*
(2) *$p \in \mathbb{P}$ has top level α, $v \in \mathbb{P}_\theta$ has top level ρ, and $v \leq_\theta p \upharpoonright \theta$;*
(3) *$A \subseteq \mathrm{dom}(p)$ is finite, $B \subseteq \kappa$ is finite, $A \subseteq B$, and $B \cap \theta \subseteq \mathrm{dom}(v)$;*
(4) *$\vec{b} = (b_0, \dots, b_{n-1})$ consists of distinct elements of T_α;*
(5) *$\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$;*
(6) *Y is a finite subset of $T_{\rho+1}$ with unique drop-downs to ρ;*
(7) *$\vec{d}^0$ and $\vec{d}^1$ are tuples above $\vec{b}$ with height $\rho + 1$;*
(8) *for all $\tau \in A \cap \theta$, $\vec{b}$ and $\vec{d}^0 \upharpoonright \rho$ are $v(\tau)$-consistent and $\vec{b}$ and $\vec{d}^1 \upharpoonright \rho$ are $v(\tau)$-consistent;*
(9) *$\{v(\tau) : \tau \in B \cap \theta\}$ is separated on $Y \upharpoonright \rho$;*
(10) *letting D^0 be the set of elements of $\vec{d}^0$ and D^1 the set of elements of $\vec{d}^1$, we have that $D^0 \upharpoonright (\alpha + 1)$, $D^1 \upharpoonright (\alpha + 1)$, and $Y \upharpoonright (\alpha + 1)$ are pairwise disjoint;*
(11) *for all $\tau \in B \cap \theta$ and $m \in \{-1, 1\}$:*
    - *for all $x \in Y \upharpoonright \rho$, $v(\tau)^m(x) \notin (D^0 \cup D^1) \upharpoonright \rho$;*
    - *for all $x \in D^0 \upharpoonright \rho$, $v(\tau)^m(x) \notin D^1 \upharpoonright \rho$;*
    - *for all $x \in D^1 \upharpoonright \rho$, $v(\tau)^m(x) \notin D^0 \upharpoonright \rho$.*

Then there exist $q \in \mathbb{P}$ and $w \in \mathbb{P}_\theta$, both with top level $\rho + 1$, satisfying:

(I) *$q \leq p$, $w \leq_\theta v$, $q \upharpoonright \theta = w$, and $B \subseteq \mathrm{dom}(q)$;*
(II) *for all $\tau \in A$, $\vec{b}$ and $\vec{d}^0$ are $q(\tau)$-consistent and $\vec{b}$ and $\vec{d}^1$ are $q(\tau)$-consistent;*
(III) *for all $\tau \in B \cap \theta$, $(Y \cup D^0 \cup D^1) \upharpoonright \rho$ and $Y \cup D^0 \cup D^1$ are $w(\tau)$-consistent;*
(IV) *$\{q(\tau) : \tau \in B\}$ is separated on $Y \cup D^0 \cup D^1$.*

Proof. Let us define a condition $\bar{p} \leq p$ with top level α such that $\mathrm{dom}(\bar{p}) = \mathrm{dom}(p) \cup B$. Let $\bar{p} \upharpoonright \mathrm{dom}(p) = p$. For all $\tau \in B \setminus \mathrm{dom}(p)$, define $\bar{p}(\tau) = v(\tau) \upharpoonright (\alpha + 1)$ if $\tau < \theta$, and let $\bar{p}(\tau)$ be the identity function on $T \upharpoonright (\alpha + 1)$ if $\tau \geq \theta$. Note that $v \leq_\theta \bar{p} \upharpoonright \theta$.

Define $\bar{D}^0 = D^0 \restriction (\alpha+1)$, $\bar{D}^1 = D^1 \restriction (\alpha+1)$, and $\bar{Y} = Y \restriction (\alpha+1)$. Applying Lemma 5.40, fix a family $\{g_\tau : \tau \in \mathrm{dom}(\bar{p}) \setminus \theta\}$ of automorphisms of $T \restriction (\alpha+2)$ satisfying:

(a) for all $\tau \in \mathrm{dom}(\bar{p}) \setminus \theta$, $\bar{p}(\tau) \subseteq g_\tau$;
(b) for all $\tau \in A \setminus \theta$, $\vec{b}$ and $\vec{d}^0 \restriction (\alpha+1)$ are g_τ-consistent and $\vec{b}$ and $\vec{d}^1 \restriction (\alpha+1)$ are g_τ-consistent;
(c) for all $x \in \bar{D}^0$: if $\tau \in A \setminus \theta$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $\bar{Y} \cup \bar{D}^1$, and if $\tau \in (B \setminus \theta) \setminus A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $\bar{D}^0 \cup \bar{D}^1 \cup \bar{Y}$;
(d) for all $x \in \bar{D}^1$: if $\tau \in A \setminus \theta$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $\bar{Y} \cup \bar{D}^0$, and if $\tau \in (B \setminus \theta) \setminus A$, then $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $\bar{D}^0 \cup \bar{D}^1 \cup \bar{Y}$;
(e) for all $x \in \bar{Y}$ and $\tau \in B \setminus \theta$, $g_\tau(x)$ and $g_\tau^{-1}(x)$ are not in $\bar{D}^0 \cup \bar{D}^1 \cup \bar{Y}$.

Define s with domain equal to $\mathrm{dom}(\bar{p}) \setminus \theta$ so that for all $\tau \in \mathrm{dom}(s)$, $s(\tau) = g_\tau$. Clearly, s is a condition with top level $\alpha+1$ and $s \le \bar{p} \restriction [\theta, \kappa)$.

By (9), we can fix an injective tuple $\vec{y}$ which enumerates Y such that $\{v(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y} \restriction \rho$. Let $\vec{x}$ be the tuple of height $\rho+1$ which consists of the concentation of the tuples $\vec{y}$, $\vec{d}^0$, and $\vec{d}^1$, in that order. So $\vec{x}$ enumerates $Y \cup D^0 \cup D^1$.

We claim that $\{s(\tau) : \tau \in B \setminus \theta\}$ is separated on $\vec{x} \restriction (\alpha+1)$. Since $\vec{d}^0 \restriction (\alpha+1)$ and $\vec{d}^1 \restriction (\alpha+1)$ are above $\vec{b}$, $s \le p \restriction [\theta, \kappa)$, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$, it follows by Lemma 5.21 (Persistence) that $\{s(\tau) : \tau \in A \setminus \theta\}$ is separated on both $\vec{d}^0 \restriction (\alpha+1)$ and $\vec{d}^1 \restriction (\alpha+1)$. By properties (c), (d), and (e) above, if a relation of the form $s(\tau)^m(x) = y$ holds, where $\tau \in B \setminus \theta$, $m \in \{-1, 1\}$, and $x, y \in \bar{D}^0 \cup \bar{D}^1 \cup \bar{Y}$, then it must be the case that $\tau \in A \setminus \theta$ and either $x, y \in \bar{D}^0$ or $x, y \in \bar{D}^1$. Based on this information, it easily follows that $\{s(\tau) : \tau \in B \setminus \theta\}$ is separated on $\vec{x} \restriction (\alpha+1)$.

Apply Lemma 5.24 (Extension) to find a condition $z \le s$ with top level $\rho+1$ with the same domain as s such that for all $\tau \in B \setminus \theta$, $\vec{x} \restriction (\alpha+1)$ and $\vec{x}$ are $z(\tau)$-consistent. By Lemma 5.21 (Persistence), $\{z(\tau) : \tau \in B \setminus \theta\}$ is separated on $\vec{x}$.

Now apply Lemma 5.39 to find a family $\{h_\tau : \tau \in \mathrm{dom}(v)\}$ of automorphisms of $T \restriction (\rho+2)$ satisfying:

(f) for all $\tau \in \mathrm{dom}(v)$, $v(\tau) \subseteq h_\tau$;
(g) for all $\tau \in B \cap \theta$, $Y \restriction \rho$ and Y are h_τ-consistent;
(h) for all $\tau \in A \cap \theta$, $\vec{d}^0 \restriction \rho$ and $\vec{d}^0$ are h_τ-consistent and $\vec{d}^1 \restriction \rho$ and $\vec{d}^1$ are h_τ-consistent;
(i) for all $\tau \in (B \cap \theta) \setminus A$, $m \in \{-1, 1\}$, and $x \in D^0 \cup D^1$, $h_\tau^m(x) \notin Y \cup D^0 \cup D^1$.

Define w with the same domain as v so that for all $\tau \in \mathrm{dom}(v)$, $w(\tau) = h_\tau$. Clearly, $w \in \mathbb{P}_\theta$, $w \le_\theta v$, and w has top level $\rho+1$. Since $\{v(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y} \restriction \rho$, by Lemma 5.21 (Persistence), $\{w(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y}$.

Finally, let $q = w \cup z$. Then $\{q(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y}$ and $\{q(\tau) : \tau \in B \setminus \theta\}$ is separated on $\vec{x}$.

Let us prove conclusions (I)-(IV). (I) is clear. (II) Let $\tau \in A$. If $\tau \in A \cap \theta$, then (II) holds by (8) and (h). If $\tau \in A \setminus \theta$, then (II) holds by (b) and the choice of z.

(III) Statement (g) implies that for all $\tau \in B \cap \theta$, $Y \restriction \rho$ and Y are $w(\tau)$-consistent. Statement (i) implies that for all $\tau \in (B \cap \theta) \setminus A$, if $x, y \in (Y \cup D^0 \cup D^1) \restriction \rho$ and $w(\tau)(x) = y$, then $x, y \in Y$. By (II), for all $\tau \in A \cap \theta$, $\vec{b}$ and $\vec{d}^0$ are $w(\tau)$-consistent and $\vec{b}$ and $\vec{d}^1$ are $w(\tau)$-consistent. This implies that for all $\tau \in A \cap \theta$, $D^0 \restriction \rho$ and D^0 are $w(\tau)$-consistent and $D^1 \restriction \rho$ and D^1 are $w(\tau)$-consistent. By (11), if $\tau \in A \cap \theta$, $x, y \in (Y \cup D^0 \cup D^1) \restriction \rho$, and $w(\tau)(x) = y$, then either $x, y \in Y \restriction \rho$, $x, y \in D^0 \restriction \rho$, or $x, y \in D^1 \restriction \rho$. Altogether this information easily implies (III).

(IV) We claim that $\{q(\tau) : \tau \in B\}$ is separated on $\vec{x}$. First, let us show that for all $\tau \in B$, $q(\tau)$ has no fixed points in $\vec{x}$. Since $\{q(\tau) : \tau \in B \setminus \theta\}$ is separated on $\vec{x}$, for all $\tau \in B \setminus \theta$, $q(\tau)$ has no fixed points in $\vec{x}$. So it suffices to consider $\tau \in B \cap \theta$. As $\{q(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y}$, for all $\tau \in B \cap \theta$, $q(\tau)$ has no fixed points in $\vec{y}$. Because $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$, by Lemma 5.21 (Persistence), $\{q(\tau) : \tau \in A\}$ is separated on $\vec{d}^0$ and separated on $\vec{d}^1$. Hence, for all $\tau \in A \cap \theta$, $q(\tau)$ has no fixed points in $\vec{d}^0$ or in $\vec{d}^1$. Finally, consider $\tau \in (B \setminus A) \cap \theta$ and we show that $q(\tau)$ has no fixed points in $\vec{d}^0$ or $\vec{d}^1$. But this follows from (i).

Consider a relation of the form $q(\tau)^m(x) = y$, where $\tau \in B$, $m \in \{-1, 1\}$, $x, y \in Y \cup D^0 \cup D^1$, and y appears earlier in the ordering of $\vec{x}$ than x. First, assume that $\tau < \theta$. By (11), the only possibilities are that $x, y \in Y$, $x, y \in D^0$, or $x, y \in D^1$. By (i), if $\tau \notin A$ then $x, y \in Y$. Secondly, assume that $\tau \geq \theta$. By (c), (d), and (e), $\tau \in A$ and either $x, y \in D^0$ or $x, y \in D^1$. So altogether, if one of x or y is in Y, then $\tau < \theta$ and x and y are both in Y. Since $\{q(\tau) : \tau \in B \cap \theta\}$ is separated on $\vec{y}$, there is at most one such relation which holds in this case. Now suppose that it is not the case that one of x or y is in Y. Then $\tau \in A$, and either $x, y \in D^0$ or $x, y \in D^1$. But since $\{q(\tau) : \tau \in A\}$ is separated on $\vec{d}^0$ and separated on $\vec{d}^1$, there is at most one such relation which holds in this case as well. □

Proposition 5.54 (Existence of Nice Conditions)**.** *Suppose that T is a free Suslin tree. Let N be suitable and let $\delta = N \cap \omega_1$. Suppose that $p \in N \cap \mathbb{P}$ has top level $\alpha < \delta$ and p forces in $\mathbb{P}$ that $\dot{\vec{b}}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. Assume that $A \subseteq \mathrm{dom}(p)$ is finite, $\vec{b}$ consists of distinct elements of T_α, and $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$. Then for any $q \leq p$ in $N \cap \mathbb{P}$, there exists some $v \in \mathbb{P}_\theta$ which is N-nice for p, $\vec{b}$, and A such that $v \leq_\theta q \upharpoonright \theta$.*

Proof. Let $q \leq p$ in $N \cap \mathbb{P}$. We will prove that there exists some $v \in \mathbb{P}_\theta$ which is N-nice for p, $\vec{b}$, and A such that $v \leq_\theta q \upharpoonright \theta$. Without loss of generality, assume that the top level of q is greater than α.

To help with the construction of v, we fix the following objects:

- an enumeration $\langle (\vec{a}^{n,0}, \vec{a}^{n,1}) : n < \omega \rangle$ of all distinct pairs of tuples above $\vec{b}$ with height δ;
- a non-decreasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \gamma_1 = \alpha$;
- an enumeration $\langle z_n : n < \omega \rangle$ of T_δ;
- an enumeration $\langle s_n : n < \omega \rangle$ of $N \cap \mathbb{P}$;
- a surjection $g : \omega \to 4 \times \omega \times (N \cap \kappa)$ such that every element of the codomain has an infinite preimage;
- an enumeration $\langle D_n : n < \omega \rangle$ of all of the dense open subsets of $\mathbb{P}$ which lie in N.

As in previous proofs, our construction of the nice condition v will take place over ω-many stages, with the function g being used for bookkeeping. In order to satisfy properties (2) and (3) of Definition 5.51 (Nice Conditions), the construction will involve building not only v, but also infinitely many total master conditions r with $r \upharpoonright \theta = v$. At any given stage n, we will define an approximation v_n of v together with finitely many approximations of the total master conditions. The first coordinate n_0 of the value $g(n)$ splits the construction into four cases. When $n_0 = 0$, we handle a case of Definition 5.51(2), and when $n_0 = 3$, we handle a case of Definition 5.51(3). When n_0 is 1 or 2, we take the usual steps for building total master conditions, namely, meeting dense sets in the first case and applying augmentation in the second case.

More specifically, we define by induction the following objects in ω-many steps:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;
- a non-decreasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ, where each X_n has unique drop-downs to δ_n;

- a decreasing sequence $\langle v_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}_\theta$;
- a sequence $\langle l_m : m < \omega \rangle$ of natural numbers;
- a family $\{r_n^{m,l} : m < \omega,\ l < l_m,\ m < n < \omega\}$ of conditions in $N \cap \mathbb{P}$ below p such that for each $m < \omega$ and $l < l_m$, $\langle r_n^{m,l} : m < n < \omega \rangle$ is a descending sequence;
- for each $m < \omega$ such that $l_m = 3$, a pair $\vec{e}^{m,0}$ and $\vec{e}^{m,1}$ of tuples above $\vec{b}$ with height δ such that $\vec{e}^{m,0} \upharpoonright (\alpha + 1)$ and $\vec{e}^{m,1} \upharpoonright (\alpha + 1)$ are disjoint, and the elements of $\vec{e}^{m,0}$ and $\vec{e}^{m,1}$ are in X_{m+1};
- for all $m < \omega$, $l < l_m$, $n > m$, and $\tau \in A_n$, an injective partial function $h_{n,\tau}^{m,l}$ from X_n to X_n.

At stage n, we will define X_n, A_n, δ_n, v_n, and when $n > 0$, l_{n-1}, $r_n^{m,l}$ and $h_{n,\tau}^{m,l}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$, and when $l_{n-1} = 3$, $\vec{c}^{n-1,0}$ and $\vec{c}^{n-1,1}$.

In addition to the properties listed above, we will maintain the following inductive hypotheses for all $m < n < \omega$ and $l < l_m$:

(1) $r_n^{m,l}$ and v_n have top level δ_n and $r_n^{m,l} \upharpoonright \theta = v_n$;
(2) for all $\tau \in A_n$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright \delta_{n+1}$ are $r_{n+1}^{m,l}(\tau)$-consistent;
(3) $A_n \subseteq \operatorname{dom}(r_n^{m,l})$ and $\{r_n^{m,l}(\tau) : \tau \in A_n\}$ is separated on $X_n \upharpoonright \delta_n$;
(4) suppose that $g(n) = (n_0, n_1, \sigma)$, where $n_0 = 0$, s_{n_1} has top level less than δ_n, $s_{n_1} \leq p$, and $v_n \leq_\theta s_{n_1} \upharpoonright \theta$; then $l_n = 1$ and $r_{n+1}^{n,0} \leq s_{n_1}$;
(5) if $g(n) = (n_0, n_1, \sigma)$, where $n_0 = 1$, then $r_{n+1}^{m,l} \in \bigcap_{k<n} D_k$ and $\sigma \in \operatorname{dom}(r_{n+1}^{m,l})$;
(6) assuming that $g(n) = (n_0, n_1, \sigma)$, where $n_0 = 2$, and $\sigma \in \bigcap\{\operatorname{dom}(r_n^{m,l}) : m < n,\ l < l_m\}$, then $\sigma \in A_{n+1}$ and for all $m < n$ and $l < l_m$, z_{n_1} is in the domain and range of $h_{n+1,\sigma}^{m,l}$;
(7) for all $\tau \in A_n$ and $x, y \in X_n$,

$$h_{n,\tau}^{m,l}(x) = y \iff r_n^{m,l}(\tau)(x \upharpoonright \delta_n) = y \upharpoonright \delta_n.$$

Stage 0: Let $X_0 = \emptyset$ and $A_0 = A$. Let δ_0 be the top level of q and let $v_0 = q \upharpoonright \theta$. Note that by our assumption about q, $\delta_0 > \alpha$.

Stage 1: Let $l_0 = 1$. Let $X_1 = X_0 = \emptyset$, $A_1 = A_0$, $\delta_1 = \delta_0$, $r_1^{0,0} = q$, and $v_1 = q \upharpoonright \theta = v_0$. Define $h_{1,\tau}^{0,0} = \emptyset$ for all $\tau \in A_1$.

Stage $n + 1$ ($n > 0$): Assume that stage n is complete, where $n < \omega$ is positive. In particular, we have defined the following objects which we assume satisfy all of the required properties: X_n, A_n, δ_n, v_n, l_m for all $m < n$, and $r_n^{m,l}$ and $h_{n,\tau}^{m,l}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$. Let $g(n) = (n_0, n_1, \sigma)$.

Case a: $n_0 = 0$. Consider s_{n_1}, which is in $N \cap \mathbb{P}$, and let γ be the top level of s_{n_1}. We will only take action in the case that $\gamma < \delta_n$, $s_{n_1} \leq p$, and $v_n \leq s_{n_1} \upharpoonright \theta$. If not, then let $l_n = 0$, $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n$, $v_{n+1} = v_n$, $r_{n+1}^{m,l} = r_n^{m,l}$ and $h_{n+1,\tau}^{m,l} = h_{n,\tau}^{m,l}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$.

Otherwise, let $l_n = 1$. Without loss of generality, we may assume that $A_n \subseteq \operatorname{dom}(s_{n_1})$, for otherwise we can easily extend s_{n_1} to have this property without increasing its top level or changing the fact that $v \leq_\theta s_{n_1} \upharpoonright \theta$. Apply Lemma 5.44 (Simple Generalized Extension) to find some $s \leq s_{n_1}$ in N with top level δ_n such that $s \upharpoonright \theta = v_n$. Applying Lemma 5.29 (Separated Conditions Are Dense) and using the fact that $\{v_n(\tau) : \tau \in A_n \cap \theta\}$ is separated on $X_n \upharpoonright \delta_n$ (by inductive hypothesis (3) and Lemma 5.9), find a separated condition $r_{n+1}^{n,0} \leq s$ with top level $\delta_n + 1$ such that for all $\tau \in A_n \cap \theta$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright (\delta_n + 1)$ are $r_{n+1}^{n,0}(\tau)$-consistent. Let $v_{n+1} = r_{n+1}^{n,0} \upharpoonright \theta$. Then in particular, for all $\tau \in A_n \cap \theta$, $X_n \upharpoonright \delta_n$ and $X_n \upharpoonright (\delta_n + 1)$ are $v_{n+1}(\tau)$-consistent. Now apply Lemma 5.45 (Generalized Extension) to find for each $m < n$ and $l < l_m$ a condition $r_{n+1}^{m,l} \leq r_n^{m,l}$

with top level $\delta_n + 1$ such that $r^{m,l}_{n+1} \restriction \theta = v_{n+1}$ and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction (\delta_n + 1)$ are $r^{m,l}_{n+1}(\tau)$-consistent.

Define $X_{n+1} = X_n$, $A_{n+1} = A_n$, and $\delta_{n+1} = \delta_n + 1$. Define $h^{m,l}_{n+1,\tau}$ for all $m \leq n$, $l < l_m$, and $\tau \in A_{n+1}$ as described in inductive hypothesis (7). It is routine to check that the inductive hypotheses hold.

Case b: $n_0 = 1$. Let $l_n = 0$, $X_{n+1} = X_n$, and $A_{n+1} = A_n$. Let D be the set of conditions r in $\bigcap_{k<n} D_k$ satisfying that $\sigma \in \mathrm{dom}(r)$ and the top level of r is at least γ_{n+1}. Then D is dense open in $\mathbb{P}$, and $D \in N$ by elementarity. Apply Lemma 5.46 (Generalized Consistent Extensions Into Dense Sets) to find an ordinal $\delta_{n+1} < \delta$, a condition $v_{n+1} \leq_\theta v_n$ in N, and for each $m < n$ and $l < l_m$, a condition $r^{m,l}_{n+1} \leq r^{m,l}_n$ in $N \cap D$ with top level δ_{n+1} satisfying that $r^{m,l}_{n+1} \restriction \theta = v_{n+1}$, and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $r^{m,l}_{n+1}(\tau)$-consistent. Define $h^{m,l}_{n+1,\tau}$ for all $\tau \in A_n$ as described in inductive hypothesis (7). It is routine to check that the inductive hypotheses hold.

Case c: $n_0 = 2$. If $\sigma \notin \bigcap\{\mathrm{dom}(r^{m,l}_n) : m < n,\ l < l_m\}$, then let $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n$, $v_{n+1} = v_n$, $l_n = 0$, and $r^{m,l}_{n+1} = r^{m,l}_n$ and $h^{m,l}_{n+1,\tau} = h^{m,l}_{n,\tau}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$.

Otherwise, fix $\gamma < \delta$ large enough so that $X_n \cup \{z_{n_1}\}$ has unique drop-downs to γ. Apply Lemma 5.45 (Generalized Extension) to find some $\bar{v}_n \in N \cap \mathbb{P}_\theta$ with top level γ, and for each $m < n$ and $l < l_m$ find a condition $\bar{r}^{m,l}_n \leq r^{m,l}_n$ in $N \cap \mathbb{P}$ with top level γ such that $\bar{r}^{m,l}_n \restriction \theta = \bar{v}_n$ and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \gamma$ are $\bar{r}^{m,l}_n(\tau)$-consistent. By Lemma 5.22 (Persistence for Sets), for all $m < n$ and $l < l_m$, $\{\bar{r}^{m,l}_n(\tau) : \tau \in A_n\}$ is separated on $X \restriction \gamma$.

Now apply Lemma 5.48 (Generalized Augmentation) to find for each $m < n$ and $l < l_m$ a condition $r^{m,l}$ in $N \cap \mathbb{P}$ with top level $\gamma + 1$, a condition $v_{n+1} \in N \cap \mathbb{P}_\theta$ with top level $\gamma + 1$, and a finite set $X_{n+1} \subseteq T_\delta$ such that $X_n \cup \{z_{n_1}\} \subseteq X_{n+1}$ and X_{n+1} has unique drop-downs to $\gamma + 1$, satisfying that for all $m < n$ and $l < l_m$:

- $r^{m,l}_{n+1} \leq \bar{r}^{m,l}_n$ and $r^{m,l}_{n+1} \restriction \theta = v_{n+1}$;
- for all $\tau \in A_n$, $X_n \restriction \gamma$ and $X_n \restriction (\gamma + 1)$ are $r^{m,l}_{n+1}(\tau)$-consistent;
- $\{r^{m,l}_{n+1}(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated on $X_{n+1} \restriction (\gamma + 1)$;
- let $h^+_{m,l,\sigma}$ be the partial injective function from X_{n+1} to X_{n+1} defined by letting, for all $x, y \in X_{n+1}$, $h^+_{m,l,\sigma}(x) = y$ iff $r^{m,l}_{n+1}(\sigma)(x \restriction (\gamma + 1)) = y \restriction (\gamma + 1)$; then z_{n_1} is in the domain and range of $h^+_{m,l,\sigma}$.

Define $X_{n+1} = X_n \cup \{z_{n_1}\}$, $A_{n+1} = A_n \cup \{\sigma\}$, $\delta_{n+1} = \gamma + 1$, and $l_n = 0$. For all $m < n$, $l < l_m$, and $\tau \in A_{n+1}$, define $h^{m,l}_{n+1,\tau}$ as described in inductive hypothesis (7). Note that $h^{m,l}_{n+1,\sigma} = h^+_{m,l,\sigma}$, and therefore z_{n_1} is in the domain and range of $h^{m,l}_{n+1,\sigma}$. The inductive hypotheses are straightforward to check.

Case d: $n_0 = 3$. Let us consider $\vec{a}^{n_1,0}$ and $\vec{a}^{n_1,1}$. For simplicity in notation, write $\vec{a}^0$ for $\vec{a}^{n_1,0}$ and $\vec{a}^1$ for $\vec{a}^{n_1,1}$. We will only take action when the elements of the tuples $\vec{a}^0$ and $\vec{a}^1$ are in X_n and for all $\tau \in A \cap \theta$ and $j < 2$, $\vec{b}$ and $\vec{a}^j \restriction \delta_n$ are $v_n(\tau)$-consistent. If not, then let $l_n = 0$, $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n$, $v_{n+1} = v_n$, $r^{m,l}_{n+1} = r^{m,l}_n$ and $h^{m,l}_{n+1,\tau} = h^{m,l}_{n,\tau}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$. Otherwise, proceed as follows.

Applying Lemma 5.47 (Generalized Separated Conditions are Dense) in N, find a family $\{\bar{r}^{m,l}_n : m < n,\ l < l_m\}$ of conditions with top level $\delta_n + 1$ and a condition $\bar{v}_n \leq_\theta v_n$ such that for all $m < n$ and $l < l_m$, $\bar{r}^{m,l}_n \leq r^{m,l}_n$, $\bar{r}^{m,l}_n \restriction \theta = \bar{v}_n$, $\bar{r}^{m,l}_n$ is separated, and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction (\delta_n + 1)$ is $\bar{r}^{m,l}_n(\tau)$-consistent.

Fix an injective tuple $\vec{z} = (z_0, \ldots, z_{\hat{n}-1})$ which enumerates X_n. Since the elements of $\vec{a}^0$ and $\vec{a}^1$ are in X_n, we can fix distinct $j_0, \ldots, j_{d-1}$ and distinct $k_0, \ldots, k_{d-1}$ in $\hat{n}$ such that $\vec{a}^0 = (z_{j_0}, \ldots, z_{j_{d-1}})$ and $\vec{a}^1 = (z_{k_0}, \ldots, z_{k_{d-1}})$. Let $x_m = z_m \restriction (\delta_n + 1)$ for all $m < \hat{n}$, and let $\vec{x} = (x_0, \ldots, x_{\hat{n}-1})$.

Define $\mathcal{X}^*$ as the set of all tuples $\vec{y} = (y_0, \ldots, y_{\hat{n}-1})$ in the derived tree $T_{\vec{x}}$ for which there exist conditions $t^0, t^1 \leq p$ with top level ξ equal to the height of $\vec{y}$ and there exists $w \in \mathbb{P}_\theta$ satisfying:

- $t^0 \restriction \theta = t^1 \restriction \theta = w \leq_\theta \bar{v}_n$;
- for all $\tau \in A_n \cap \theta$, $\vec{x}$ and $\vec{y}$ are $w(\tau)$-consistent;
- for $j < 2$, $A_n \subseteq \mathrm{dom}(t^j)$;
- for $j < 2$, for any finite $Y \subseteq T_\xi$ with unique drop-downs to δ_n, $\{t^j(\tau) : \tau \in A_n\}$ is separated on Y;
- for all $\tau \in A$, $\vec{b}$ and $(y_{j_0}, \ldots, y_{j_{d-1}})$ are $t^0(\tau)$-consistent, and $\vec{b}$ and $(y_{k_0}, \ldots, y_{k_{d-1}})$ are $t^1(\tau)$-consistent;

Now let $\mathcal{X}$ be the set of all $\vec{y}$ in $T_{\vec{x}}$ such that either $\vec{y} \in \mathcal{X}^*$, or else for all $\vec{z} \geq \vec{y}$, $\vec{z} \notin \mathcal{X}^*$. Obviously, $\mathcal{X}$ is dense in $T_{\vec{x}}$. Using Lemma 5.45 (Generalized Extension) and Lemma 5.22 (Persistence for Sets) it is easy to show that $\mathcal{X}^*$ is open. Also, $\mathcal{X} \in N$ by elementarity. Since T is a free Suslin tree, $T_{\vec{x}}$ is Suslin. So we can fix some $\xi < \delta$ greater than γ such that every member of $T_{\vec{x}}$ of height at least ξ is in $\mathcal{X}$.

We consider two cases. First, assume that $\vec{z} \restriction \xi \notin \mathcal{X}^*$. Since $\vec{z} \restriction \xi \in \mathcal{X}$, it follows that $\vec{z}$ is not in $\mathcal{X}^*$ either. In this case, let $l_n = 0$, $X_{n+1} = X_n$, $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n + 1$, $v_{n+1} = \bar{v}_n$, $r^{m,l}_{n+1} = \bar{r}^{m,l}_n$ and $h^{m,l}_{n+1,\tau} = h^{m,l}_{n,\tau}$ for all $m < n$, $l < l_m$, and $\tau \in A_n$.

Secondly, assume that $\vec{z} \restriction \xi \in \mathcal{X}^*$. Fix t^0, t^1, and w in N satisfying the five statements listed in the definition of $\mathcal{X}^*$. Let $l_n = 3$. Apply Lemma 5.31 (Generalized Key Property) to find tuples $\vec{c}^0$ and $\vec{c}^1$ above $\vec{b}$ with height ξ satisfying:

- for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{c}^j$ are $t^j(\tau)$-consistent;
- $\vec{c}^0 \restriction (\alpha + 1)$, $\vec{c}^1 \restriction (\alpha + 1)$, and $X_n \restriction (\alpha + 1)$ are pairwise disjoint;
- for all $x \in X_n \restriction \xi$, $\tau \in A_n \cap \theta$, and $m \in \{-1, 1\}$, $w(\tau)^m(x)$ is not in $\vec{c}^0$ or in $\vec{c}^1$;
- for all $\tau \in A_n \cap \theta$, $m \in \{-1, 1\}$, and $j < 2$, if x is in $\vec{c}^j$ then $w(\tau)^m(x)$ is not in $\vec{c}^{1-j}$.

Pick arbitrary tuples $\vec{d}^0$ and $\vec{d}^1$ above $\vec{c}^0$ and $\vec{c}^1$ respectively of height $\xi + 1$. Let D^0 be the set of elements in $\vec{d}^0$ and let D^1 the set of elements in $\vec{d}^1$. Apply Lemma 5.53 in N to find conditions $r^{n,2}_{n+1} \in N \cap \mathbb{P}$ and $v_{n+1} \in N \cap \mathbb{P}_\theta$, both with top level $\xi + 1$, satisfying:

- $r^{n,2}_{n+1} \leq p$, $v_{n+1} \leq_\theta w$, $r^{n,2}_{n+1} \restriction \theta = v_{n+1}$, and $A_n \subseteq \mathrm{dom}(r^{n,2}_{n+1})$;
- for all $\tau \in A$, $\vec{b}$ and $\vec{d}^0$ are $r^{n,2}_{n+1}(\tau)$-consistent and $\vec{b}$ and $\vec{d}^1$ are $r^{n,2}_{n+1}(\tau)$-consistent;
- for all $\tau \in A_n \cap \theta$, $(X_n \restriction \xi) \cup (D^0 \restriction \xi) \cup (D^1 \restriction \xi)$ and $(X_n \restriction (\xi + 1)) \cup D^0 \cup D^1$ are $v_{n+1}(\tau)$-consistent;
- $\{r^{n,2}_{n+1}(\tau) : \tau \in A_n\}$ is separated on $(X_n \restriction (\xi + 1)) \cup D^0 \cup D^1$.

Apply Lemma 5.45 (Generalized Extension) to find $r^{n,0}_{n+1} \leq t^0$ and $r^{n,1}_{n+1} \leq t^1$ with top level $\xi + 1$ such that for each $j < 2$, $r^{n,j}_{n+1} \restriction \theta = v_{n+1}$ and for all $\tau \in A_n$, $(X_n \restriction \xi) \cup (D^0 \restriction \xi) \cup (D^1 \restriction \xi)$ and $(X_n \restriction (\xi + 1)) \cup D^0 \cup D^1$ are $r^{n,j}_{n+1}(\tau)$-consistent. It follows that for $j < 2$, for all $\tau \in A$, $\vec{b}$ and $\vec{d}^j$ are $r^{n,j}_{n+1}(\tau)$-consistent. For each $j < 2$, since $\delta_n > \alpha$, the set $(X_n \restriction \xi) \cup (D^0 \restriction \xi) \cup (D^1 \restriction \xi)$ has unique drop-downs to δ_n, so $\{t^j(\tau) : \tau \in A_n\}$ is separated on it; by Lemma 5.22 (Persistence for Sets), $\{r^{n,j}_{n+1}(\tau) : \tau \in A_n\}$ is separated on $(X_n \restriction (\xi + 1)) \cup D^0 \cup D^1$.

Apply Lemma 5.45 (Generalized Extension) to find a family $\{r^{m,l}_{n+1} : m < n,\ l < l_m\}$ of conditions with top level $\xi + 1$ such that for all $m < n$ and $l < l_m$, $r^{m,l}_{n+1} \leq \bar{r}^{m,l}_n$, $r^{m,l}_{n+1} \restriction \theta = v_{n+1}$, and for all $\tau \in A_n$, $X_n \restriction (\delta_n + 1)$ and $X_n \restriction (\xi + 1)$ are $r^{m,l}_{n+1}(\tau)$-consistent. Since each $\bar{r}^{m,l}_n$ is separated and $(X_n \restriction (\xi+1)) \cup D^0 \cup D^1$ has unique drop-downs to $\delta_n + 1$, by Lemma 5.22 (Persistence for Sets), $\{r^{m,l}_{n+1}(\tau) : \tau \in A_n\}$ is separated on $(X_n \restriction (\xi+1)) \cup D^0 \cup D^1$.

Define $A_{n+1} = A_n$ and $\delta_{n+1} = \xi + 1$. Fix arbitrary tuples $\vec{e}^{\,n,0}$ and $\vec{e}^{\,n,1}$ above $\vec{d}^0$ and $\vec{d}^1$ respectively with height δ. Define X_{n+1} by adding to X_n the elements of $\vec{e}^{\,n,0}$ and $\vec{e}^{\,n,1}$. Finally, define $h^{m,l}_{n+1,\tau}$ for each $m < n$, $l < l_m$, and $\tau \in A_{n+1}$ as described in inductive hypothesis (7). It is straightforward to check that the required properties are satisfied.

To make the verification of (3) below easier to check, let us highlight the following facts which we have proven:

- for all $\tau \in A$, $\vec{b}$ and $\vec{d}^0$ are $r^{n,2}_{n+1}(\tau)$-consistent and $r^{n,0}_{n+1}(\tau)$-consistent;
- for all $\tau \in A$, $\vec{b}$ and $\vec{d}^1$ are $r^{n,2}_{n+1}(\tau)$-consistent and $r^{n,1}_{n+1}(\tau)$-consistent;
- $\vec{d}^0 < \vec{e}^0$, $\vec{d}^1 < \vec{e}^1$, and the elements of $\vec{e}^0$ and $\vec{e}^1$ are in X_{n+1}.

This completes the construction. For all $m < \omega$ and $l < l_m$, define $r^{m,l} \in \mathbb{P}$ with domain $N \cap \kappa$ so that for all $\tau \in N \cap \kappa$,

$$r^{m,l}(\tau) = \bigcup\{r^{m,l}_n(\tau) : m < n < \omega,\ \tau \in A_n\} \cup \bigcup\{h^{m,l}_{n,\tau} : m < n < \omega,\ \tau \in A_n\}.$$

Also, define $v = r^{m,l} \restriction \theta$ for some (any) any $m < \omega$ and $l < l_m$. By Lemma 5.32 (Constructing Total Master Conditions), each $r^{m,l}$ is a total master condition over N which is a lower bound of the sequence $\langle r^{m,l}_n : m < n < \omega\rangle$, and for all $n < \omega$ and for all $\tau \in A_n$, $X_n \restriction \delta_n$ and X_n are $r^{m,l}(\tau)$-consistent. By Lemma 5.35, each $r^{m,l}$ is separated.

This completes the construction. Note that by what we did at stage 0, $v \le q \restriction \theta$. So it remains to prove that v is N-nice for p, $\vec{b}$, and A. We verify property (1)-(3) of Definition 5.51 (Nice Conditions).

(1) Clearly, $v \le p \restriction \theta$ and v has top level δ. Since, for example, $r^{0,0}$ is a total master condition for $\mathbb{P}$ over N, by a standard argument it follows that $v = r^{0,0} \restriction \theta$ is a total master condition for $\mathbb{P}_\theta$ over N. So v decides $\dot{U} \restriction \delta$.

(2) Let $s \in N \cap \mathbb{P}$ have top level γ such that $s \le p$ and $v \le_\theta s \restriction \theta$. It suffices to show that for some n, $r^{n,0} \le s$. Fix n_1 such that $s = s_{n_1}$. Pick n' large enough so that $\mathrm{dom}(s) \cap \theta \subseteq \mathrm{dom}(v_{n'})$ and $\delta_{n'} > \gamma$. Note that for any $n \ge n'$, $v_n \le_\theta s \restriction \theta$. Now find $n \ge n'$ such that for some $\sigma \in N \cap \kappa$, $g(n) = (0, n_1, \sigma)$. By inductive hypothesis (4), $r^{n,0} \le s$.

(3) Suppose that $\vec{a}^0$ and $\vec{a}^1$ are distinct tuples above $\vec{b}$ with height δ, and $q_0, q_1 \le p$ have top level δ and satisfy: $q_0 \restriction \theta = q_1 \restriction \theta = v$, $N \cap \kappa \subseteq \mathrm{dom}(q_0) \cap \mathrm{dom}(q_1)$, q_0 and q_1 are separated, q_0 and q_1 decide $\dot{b} \cap \delta$, and for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{a}^j$ are $q_j(\tau)$-consistent. In particular, for all $\tau \in A \cap \theta$ and $j < 2$, $\vec{b}$ and $\vec{a}^j$ are $v(\tau)$-consistent. Fix n' large enough so that the elements of $\vec{a}^0$ and $\vec{a}^1$ are in $X_{n'}$. Fix n_1 so that $(\vec{a}^{n_1,0}, \vec{a}^{n_1,1}) = (\vec{a}^0, \vec{a}^1)$. Find $n \ge n'$ such that for some $\sigma \in N \cap \kappa$, $g(n) = (3, n_1, \sigma)$. Observe that for all $\tau \in A \cap \theta$ and $j < 2$, the fact that $\vec{b}$ and $\vec{a}^j$ are $v(\tau)$-consistent implies that $\vec{b}$ and $\vec{a}^j \restriction \delta_n$ are $v_n(\tau)$-consistent.

So the requirements described in the first paragraph of Case d are met. Letting $\vec{z}$ be the enumeration of X_n given in Case d, clearly $\vec{z} \in \mathcal{X}^*$ as witnessed by q_0, q_1, and v. Using the information which we highlighted at the end of Case d and the fact that for all $j < 3$ and for all $\tau \in A_{n+1}$, $X_{n+1} \restriction \delta_{n+1}$ and X_{n+1} are $r^{n,j}(\tau)$-consistent, it is routine to check that $r^{n,0}$, $r^{n,1}$, $r^{n,2}$, $\vec{e}^{\,n,0}$, and $\vec{e}^{\,n,1}$ satisfy the properties described of q^*_0, q^*_1, r, $\vec{e}^0$, and $\vec{e}^1$ in (3) in Definition 5.51 (Nice Conditions). □

5.9. **The Automorphism Forcing Adds No New Cofinal Branches.** In this subsection we complete the proof of Theorem 5.49 (No New Cofinal Branches).

Proposition 5.55. *Suppose that $\bar{p} \in \mathbb{P}$ has top level β and $\bar{p}$ forces that $\dot{b}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. Assume that $A \subseteq \mathrm{dom}(\bar{p})$ is finite, $\vec{x} = (x_0, \ldots, x_{n-1})$ consists of distinct elements of T_β, and $\{\bar{p}(\tau) : \tau \in A\}$ is separated on $\vec{x}$. Define $\mathcal{X}$ as the set of all tuples $\vec{b} = (b_0, \ldots, b_{n-1})$ in the derived tree $T_{\vec{x}}$ for which there exist $q_0, q_1 \leq \bar{p}$ with top level equal to the height γ of $\vec{b}$ such that:*

1. *$q_0 \restriction \theta = q_1 \restriction \theta$;*
2. *for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;*
3. *there exists some $\zeta < \gamma$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.*

Then $\mathcal{X}$ is dense open in $T_{\vec{x}}$.

Proof. To prove that $\mathcal{X}$ is open, assume that $\vec{b} \in \mathcal{X}$ has height γ and $\vec{c} > \vec{b}$ has height ξ. Fix $q_0, q_1 \in \mathbb{P}$ with top level γ which witness that $\vec{b} \in \mathcal{X}$, and let $v = q_0 \restriction \theta = q_1 \restriction \theta$. By Lemma 5.45 (Generalized Extension), extend q_0 and q_1 to r_0 and r_1 respectively with top level ξ such that $r_0 \restriction \theta = r_1 \restriction \theta$ and for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{c}$ are $r_j(\tau)$-consistent. Then r_0 and r_1 witness that $\vec{c} \in \mathcal{X}$.

Now we prove that $\mathcal{X}$ is dense. Suppose for a contradiction that $\vec{b} \in T_{\vec{x}}$ and for all $\vec{c} \geq \vec{b}$, $\vec{c} \notin \mathcal{X}$. Let α be the height of $\vec{b}$. Apply Lemma 5.24 (Extension) to find some $p \leq \bar{p}$ with top level α such that for all $\tau \in A$, $\vec{x}$ and $\vec{b}$ are $p(\tau)$-consistent. Since $\{\bar{p}(\tau) : \tau \in A\}$ is separated on $\vec{x}$, $\{p(\tau) : \tau \in A\}$ is separated on $\vec{b}$.

Let $\mathcal{C}$ be the collection of all suitable sets. Fix a $\in$-increasing and continuous chain $\langle N_i : i < \omega_1 \rangle$ of suitable sets which are elementary substructures of $(H(\lambda), \in, \trianglelefteq, \mathcal{C})$ such that p is in N_0. Let $\delta_\gamma = N_\gamma \cap \omega_1$ for all $\gamma < \omega_1$. Observe that for all $\gamma_1 < \gamma_2 < \omega_1$, the property of a condition being N_{γ_1}-nice for $p, \vec{b}$, and A is definable in N_{γ_2}.

We define a function F which takes as inputs any pair (v, r) satisfying that for some $\gamma < \omega_1$:

1. v is N_γ-nice for $p, \vec{b}$, and A;
2. $r \leq p$, r has top level δ_γ, $r \restriction \theta = v$, $N_\gamma \cap \kappa \subseteq \mathrm{dom}(r)$, r is separated, and r decides $\dot{b} \cap \delta_\gamma$.

Let $F(v, r)$ be the unique set b_r such that $r \Vdash_{\mathbb{P}} \dot{b} \cap \delta_\gamma = b_r$.

<u>Claim 1:</u> For any $\gamma < \omega_1$ and any v which is N_γ-nice for $p, \vec{b}$, and A, there exists some r such that $(v, r) \in \mathrm{dom}(F)$. Proof: This follows from (2) of Definition 5.51 (Nice Conditions).

<u>Claim 2:</u> For any $\gamma < \omega_1$ and any v which is N_γ-nice for $p, \vec{b}$, and A, if (v, q_0) and (v, q_1) are both in the domain of F, then $F(v, q_0) = F(v, q_1)$. Proof: Using Lemma 5.25 (Key Property) twice, fix disjoint tuples $\vec{a}^0$ and $\vec{a}^1$ above $\vec{b}$ with height δ_γ such that for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{a}^j$ are $q_j(\tau)$-consistent.

Note that q_0, q_1, $\vec{a}^0$, and $\vec{a}^1$ satisfy the properties listed in (3) of Definition 5.51 (Nice Conditions). Since v is N_γ-nice, there exist q_0^* and q_1^* satisfying the same properties of q_0 and q_1 which are listed in (3) of Definition 5.51 (Nice Conditions), there exists $r \leq p$ with top level δ_γ, and there exist disjoint tuples $\vec{e}^0$ and $\vec{e}^1$ above $\vec{b}$ with height δ_γ satisfying: $r \restriction \theta = v$, $N_\gamma \cap \kappa \subseteq \mathrm{dom}(r)$, r is separated, r decides $\dot{b} \cap \delta_\gamma$, and for all $\tau \in A$ and $j < 2$, $\vec{b}$ and $\vec{e}^j$ are $q_j^*(\tau)$-consistent and $r(\tau)$-consistent.

Observe that (v, q_0^*), (v, q_1^*), and (v, r) are in the domain of F. Recall that for all $\vec{c} \geq \vec{b}$, $\vec{c} \notin \mathcal{X}$. In particular, $\vec{e}^0$, $\vec{e}^1$, $\vec{a}^0$, and $\vec{a}^1$ are not in $\mathcal{X}$. If $F(v, q_0^*) \neq F(v, r)$, then q_0^* and r would witness that $\vec{e}^0 \in \mathcal{X}$. Hence, $F(v, q_0^*) = F(v, r)$. If $F(v, r) \neq F(v, q_1^*)$, then $\vec{e}^1$ would be in $\mathcal{X}$. So $F(v, r) = F(v, q_1^*)$. Similarly, the fact that $\vec{a}^0$ and $\vec{a}^1$ are not in $\mathcal{X}$ implies that $F(v, q_0) = F(v, q_0^*)$

and $F(v, q_1) = F(v, q_1^*)$. So $F(v, q_0) = F(v, q_0^*) = F(v, r) = F(v, q_1^*) = F(v, q_1)$. This completes the proof of claim 2.

Define $F(v)$ to be equal to $F(v, r)$ for any r such that (v, r) is in the domain of F. By claim 1, $F(v)$ is defined for any v which is N_γ-nice for p, $\vec{b}$, and A for some $\gamma < \omega_1$. By claim 2, $F(v)$ is well-defined.

Claim 3: Suppose that $\gamma_1 < \gamma_2$, $v_1 \in N_{\gamma_2}$ is N_{γ_1}-nice for p, $\vec{b}$ and A, v_2 is N_{γ_2}-nice for p, $\vec{b}$, and A, and $v_2 \leq_\theta v_1$. Then $F(v_1) = F(v_2) \cap \delta_{\gamma_1}$. Proof: By (2) of Definition 5.51 (Nice Conditions) and elementarity, fix some $r_1 \leq p$ in N_{γ_2} with top level δ_{γ_1} such that $r_1 \restriction \theta = v_1$, $N_{\gamma_1} \cap \kappa \subseteq \mathrm{dom}(r_1)$, r_1 is separated, and r_1 decides $\dot{b} \cap \delta_{\gamma_1}$ as some set b_1. Then $F(v_1) = b_1$. Since $v_2 \leq_\theta v_1 = r_1 \restriction \theta$ and v_2 is N_{γ_2}-nice for p, $\vec{b}$, and A, by (2) of Definition 5.51 (Nice Conditions) we can find some $r_2 \leq r_1$ with top level δ_{γ_2} such that $r_2 \restriction \theta = v_2$, $N_{\gamma_2} \cap \kappa \subseteq \mathrm{dom}(r_2)$, r_2 is separated, and r_2 decides $\dot{b} \cap \delta_{\gamma_2}$ as some set b_2. Then $F(v_2) = b_2$. Since $r_2 \leq r_1$, $b_1 = b_2 \cap \delta_{\gamma_1}$, which completes the proof of claim 3.

Claim 4: If $\gamma \leq \xi < \omega_1$, v is N_γ-nice for p, $\vec{b}$, and A, w is N_ξ-nice for p, $\vec{b}$, and A, and v and w are compatible in $\mathbb{P}_\theta$, then $F(v) = F(w) \cap \delta_\gamma$. In particular, if $\gamma = \xi$ then $F(v) = F(w)$. Proof: If not, then since $N_\xi \in N_{\xi+1}$, by elementarity we can find counter-examples v and w in $N_{\xi+1}$. Now find $z \leq_\theta v, w$ in $N_{\xi+1}$. Apply Lemma 5.44 (Simple Generalized Extension) to find $q \leq p$ in $N_{\xi+1}$ such that $q \restriction \theta = z$. By Proposition 5.54 (Existence of Nice Conditions), fix $v_2 \leq_\theta q \restriction \theta = z$ such that v_2 is $N_{\xi+1}$-nice for p, $\vec{b}$, and A. Then by claim 3, $F(v) = F(v_2) \cap \delta_\gamma$ and $F(w) = F(v_2) \cap \delta_\xi$, so $F(v) = (F(v_2) \cap \delta_\xi) \cap \delta_\gamma = F(w) \cap \delta_\gamma$, which is a contradiction. This completes the proof of claim 4.

For each $\gamma < \omega_1$, let $\dot{c}_\gamma$ be a $\mathbb{P}_\theta$-name for the unique set which is equal to $F(v)$, where $v \in \dot{G}_\theta$ is N_γ-nice for p, $\vec{b}$, and A, and is equal to the emptyset if there is no such v. Note that by claim 4, $\dot{c}_\gamma$ is well-defined. By elementarity, we can choose the name $\dot{c}_\gamma$ to be in $N_{\gamma+1}$. Now let $\dot{c}$ be a $\mathbb{P}_\theta$-name for the union $\bigcup\{\dot{c}_\gamma : \gamma < \omega_1\}$. Note that the names $\dot{c}_\gamma$ and the name $\dot{c}$ are also $\mathbb{P}$-names since $\mathbb{P}_\theta$ is a regular suborder of $\mathbb{P}$.

Claim 5: p forces in $\mathbb{P}$ that $\dot{c}$ is a chain in $\dot{U}$. Proof: Clearly, for all $\gamma < \omega_1$, p forces that $\dot{c}_\gamma$ is a chain in $\dot{U}$. So it suffices to show that p forces that for all $\gamma < \xi$ such that $\dot{c}_\gamma$ and $\dot{c}_\xi$ are non-empty, $\dot{c}_\gamma = \dot{c}_\xi \cap \delta_\gamma$. So let G be a generic filter on $\mathbb{P}$ which contains p and let $G_\theta = G \cap \mathbb{P}_\theta$. Let $c_\gamma = \dot{c}_\gamma^{G_\theta}$ and $c_\xi = \dot{c}_\xi^{G_\theta}$, and assume that c_γ and c_ξ are non-empty. Fix $v \in G_\theta$ which is N_γ-nice for p, $\vec{b}$, and A, and fix $w \in G_\theta$ which is N_ξ-nice for p, $\vec{b}$, and A, which exist because c_γ and c_ξ are non-empty. Then $c_\gamma = F(v)$ and $c_\xi = F(w)$. Since v and w are in G_θ, they are compatible in $\mathbb{P}_\theta$, so by claim 4, $c_\gamma = F(v) = F(w) \cap \delta_\gamma = c_\xi$. This completes the proof of claim 5.

Claim 6: p forces in $\mathbb{P}$ that $\dot{c}$ is equal to $\dot{b}$, and hence that $\dot{b} \in V^{\mathbb{P}_\theta}$. Since $p \leq \bar{p}$, this claim gives a contradiction to our initial assumptions. Proof: Since p forces that $\dot{c}$ is a chain in $\dot{U}$ and $\dot{b}$ is an uncountable branch of $\dot{U}$, it suffices to show that p forces that $\dot{b} \subseteq \dot{c}$.

Suppose for a contradiction that there exists some $q \leq p$ and some $\zeta < \omega_1$ such that $q \Vdash_{\mathbb{P}} \zeta \in \dot{b} \setminus \dot{c}$. Fix a $\in$-increasing and continuous sequence $\langle M_\gamma : \gamma < \omega_1 \rangle$ of suitable sets such that M_0 contains as members the objects p, q, ζ, $\langle N_i : i < \omega_1 \rangle$, and $\langle \dot{c}_i : i < \omega_1 \rangle$. Let $D \subseteq \omega_1$ be a club such that for all $\gamma \in D$, $N_\gamma \cap \omega_1 = M_\gamma \cap \omega_1$.

Fix $\gamma \in D$. Apply Proposition 5.54 (Existence of Nice Conditions) to find some v which is M_γ-nice for p, $\vec{b}$, and A such that $v \leq_\theta q \restriction \theta$. By (2) of Definition 5.51 (Nice Conditions), fix $r \leq q$ with top level δ_γ such that $r \restriction \theta = v$, $N \cap \kappa \subseteq \mathrm{dom}(r)$, r is separated, and r decides $\dot{b} \cap \delta_\gamma$ as some set b_r. By Lemma 5.52, v is also N_γ-nice. Hence, $F(v) = b_r$. Since $q \Vdash_{\mathbb{P}} \zeta \in \dot{b}$, $r \leq q$, and $\zeta < \delta_\gamma$, r forces that $\zeta \in \dot{b} \cap \delta_\gamma = b_r$. Hence, $\zeta \in F(v)$. Now r forces that $v \in \dot{G}_\theta$, so r forces that $\dot{c}_\gamma = F(v)$, and hence r forces that $\zeta \in \dot{c}_\gamma$. But p forces that $\dot{c}_\gamma$ is a subset of $\dot{c}$. So $r \leq q$ and $r \Vdash_{\mathbb{P}} \zeta \in \dot{c}$, which is a contradiction. □

Lemma 5.56. *Suppose that T is a free Suslin tree. Let N be suitable and let $\delta = N \cap \omega_1$. Suppose that $p_0, \ldots, p_{l-1}$ are in $N \cap \mathbb{P}$ and $v \in N \cap \mathbb{P}_\theta$, all of which have top level β. Assume that for all $k < l$, $p_k \upharpoonright \theta = v$ and p_k forces that $\dot{b}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. Let $X \subseteq T_\delta$ be finite with unique drop-downs to β, $A \subseteq \bigcap_{k<l} \mathrm{dom}(p_k)$ is finite, and suppose that for all $k < l$, $\{p_k(\tau) : \tau \in A\}$ is separated on $X \upharpoonright \beta$.*

Then there exist $\gamma < \delta$, $w \in N \cap \mathbb{P}_\theta$, and for all $k < l$ conditions $q_{k,0}, q_{k,1}$ in $N \cap \mathbb{P}$, all with top level γ, satisfying:

1. *for each $j < 2$, $q_{k,j} \leq p_k$ and $q_{k,j} \upharpoonright \theta = w$;*
2. *for each $j < 2$ and $\tau \in A$, $X \upharpoonright \beta$ and $X \upharpoonright \gamma$ are $q_{k,j}(\tau)$-consistent;*
3. *there exists some $\zeta < \gamma$ such that $q_{k,0} \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_{k,1} \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.*

Proof. The proof is by induction on l. Let N and δ be as above and let $X \subseteq T_\delta$ be finite.

Base case: Suppose that $p \in N \cap \mathbb{P}$ and $v \in N \cap \mathbb{P}_\theta$ have top level β, $p \upharpoonright \theta = v$, and p forces that $\dot{b}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. Assume that X has unique drop-downs to β, $A \subseteq \mathrm{dom}(p)$ is finite, and $\{p(\tau) : \tau \in A\}$ is separated on $X \upharpoonright \beta$. Fix an injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which enumerates X so that $\{p(\tau) : \tau \in A\}$ is separated on $\vec{a} \upharpoonright \beta$. Let $\vec{x} = \vec{a} \upharpoonright \beta$.

Define $\mathcal{X}$ as the set of all tuples $\vec{b} = (b_0, \ldots, b_{n-1})$ in the derived tree $T_{\vec{x}}$ for which there exist $q_0, q_1 \leq p$ with top level equal to the height ρ of $\vec{b}$ such that:

- $q_0 \upharpoonright \theta = q_1 \upharpoonright \theta$;
- for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;
- there exists some $\zeta < \rho$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.

By Proposition 5.55, $\mathcal{X}$ is dense open in $T_{\vec{x}}$, and $\mathcal{X} \in N$ by elementarity. Since T is a free Suslin tree, $T_{\vec{x}}$ is Suslin. So by elementarity we can find some $\gamma < \delta$ greater than β such that every member of $T_{\vec{x}}$ with height at least γ is in $\mathcal{X}$. In particular, $\vec{a} \upharpoonright \gamma \in \mathcal{X}$. Fix $q_0, q_1 \leq p$ which witness that $\vec{a} \upharpoonright \gamma \in \mathcal{X}$. Then γ, $q_0 \upharpoonright \theta$, q_0, and q_1 satisfy conclusions (1)-(3).

Inductive Step: Let $l > 0$ be given and assume that the statement is true for l. We will prove that it is true for $l + 1$. Suppose that $p_0, \ldots, p_l$ are in $N \cap \mathbb{P}$ and $v \in N \cap \mathbb{P}_\theta$, all of which have top level β. Assume that for all $k \leq l$, $p_k \upharpoonright \theta = v$ and p_k forces that $\dot{b}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. Suppose that X has unique drop-downs to β, $A \subseteq \bigcap_{k \leq l} \mathrm{dom}(p_k)$ is finite, and for all $k \leq l$, $\{p_k(\tau) : \tau \in A\}$ is separated on $X \upharpoonright \beta$.

By the inductive hypothesis, we can fix $\gamma < \delta$, $w \in N \cap \mathbb{P}_\theta$, and conditions $q_{k,0}, q_{k,1} \leq p_k$ in $N \cap \mathbb{P}$ for all $k < l$ satisfying conclusions (1)-(3). By Lemma 5.44 (Simple Generalized Extension), find $q \leq p_l$ with top level γ such that $q \upharpoonright \theta = w$ and for all $\tau \in A$, $X \upharpoonright \beta$ and $X \upharpoonright \gamma$ are $q(\tau)$-consistent.

Since $\{p_l(\tau) : \tau \in A\}$ is separated on $X \upharpoonright \beta$, $\{q(\tau) : \tau \in A\}$ is separated on $X \upharpoonright \gamma$ by Lemma 5.21 (Persistence). Fix an injective tuple $\vec{a} = (a_0, \ldots, a_{n-1})$ which enumerates X so that $\{q(\tau) : \tau \in A\}$ is separated on $\vec{a} \upharpoonright \gamma$. Let $\vec{x} = \vec{a} \upharpoonright \gamma$.

Define $\mathcal{X}$ as the set of all tuples $\vec{b} = (b_0, \ldots, b_{n-1})$ in the derived tree $T_{\vec{x}}$ for which there exist $q_0, q_1 \leq q$ with top level equal to the height ρ of $\vec{b}$ such that:

1. $q_0 \upharpoonright \theta = q_1 \upharpoonright \theta$;
2. for all $\tau \in A$ and $j < 2$, $\vec{x}$ and $\vec{b}$ are $q_j(\tau)$-consistent;
3. there exists some $\zeta < \rho$ such that $q_0 \Vdash_{\mathbb{P}} \zeta \in \dot{b}$ and $q_1 \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.

By Proposition 5.55, $\mathcal{X}$ is dense open in $T_{\vec{x}}$, and $\mathcal{X} \in N$ by elementarity. Since T is a free Suslin tree, $T_{\vec{x}}$ is Suslin. So by elementarity we can find some $\xi < \delta$ greater than γ such that every member of $T_{\vec{x}}$ with height at least ξ is in $\mathcal{X}$. In particular, $\vec{a} \restriction \xi \in \mathcal{X}$.

Fix $\bar{q}_{l,0}, \bar{q}_{l,1} \leq q$ which witness that $\vec{a} \restriction \xi \in \mathcal{X}$. Let $z = \bar{q}_{l,0} \restriction \theta$. Now apply Lemma 5.45 (Generalized Extension) in N to find, for each $k < l$ and $j < 2$, a condition $\bar{q}_{k,j} \leq q_{k,j}$ in N with top level ξ such that $\bar{q}_{k,j} \restriction \theta = z$ and for all $\tau \in A$, $X \restriction \gamma$ and $X \restriction \xi$ are $\bar{q}_{k,j}(\tau)$-consistent. Then ξ, z, and $\bar{q}_{k,j}$ for all $k < l$ and $j < 2$ are as required. □

Proof of Theorem 5.49 (No New Cofinal Branches). Suppose for a contradiction that there exists a condition $p \in \mathbb{P}$ which forces in $\mathbb{P}$ that $\dot{b}$ is a cofinal branch of $\dot{U}$ which is not in $V^{\mathbb{P}_\theta}$. We will find some $v \leq_\theta p \restriction \theta$ which forces in $\mathbb{P}_\theta$ that $\dot{U}$ has an uncountable level, which contradicts that $\dot{U}$ is a $\mathbb{P}_\theta$-name for an ω_1-tree. Let α be the top level of p.

Fix a suitable set N such that $p \in N$ and let $\delta = N \cap \omega_1$. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals cofinal in δ with $\gamma_0 = \alpha$, and fix an enumeration $\langle D_n : n < \omega \rangle$ of all dense open subsets of $\mathbb{P}$ which lie in N. Let $g : \omega \to 2 \times T_\delta \times (N \cap \kappa)$ be a surjection such that every element of the codomain has an infinite preimage.

We will define by induction the following objects in ω-many steps:

- a subset-increasing sequence $\langle X_n : n < \omega \rangle$ of finite subsets of T_δ with union equal to T_δ;
- a subset-increasing sequence $\langle A_n : n < \omega \rangle$ of finite subsets of $N \cap \kappa$ with union equal to $N \cap \kappa$;
- an increasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals cofinal in δ;
- a decreasing sequence $\langle v_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}_\theta$;
- a family of conditions $\{r^s : s \in {}^{<\omega}2\} \subseteq N \cap \mathbb{P}$ and ordinals $\{\zeta_s : s \in {}^{<\omega}2\} \subseteq \delta$;
- for all $n < \omega$, $s \in {}^n 2$, and $\tau \in A_n$, an injective partial function h^s_τ from X_n to X_n.

We will maintain the following inductive hypotheses for all $n < \omega$ and $s \in {}^n 2$:

(1) X_n has unique drop-downs to δ_n;
(2) δ_n is the top level of r^s;
(3) $r^s \leq p$, and if $m > n$, $t \in {}^m 2$, and $s \subseteq t$, then $r^t \leq r^s$;
(4) $r^s \restriction \theta = v_n$;
(5) $A_n \subseteq \operatorname{dom}(r^s)$;
(6) $\zeta_s < \delta_{n+1}$, $r^{s^\frown 0} \Vdash \zeta_s \in \dot{b}$, and $r^{s^\frown 1} \Vdash \zeta_s \notin \dot{b}$;
(7) for all $\tau \in A_n$ and $j < 2$, $X_n \restriction \delta_n$ and $X_n \restriction \delta_{n+1}$ are $r^{s^\frown j}(\tau)$-consistent;
(8) $r^{s^\frown 0}$ and $r^{s^\frown 1}$ are in D_n;
(9) $\{r^s(\tau) : \tau \in A_n\}$ is separated on $X_n \restriction \delta_n$;
(10) for all $\tau \in A_n$ and $x, y \in X_n$,
$$h^s_\tau(x) = y \iff r^s(\tau)(x \restriction \delta_n) = y \restriction \delta_n.$$

Stage 0: Let $X_0 = \emptyset$, $A_0 = \emptyset$, $\delta_0 = \alpha$, $v_0 = p \restriction \theta$, $r^\emptyset = p$.

Stage $n+1$: Let $n < \omega$ and assume that we have completed stage n. In particular, we have defined the following objects which satisfy the required properties: X_n, A_n, δ_n, v_n, r^s, ζ_s, and h^s_τ for all $s \in {}^n 2$ and $\tau \in A_n$. Let $g(n) = (n_0, z, \sigma)$.

Fix $\rho < \delta$ larger than δ_n and γ_{n+1} such that $X_n \cup \{z\}$ has unique drop-downs to ρ. Let D be the set of conditions r in D_n which have some top level $\xi \geq \rho$ such that $A_n \cup \{\sigma\} \subseteq \operatorname{dom}(r)$. Then $D \in N$ and D is dense open in $\mathbb{P}$.

Apply Lemma 5.46 (Generalized Consistent Extensions Into Dense Sets) to find a family $\{\bar{r}^s : s \in {}^n 2\}$ of conditions in $N \cap D$, a condition $\bar{v}_n \in N \cap \mathbb{P}_\theta$, and $\gamma < \delta$ so that for each $s \in {}^n 2$:

(a) $\bar{r}^s \leq r^s$, $\bar{r}^s$ has top level γ, and $\bar{r}^s \restriction \theta = \bar{v}_n$;
(b) for all $\tau \in A_n$, $X_n \restriction \delta_n$ and $X_n \restriction \gamma$ are $\bar{r}^s(\tau)$-consistent.

Apply Lemma 5.48 (Generalized Augmentation) to find a family $\{\hat{r}^s : s \in {}^n2\}$ of conditions in $N \cap \mathbb{P}$ and a condition $\hat{v}_n \in N \cap \mathbb{P}_\theta$, all with top level $\gamma + 1$, and a finite set $Y \subseteq T_\delta$ such that $X_n \cup \{z\} \subseteq Y$ and Y has unique drop-downs to $\gamma + 1$, satisfying that for all $s \in {}^n2$:

(c) $\hat{r}^s \leq \bar{r}^s$ and $\hat{r}^s \restriction \theta = \hat{v}_n$;
(d) for all $\tau \in A_n$, $X_n \restriction \gamma$ and $X_n \restriction (\gamma+1)$ are $\hat{r}^s(\tau)$-consistent;
(e) $\{\hat{r}^s(\tau) : \tau \in A_n \cup \{\sigma\}\}$ is separated on $Y \restriction (\gamma+1)$;
(f) let $h^+_{s,\sigma}$ be the partial injective function from Y to Y defined by letting, for all $x, y \in Y$, $h^+_{s,\sigma}(x) = y$ iff $\hat{r}^s(\sigma)(x \restriction (\gamma+1)) = y \restriction (\gamma+1)$; then z is in the domain and range of $h^+_{s,\sigma}$.

Define $X_{n+1} = Y$ and $A_{n+1} = A_n \cup \{\sigma\}$. So for all $s \in {}^n2$, $\{\hat{r}^s(\tau) : \tau \in A_{n+1}\}$ is separated on $X_{n+1} \restriction (\gamma+1)$.

Now apply Lemma 5.56 to find $\delta_{n+1} < \delta$, $v_{n+1} \in N \cap \mathbb{P}_\theta$, and for all $s \in {}^n2$, conditions $r^{s\frown 0}, r^{s\frown 1} \leq \hat{r}^s$ in $N \cap \mathbb{P}$ satisfying that for all $s \in {}^n2$:

(g) for each $j < 2$, $r^{s\frown j}$ has top level δ_{n+1} and $r^{s\frown j} \restriction \theta = v_{n+1}$;
(h) for each $j < 2$ and $\tau \in A_{n+1}$, $X_{n+1} \restriction (\gamma+1)$ and $X_{n+1} \restriction \delta_{n+1}$ are $r^{s\frown j}(\tau)$-consistent;
(i) there exists some $\zeta_s < \delta_{n+1}$ such that $r^{s\frown 0} \Vdash_{\mathbb{P}} \zeta_s \in \dot{b}$ and $r^{s\frown 1} \Vdash_{\mathbb{P}} \zeta \notin \dot{b}$.

For each $s \in {}^n2$, $j < 2$, and $\tau \in A_{n+1}$, define a partial injective function $h^{s\frown j}_\tau$ from X_{n+1} to X_{n+1} by letting, for all $x, y \in X_{n+1}$, $h^{s\frown j}_\tau(x) = y$ iff $r^{s\frown j}(\tau)(x \restriction \delta_{n+1}) = y \restriction \delta_{n+1}$. Observe that by (f) and (h), we have:

(j) for all $s \in {}^n2$ and $j < 2$, $h^+_{s,\sigma} = h^{s\frown j}_\sigma$, and hence z is in the domain and range of $h^{s\frown j}_\sigma$.

This completes stage $n+1$. It is easy to check that the inductive hypotheses are satisfied.

This completes the construction. For each $f \in {}^\omega 2$, define a condition r_f with domain equal to $N \cap \kappa$ as follows. For any $\tau \in N \cap \kappa$, define

$$r_f(\tau) = \bigcup\{r^{f\restriction n}(\tau) : n < \omega,\ \tau \in A_n\} \cup \bigcup\{h^{f\restriction n}_\tau : n < \omega,\ n \in A_n\}.$$

By Lemma 5.33 (Constructing Total Master Conditions), each r_f is a total master condition for $\mathbb{P}$ over N. Define $v = r_f \restriction \theta$ for some (any) $f \in {}^\omega\omega$.

For each $f \in {}^\omega 2$, let b_f be such that $r_f \Vdash_{\mathbb{P}} \dot{b} \cap \delta = b_f$. Due to our assumption about the levels of $\dot{U}$, it is easy to argue that r_f forces that b_f is a cofinal branch of $\dot{U} \restriction \delta$. Suppose that $f \neq g$. Let n be least such that $f(n) \neq g(n)$, and assume without loss of generality that $f(n) = 0$ and $g(n) = 1$. Let $s = f \restriction n = g \restriction n$. Then $r_f \leq r^{s\frown 0}$ and $r_g \leq r^{s\frown 1}$. So $r_f \Vdash_{\mathbb{P}} \zeta_s \in \dot{b} \restriction \delta$ and $r_g \Vdash_{\mathbb{P}} \zeta_s \notin \dot{b} \restriction \delta$. Hence, $b_f \neq b_g$.

Now we are ready to get a contradiction. Let G_θ be a generic filter for $\mathbb{P}_\theta$ such that $v \in G_\theta$. In $V[G_\theta]$, let $\mathbb{P}/G_\theta$ be the suborder of $\mathbb{P}$ consisting of all $q \in \mathbb{P}$ such that $q \restriction \theta \in G_\theta$. Let $U = \dot{U}^{G_\theta}$. In $V[G_\theta]$, each b_f is a cofinal branch of $U \restriction \delta$, and r_f forces in $\mathbb{P}/G_\theta$ that $b_f = \dot{b} \cap \delta$. Hence r_f forces that $\dot{b}(\delta)$ is an upper bound of b_f. Since having an upper bound in U is absolute between $V[G_\theta]$ and any generic extension by $\mathbb{P}/G_\theta$, b_f does in fact have an upper bound in U_δ, which we will denote by x_f. By construction, if $f \neq g$ then $b_f \neq b_g$, so $x_f \neq x_g$. Hence, U_δ is uncountable, which contradicts that U is an ω_1-tree in $V[G_\theta]$. □

6. The Main Result

We are now prepared to prove the main result of the article.

Theorem 6.1 (Main Theorem)**.** *Suppose that there exists an inaccessible cardinal κ and an infinitely splitting normal free Suslin tree T. Then there exists a forcing poset $\mathbb{P}$ satisfying that the product forcing* $\mathrm{Col}(\omega_1, {<}\kappa) \times \mathbb{P}$ *forces:*

(1) $\kappa = \omega_2$;
(2) *GCH holds;*
(3) T *is a Suslin tree;*
(4) *there exists an almost disjoint family* $\{f_\tau : \tau < \omega_2\}$ *of automorphisms of* T*;*
(5) *there does not exist a Kurepa tree.*

Proof. Let V be the ground model in which T and κ are as above. For simplicity in notation, let $\mathbb{Q} = \mathrm{Col}(\omega_1, <\kappa)$. Note that $\mathbb{Q} \times \mathbb{P}$ has size κ. Let $\mathbb{P}$ be the forcing poset of Definition 5.17 in V. Since ω_1-closed forcings preserve Suslin trees, T is still free in $V^{\mathbb{Q}}$. Also, the definition of $\mathbb{P}$ is easily seen to be absolute between V and $V^{\mathbb{Q}}$. So $\mathbb{Q} \times \mathbb{P}$ is forcing equivalent to the two-step forcing iteration of $\mathbb{Q}$ followed by the forcing of Definition 5.17 (with $\kappa = \omega_2$). Since CH holds in $V^{\mathbb{Q}}$, in $V^{\mathbb{Q}}$ we have that $\mathbb{P}$ is ω_2-c.c. Consequently, $\mathbb{Q} \times \mathbb{P}$ is κ-c.c. As $\mathbb{Q}$ is ω_1-closed and hence totally proper, and $\mathbb{Q}$ forces that $\mathbb{P}$ is totally proper and preserves the fact that T is Suslin, $\mathbb{Q} \times \mathbb{P}$ is totally proper and forces that T is Suslin. Statements (1)-(4) are now clear.

For (5), let $\dot{U}$ be a nice $(\mathbb{Q} \times \mathbb{P})$-name for an ω_1-tree (with underlying set ω_1). By the κ-c.c. property and the fact that conditions in $\mathbb{Q} \times \mathbb{P}$ have countable domain, there exists some $\theta < \kappa$ such that $\dot{U}$ is a $(\mathrm{Col}(\omega_1, <\theta) \times \mathbb{P}_\theta)$-name. Let $G \times H$ be a generic filter on $\mathbb{Q} \times \mathbb{P}$. By the usual factor analysis for product forcings, we can write $V[G] = V[G_\theta][G^\theta][H_\theta][H]$, where $G_\theta = G \cap \mathrm{Col}(\omega_1, <\theta)$, $G^\theta = G \cap \mathrm{Col}(\omega_1, [\theta, \kappa))$, $H_\theta = H \cap \mathbb{P}_\theta$, and we consider H to be a $V[G][H_\theta]$-generic filter on the quotient forcing $\mathbb{P}/H_\theta$.

Let $U = \dot{U}^{G_\theta \times H_\theta}$. Then U is in $V[G_\theta][H_\theta]$, and hence U is in $V[G][H_\theta]$. Consider a cofinal branch b of U in $V[G][H]$. Applying Theorem 5.49 in $V[G]$, b is in the model $V[G][H_\theta]$, which by the product lemma is equal to the model $V[G_\theta][H_\theta][G^\theta]$. Since $\mathrm{Col}(\omega_1, [\theta, \kappa))$ is ω_1-closed in $V[G_\theta][H_\theta]$ and ω_1-closed forcings do not add new cofinal branches of ω_1-trees, b is in $V[G_\theta][H_\theta]$. As κ is inaccessible and $\theta < \kappa$, κ is also inaccessible in $V[G_\theta][H_\theta]$. Because every cofinal branch of U in $V[G][H]$ is in $V[G_\theta][H_\theta]$, there are fewer than κ many cofinal branches of U in $V[G][H]$. So U is not a Kurepa tree in $V[G][H]$. □

It remains to verify the following claim from Section 1.

Proposition 6.2. *Let* κ *be an inaccessible cardinal, let* $\mathbb{Q}$ *be Jech's forcing for adding a Suslin tree, and let* $\dot{\mathbb{P}}$ *be a* $\mathbb{Q}$*-name for the forcing of Definition 5.17 using the generic Suslin tree. Then* $\mathbb{Q} * \dot{\mathbb{P}}$ *is forcing equivalent to some* ω_1*-closed forcing.*

Proof. In V define a forcing poset $\mathbb{A}$ as follows. A condition in $\mathbb{A}$ is a pair (t, f) satisfying:

- $t \in \mathbb{Q}$;
- f is a function whose domain is a countable subset of κ and whose range is a set of automorphisms of t.

Let $(u, g) \leq (t, f)$ if $u \leq_{\mathbb{Q}} t$, $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, and for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \subseteq g(\alpha)$. It is easy to check that $\mathbb{A}$ is ω_1-closed. (The forcing $\mathbb{A}$ is similar to [Jec72, Theorem 3], but we are not requiring that f be injective nor that the range of f be a group).

Let $\mathbb{A}'$ be the suborder of $\mathbb{A}$ consisting of all conditions (t, f) such that t has successor height. We claim that $\mathbb{A}'$ is dense in $\mathbb{A}$. So let $(t, f) \in \mathbb{A}$ be such that t has height a limit ordinal δ. Enumerate t as $\langle x_n : n < \omega \rangle$ and $\mathrm{dom}(f)$ as $\langle \gamma_n : n < \omega \rangle$. For each $n < \omega$ fix a cofinal branch b_n of t with $x_n \in b_n$. For purposes of bookkeeping, fix a surjection

$$h : \omega \to 2 \times \{-1, 1\} \times \omega \times \omega$$

such that: for all $n < \omega$, letting $h(n) = (i, m, k, l)$, if $n = 0$ then $i = 0$, and if $n > 0$ then $k < n$.

We build a sequence $\langle c_n : n < \omega \rangle$ of cofinal branches of t in ω-many stages as follows. Assuming that $n < \omega$ and we have completed stages before n, let $h(n) = (i, m, k, l)$. If $i = 0$, let $c_n =$

$f(\gamma_l)^m[b_k]$. If $i = 1$, let $c_n = f(\gamma_l)^m[c_k]$. This completes the construction. Now let $u = t \cup \{b_n : n < \omega\} \cup \{c_n : n < \omega\}$. Define g with domain equal to $\mathrm{dom}(f)$ so that for all $\tau \in \mathrm{dom}(f)$, $g(\tau) \restriction t = f(\tau)$ and for all b in $u \setminus t$, $g(\tau)(b) = f(\tau)[b]$. By our bookkeeping, it is straightforward to check that $(u, g) \in \mathbb{A}'$ and $(u, g) \leq (t, f)$.

Now the map from $\mathbb{A}'$ to $\mathbb{Q} * \dot{\mathbb{P}}$ given by $(t, f) \mapsto (t, \check{f})$ is clearly an embedding. To see that the range is dense, consider $(t, \dot{f}) \in \mathbb{Q} * \dot{\mathbb{P}}$. Extend t to u in $\mathbb{Q}$ which decides $\dot{f}$ as f and has successor height greater than the height of t. Now applying Proposition 5.15 (replacing $T \restriction (\alpha + 1)$ with u, letting $X = \emptyset$, and ignoring (3)), there exists some g such that $\mathrm{dom}(f) = \mathrm{dom}(g)$ and for all $\tau \in \mathrm{dom}(f)$, $g(\tau)$ is an automorphism of u such that $f(\tau) \subseteq g(\tau)$. Then $(u, g) \in \mathbb{A}'$ and $(u, \check{g}) \leq (t, \dot{f})$ in $\mathbb{Q} * \dot{\mathbb{P}}$. □

By the arguments given in Section 1, we have the following consequences of Theorem 6.1 (Main Theorem).

Corollary 6.3 (Almost Kurepa Suslin Tree $+ \neg$KH)**.** *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, CH holds, there exists an almost Kurepa Suslin tree, and there does not exist a Kurepa tree.*

Corollary 6.4 (T is Suslin $+ \sigma(T) = \omega_2 + \diamondsuit + \neg$KH)**.** *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, $\diamondsuit$ holds, there exists a normal Suslin tree with ω_2-many automorphisms, and there does not exist a Kurepa tree.*

Corollary 6.5 (Non-Saturated Aronszajn Tree $+ \neg$KH)**.** *Assume that there exists an inaccessible cardinal κ. Then there exists a generic extension in which κ equals ω_2, there exists a non-saturated Aronszajn tree, and there does not exist a Kurepa tree.*

Acknowledgements

Some of the work for this article was completed while the first author was visiting the second author at the Department of Logic at Charles University during August and October of 2023. The first author thanks the second author and Radek Honzík for their hospitality during his visits. The first author thanks Justin Moore for pointing out that an almost Kurepa Suslin tree implies the existence of a non-saturated Aronszajn tree. The second author thanks Assaf Rinot for bringing Problem 3 to her attention while she was a postdoc at Bar-Ilan University in 2022. The first author acknowledges support from the Simons Foundation under Grant 631279.

References

[AS81] U. Abraham and S. Shelah, *Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel J. Math. **38** (1981), no. 1-2, 161–176.

[AS85] ———, *Isomorphism types of Aronszajn trees*, Israel J. Math. **50** (1985), no. 1-2, 75–113.

[AS93] ———, *A Δ^2_2-well order of the reals and incompactness of $L(Q^{MM})$*, Ann. Pure Appl. Logic **59** (1993), no. 1, 1–32.

[Bau70] J. E. Baumgartner, *Results and independence proofs in combinatorial set theory*, Ph.D. thesis, University of California, Berkeley, 1970.

[Bau85] ———, *Bases for Aronszajn trees*, Tsukuba J. Math. **9** (1985), no. 1, 31–40.

[BMR70] J. E. Baumgartner, J. Malitz, and W. Reinhardt, *Embedding trees in the rationals*, Proc. Nat. Acad. Sci. **67** (1970), no. 4, 1748–1753.

[Bil89] S. Bilaniuk, *Some results on Souslin trees*, Ph.D. thesis, Dartmouth College, 1989.

[Buk66] L. Bukovský, *Consistency theorems connected with some combinatorial problems*, Comment. Math. Univ. Carolin. **7** (1966), 495–499.

[Coh66] P. Cohen, *Set theory and the continuum hypothesis*, W. A. Benjamin, Inc., New York-Amsterdam, 1966.

[DJ74] K. Devlin and H. Johnsbråten, *The Souslin problem*, Lecture Notes in Mathematics, vol. 405, Springer-Verlag, Berlin-New York, 1974.

[FH09] G. Fuchs and J. Hamkins, *Degrees of rigidity for Souslin trees*, J. Symbolic Logic **74** (2009), no. 2, 423–454.
[Göd40] K. Gödel, *The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory*, Annals of Mathematics Studies, vol. 3, Princeton University Press, Princeton, N.J., 1940.
[Jec67] T. Jech, *Nonprovability of Souslin's hypothesis*, Comment. Math. Univ. Carolinae **8** (1967), 291–305.
[Jec72] ———, *Automorphisms of ω_1-trees*, Trans. Amer. Math. Soc. **173** (1972), 57–70.
[Jena] R. Jensen, $\square$ *implies GKH*, handwritten notes.
[Jenb] ———, *The generic Kurepa hypothesis I + II*, handwritten notes.
[JS90] R. Jensen and K. Schlechta, *Results on the generic Kurepa hypothesis*, Arch. Math. Logic **30** (1990), 13–27.
[JS97] R. Jin and S. Shelah, *Can a small forcing create Kurepa trees*, Ann. Pure Appl. Logic **85** (1997), no. 1, 47–68.
[KLMV08] B. König, P. Larson, J. T. Moore, and B. Veličković, *Bounding the consistency strength of a five element linear basis*, Israel J. Math. **164** (2008), 1–18.
[Kru] J. Krueger, *Suslin tree preservation and club isomorphisms*, J. Symbolic Logic, to appear.
[Kru20] ———, *Entangledness in Suslin lines and trees*, Topology Appl. **275** (2020).
[Kur35] Đ Kurepa, *Ensembles ordonnés et ramijiés*, Publ. Math. Univ. Belgrade **4** (1935), 1–138.
[Kur36] ———, *L'hypothese de ramification*, Comptes Rendus Hebdomadaires des Seances de l'Academie des Sciences, A et B, vol. 202, 1936, pp. 185–187.
[Kur38] ———, *Ensembles linéaires et une classe de tableaux ramifiés (Tableaux ramifiés de M. Aronszajn*, Publ. Math. Univ. Belgrade **6/7** (1937-1938), 129–160.
[Kur42] ———, *A propos d'une généralisation de la notion d'ensembles bien ordonnés*, Acta Math. **75** (1942), 139–150.
[Lar99] P. Larson, *An $\mathbb{S}_{max}$ variation for one Souslin tree*, J. Symbolic Logic **64** (1999), no. 1, 81–98.
[LT02] P. Larson and S. Todorčević, *Katětov's problem*, Trans. Amer. Math. Soc. **354** (2002), 1783–1791.
[MS70] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178.
[Mil43] E. Miller, *A note on Souslin's problem*, Amer. J. Math **65** (1943), no. 4, 673–678.
[Moo06] J. T. Moore, *A five element basis for the uncountable linear orders*, Ann. of Math. (2) **163** (2006), no. 2, 669–688.
[Moo08] ———, *Structural analysis of Aronszajn trees*, Lect. Notes Log., vol. 28, Assoc. Symbol. Logic, Urbana, IL, 2008, pp. 85–106.
[Row64] F. Rowbottom, *Large cardinals and small constructible sets*, Ph.D. thesis, University of Wisconsin, Madison, 1964.
[SF10] G. Scharfenberger-Fabian, *Optimal matrices of partitions and an application to Souslin trees*, Fund. Math. **210** (2010), no. 2, 111–131.
[She82] S. Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982.
[She84] ———, *Can you take Solovay's inaccessible away?*, Israel J. Math. **48** (1984), no. 1, 1–47.
[She98] ———, *Proper and improper forcing*, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.
[SZ99] S. Shelah and J. Zapletal, *Canonical models for $\aleph_1$-combinatorics*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 217–259.
[Sie48] W. Sierpiński, *Sur un probleme de la théorie générale des ensembles equivalent un problème de Souslin*, Fund. Math. **35** (1948), 165–174.
[Sil71] J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Axiomatic Set Theory (D. S. Scott, ed.), Proc. Sympos. Pure Math., vol. XIII, Part 1, Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390.
[ST71] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. (2) **94** (1971), 201–245.
[Ste66] D. H. Stewart, *The consistency of the Kurepa hypothesis with the axioms of set theory*, Master's thesis, Bristol, 1966.
[Sus20] M. Suslin, *Probléme 3*, Fund. Math. **1** (1920), 223.
[Ten68] S. Tennenbaum, *Souslin's problem*, Proc. Nat. Acad. Sci. USA **59** (1968), 60–63.
[Tod] S. Todorčević, *Forcing with a coherent Souslin tree*, preprint.
[Tod87] ———, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), no. 3-4, 261–294.
[Vel92] B. Veličković, *Forcing axioms and stationary sets*, Adv. Math. **94** (1992), no. 2, 256–284.
[Woo99] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, De Gruyter Series in Logic and its Applications, Walter de Gruyter and Co., Berlin, 1999.

JOHN KRUEGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE #311430, DENTON, TX 76203, USA
Email address: jkrueger@unt.edu

ŠÁRKA STEJSKALOVÁ, DEPARTMENT OF LOGIC, CHARLES UNIVERSITY, CELETNÁ 20, PRAGUE 1, 116 42, CZECH REPUBLIC

Email address: sarka.stejskalova@ff.cuni.cz