

EASTON'S THEOREM FOR THE TREE PROPERTY BELOW \aleph_ω

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Abstract. Starting with infinitely many supercompact cardinals, we show that the tree property at every cardinal \aleph_n , $1 < n < \omega$, is consistent with an arbitrary continuum function below \aleph_ω which satisfies $2^{\aleph_n} > \aleph_{n+1}$, $n < \omega$. Thus the tree property has no provable effect on the continuum function below \aleph_ω except for the restriction that the tree property at κ^{++} implies $2^\kappa > \kappa^+$ for every infinite κ .

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1 INTRODUCTION

Recall that the continuum function is the function which maps an infinite cardinal κ to 2^κ . It is well known that at regular cardinals the continuum function is very easily changed by forcing, as was shown by Easton [8]. The case of singular cardinals, or regular limit cardinals whose “largeness” we wish to preserve, is more difficult and gave rise to several results which generalize Easton’s theorem in this direction (see for instance [19], [10], [3] or [4]).

In this paper we study yet another generalization of Easton’s theorem in which we require that some successor cardinals should retain their largeness in terms of a certain compactness property. If λ is a regular uncountable cardinal, we say that λ has *the tree property*, and we denote it by $\text{TP}(\lambda)$, if all λ -trees have a cofinal branch. It is known that if the tree property holds at κ^{++} , then $2^\kappa > \kappa^+$. In other words the tree property has a non-trivial effect on the continuum function. It seems natural to ask whether the tree property at κ^{++} puts more restrictions on the continuum function in addition to $2^\kappa > \kappa^+$ (and the usual restrictions which the continuum function needs to satisfy); or equivalently, which continuum functions are compatible with the tree property. Since it is still open how to get the tree property at a long interval of cardinals (for more information see [21]), any Easton’s theorem for the tree property is at the moment limited to countable intervals of cardinals.

As should be expected, the difficulty of this question increases if we wish to have (A) the tree property at consecutive cardinals or (B) at cardinals which are the successors or double successors of singular cardinals. We deal with the type (A) in this paper.

The first partial answer to (A) was given by Unger ([22]) who showed that the tree property at \aleph_2 is consistent with 2^{\aleph_0} arbitrarily large.¹ We generalized this result in [14] for all cardinals below \aleph_ω for the weak tree property (no special Aronszajn trees) and for all even cardinals \aleph_{2n} for the full tree property. The argument used infinitely many weakly compact cardinals which is optimal for the result. In [14], we left open the natural question whether having the tree property at every \aleph_n for $2 \leq n < \omega$ is consistent with any continuum function which violates GCH below \aleph_ω . Unlike the argument in [14], this requires much larger cardinals because it is known that consecutive cardinals with the tree property imply the consistency of at least a Woodin cardinal (see [9]). In this paper we provide the affirmative answer to this question, i.e. we show that if there are infinitely many supercompact cardinals, then it is consistent that the tree property holds at every \aleph_n for $2 \leq n < \omega$, and the continuum function below \aleph_ω is anything not outright inconsistent with the tree property.²

The argument is based on the construction in the paper by Cummings and Foreman [6], extended to obtain the right continuum function. We outline the argument in Section 1.1.

Although it is not the focus of this paper, let us say a few words about the type

¹The result can be easily generalized to an arbitrary regular cardinal κ with the tree property at κ^{++} .

²There is nothing specific about the \aleph_n ’s; the final consecutive sequence $\langle \kappa_n \mid n < \omega \rangle$ of regular cardinals with the tree property can live much higher.

(B). We showed in [13] that the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality does not put any restrictions on the value of 2^κ apart from the trivial ones.³ In [12] we followed up with the result that 2^{\aleph_ω} can be equal to $\aleph_{\omega+2+n}$ for any $n < \omega$ with the tree property holding at $\aleph_{\omega+2}$.

1.1 AN OUTLINE OF THE ARGUMENT

Let us briefly outline the structure of the argument for a reader roughly familiar with the papers of Abraham [1] and Cummings and Foreman [6]. Let κ_n , $1 < n < \omega$, be an increasing sequence of supercompact cardinals with $\kappa_0 = \aleph_0$ and $\kappa_1 = \aleph_1$. For forcing the tree property at $\kappa = \kappa_{n+2}$ for $n \geq 1$ over some model V_{n-1} , we are going to use a variant of the Mitchell forcing as it was defined in [6]; this forcing contains the Cohen forcing at κ_n . If we define this Cohen forcing in V_{n-1} , V_{n-1} must satisfy $\kappa_n^{<\kappa_n} = \kappa_n$ otherwise some cardinals above κ_n will be unintentionally collapsed. κ_n is either ω_1 or an inaccessible cardinal in the ground model V , but in either case it will be a successor cardinal in V_{n-1} , in fact it will be the successor of κ_{n-1} (more to the point, it will be the \aleph_n of V_{n-1}). It follows that for forcing the tree property at κ over V_{n-1} , the Cohen forcing at κ_n must come from a model where $2^{\kappa_{n-1}} \leq \kappa_n$. Since by the inductive construction for the tree property we will necessarily have $2^{\kappa_{n-1}} > \kappa_n$ in V_{n-1} , the Cohen forcing cannot come from V_{n-1} , but should come from some earlier model.⁴ Cummings and Foreman solved this problem by postulating the the Cohen forcing at κ_n comes from the model V_{n-2} , which works provided that $2^{\kappa_{n-1}} = \kappa_n$ in V_{n-2} . Unless we manipulate the continuum function further, this will leave us with gap 2 below \aleph_ω : $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$.

In order to realize an arbitrary Easton function below \aleph_ω (which satisfies $2^{\aleph_n} \geq \aleph_{n+2}$ for all $n < \omega$) we need to modify the construction of Cummings and Foreman in some way. There seem to be essentially two options: (i) modify the construction in Cummings and Foreman directly and add the required number of subsets of κ_n by a Cohen forcing which lives in V_{n-2} , or in some earlier model, perhaps even the ground model V , or (ii) leave the inductive construction for the tree property as it is in Cummings and Foreman (which gives gap 2 for the continuum function) and increase the powersets as required in the next step.

The option (i) may seem cleaner at the first sight, but it causes technical complications⁵ because both tasks – ensuring the tree property and the right continuum function – are mixed into a single iteration. The option (ii) deals with the two tasks

³An easier proof of this theorem can be found in [15]; the proof is based on an application of the indestructibility of the tree property under certain κ^+ -cc forcing notions. The advantage of the new proof is that it can be directly generalized to singular cardinals with an uncountable cofinality (it does not use any of the properties of the Prikry-type forcing notions except the chain condition).

⁴We should add that this implies that the Cohen forcing will no longer be κ_n -closed in V_{n-1} so an additional argument must be provided for not collapsing below κ_n .

⁵Roughly speaking, it is hard to argue for the distributivity of the tail of the Mitchell iteration (i.e. a tail of \mathbb{R}_ω in (1.1)). In option (ii), the distributivity is ensured by closure in a suitable submodel (essentially an application of Easton's lemma).

separately, but one needs to make sure that forcing the right continuum function does not “undo” the tree property part.

We have opted for the option (ii) and defined a certain forcing \mathbb{Z} so that

$$(1.1) \quad \mathbb{Z} = \mathbb{R}_\omega * \dot{\mathbb{E}},$$

where \mathbb{R}_ω is exactly the forcing from Cummings and Foreman paper and $\dot{\mathbb{E}}$ is a full-support product of Cohen forcings to obtain the desired continuum function. The Cohen forcings in $\dot{\mathbb{E}}$ are chosen from appropriate inner models of the extension $V[\mathbb{R}_\omega]$ in order to satisfy the restrictions described in previous paragraphs (more precisely, the Cohen forcing at some κ_n in $\dot{\mathbb{E}}$ comes from the same inner model as the Cohen at κ_n which is the part of the Mitchell forcing in the iteration \mathbb{R}_ω).

The present paper is structured as follows. In Section 2 we provide some background information to make the paper self-contained. First we review some basic forcing properties which deal with the interactions of the chain condition and the closure between different models (Section 2.1), then we discuss forcing conditions for not adding cofinal branches to certain trees (Section 2.2), and finally we review the Mitchell forcing and the argument of Cummings and Foreman from [6].

In Section 3 we prove our theorem. The argument is divided into two sections: In Section 3.2 we show that the forcing \mathbb{Z} collapses only the intended cardinals and moreover forces the right continuum function. In Section 3.3 we show that \mathbb{Z} forces the tree property at every \aleph_n , $2 \leq n < \omega$, which finishes the argument.

In the final section we discuss open questions and further research.

2 PRELIMINARIES

2.1 SOME BASIC PROPERTIES OF FORCING NOTIONS

In this section we review some basic properties which we will use later in the paper.

Definition 2.1 *Let P be a forcing notion and let $\kappa > \aleph_0$ be a regular cardinal. We say that P is:*

- κ -cc if every antichain of P has size less than κ (we say that P is ccc if it is \aleph_1 -cc).
- κ -Knaster if for every $X \subseteq P$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of Y are pairwise compatible.
- κ -closed if every decreasing sequence of conditions in P of size less than κ has a lower bound.
- κ -distributive if P does not add new sequences of ordinals of length less than κ .

It is easy to check that all these properties – except for the κ -closure – are invariant under forcing equivalence⁶. Regarding the closure, note that for every non-trivial forcing notion P which is κ -closed there exists a forcing-equivalent forcing notion which is not even \aleph_1 -closed (e.g. the Boolean completion of P).

Lemma 2.2 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P is a forcing notion and \dot{Q} is a P -name for a forcing notion. Then the following hold:*

- (i) *P is κ -closed and P forces \dot{Q} is κ -closed if and only if $P * \dot{Q}$ is κ -closed.*
- (ii) *P is κ -distributive and P forces \dot{Q} is κ -distributive if and only if $P * \dot{Q}$ is κ -distributive.*
- (iii) *P is κ -cc and P forces \dot{Q} is κ -cc if and only if $P * \dot{Q}$ is κ -cc.*
- (iv) *If P is κ -Knaster and P forces \dot{Q} κ -Knaster then $P * \dot{Q}$ is κ -Knaster*

PROOF. The proofs are routine; for more details see [16] or [18]. □

If Q is in the ground model, $P * \check{Q}$ is equivalent to $P \times Q$. We state some properties which the product forcing has with respect to the chain condition.

Lemma 2.3 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions. Then the following hold:*

- (i) *If P and Q are κ -Knaster, then $P \times Q$ is κ -Knaster.*
- (ii) *If P is κ -Knaster and Q is κ -cc, then $P \times Q$ is κ -cc.*

PROOF. The proofs are routine. □

The following lemma summarises some of the more important forcing properties of a product $P \times Q$ regarding the chain condition.

Lemma 2.4 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions such that P is κ -Knaster and Q is κ -cc. Then the following hold:*

- (i) *P forces that Q is κ -cc.*
- (ii) *Q forces that P is κ -Knaster.*

PROOF. (i). This is an easy consequence of Lemmas 2.2(iii) and 2.3(ii).

(ii). A proof (attributed to Magidor) can be found in [5]. □

The following lemma summarises some of the more important properties of the product $P \times Q$ regarding the distributivity and closure.

Lemma 2.5 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ -closed and Q is κ -distributive. Then the following hold:*

- (i) *P forces that Q is κ -distributive.*

⁶We say that (P, \leq_P) and (Q, \leq_Q) are forcing equivalent if their Boolean completions are isomorphic.

(ii) Q forces that P is κ -closed.

PROOF. The proof is routine. □

We can also formulate some results for the product of two forcing notions with respect to preservation of the chain condition and distributivity at the same time. The following lemma appeared in [8].

Lemma 2.6 (Easton) *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ -cc and Q is κ -closed. Then the following hold:*

(i) P forces that Q is κ -distributive.

(ii) Q forces that P is κ -cc.

PROOF. For the proof of (i), see [16, Lemma 15.19], (ii) is easy. □

2.2 TREES AND FORCING

An essential step in standard arguments that a certain partial order forces the tree property is to argue that its quotient does not add cofinal branches to certain trees. Fact 2.7 is due to Baumgartner (see [2]) and Fact 2.8 is due to Silver (see [1] for more details; a proof with $\lambda = \aleph_0$ is in [18, Chapter VIII, Section 3]) .

Fact 2.7 *Let κ be a regular cardinal and assume that P is a κ -Knaster forcing notion. If T is a tree of height κ , then forcing with P does not add cofinal branches to T .*

Fact 2.8 *Let κ, λ be regular cardinals and $2^\kappa \geq \lambda$. Assume that P is a κ^+ -closed forcing notion. If T is a λ -tree, then forcing with P does not add cofinal branches to T .*

These facts can be generalized as follows (for the first fact see [23]; the first statement of the second fact appeared in [17] with $\kappa = \aleph_0$ and $\lambda = \aleph_1$, the general version is due to Unger in [22]).

Fact 2.9 *Let κ be a regular cardinal and assume that P is a forcing notion such that square of P , $P \times P$, is κ -cc. If T is a tree of height κ , then forcing with P does not add cofinal branches to T .*

Fact 2.10 *Let $\kappa < \lambda$ be regular cardinals and $2^\kappa \geq \lambda$. Assume that P and Q are forcing notions such that P is κ^+ -cc and Q is κ^+ -closed. If T is a λ -tree in $V[P]$, then forcing with Q over $V[P]$ does not add cofinal branches to T .*

2.3 MITCHELL FORCING

Mitchell forcing was defined by Mitchell in [20]. In this section we review several variants of the Mitchell forcing, which can be found in papers [1] and [6]. All proofs of facts stated below can be found in these papers as well. If κ is a regular cardinal and α a limit ordinal, let $\text{Add}(\kappa, \alpha)$ be the set of all partial functions of size $< \kappa$ from $\text{SuccOrd}(\alpha)$ to 2, ordered by reverse inclusion, where $\text{SuccOrd}(\alpha)$ is the set of all successor ordinals below α .⁷ It is easy to see that this forcing is isomorphic to the usual Cohen forcing for adding α -many subsets of κ . It follows that if $\beta < \alpha$ and $p \in \text{Add}(\kappa, \alpha)$, then $p \restriction \beta$ is in $\text{Add}(\kappa, \beta)$.

Definition 2.11 *Let κ be a regular cardinal and $\lambda > \kappa$ an inaccessible cardinal. The Mitchell forcing at κ of length λ , denoted by $\mathbb{M}(\kappa, \lambda)$, is the set of all pairs (p, q) such that p is in Cohen forcing $\text{Add}(\kappa, \lambda)$ and q is a function with $\text{dom}(q) \subseteq \lambda$ of size at most κ and for every $\alpha \in \text{dom}(q)$, α is a successor cardinal and it holds:*

$$(2.2) \quad 1_{\text{Add}(\kappa, \alpha)} \Vdash q(\alpha) \in \dot{\text{Add}}(\kappa^+, 1),$$

where $\dot{\text{Add}}(\kappa^+, 1)$ is the canonical $\text{Add}(\kappa, \alpha)$ -name for Cohen forcing at κ^+ . A condition (p, q) is stronger than (p', q') if

- (i) $p \leq p'$,
- (ii) $\text{dom}(q) \supseteq \text{dom}(q')$ and for every $\alpha \in \text{dom}(q')$, $p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

Assuming that $\kappa < \lambda$, κ is regular, and λ is inaccessible, Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ is λ -Knaster and κ -closed. Moreover if $\kappa^{<\kappa} = \kappa$, $\mathbb{M}(\kappa, \lambda)$ preserves κ^+ (by a product analysis of Abraham [1]), collapses cardinals exactly in the open interval (κ^+, λ) and forces $2^\kappa = \lambda = \kappa^{++}$.

Theorem 2.12 (Mitchell) *Assume $\kappa^{<\kappa} = \kappa$. If λ is a weakly compact cardinal, then $\mathbb{M}(\kappa, \lambda)$ forces the tree property at $\lambda = \kappa^{++}$.*

We modify the definition of Mitchell forcing in two steps. In the first step we define the variation of Mitchell forcing where the Cohen part of Mitchell forcing is taken from some suitable inner model of our universe. In the second step we add a third coordinate which will prepare the universe for a further lifting of an appropriate embedding.

Definition 2.13 *Let $V \subseteq W$ be two inner models of ZFC with the same ordinals, κ be a regular cardinal and $\lambda > \kappa$ inaccessible in W . Suppose that $\text{Add}(\kappa, \lambda)^V$ is in W κ^+ -cc and κ -distributive. In W , the Mitchell forcing at κ of length λ , denoted by $\mathbb{M}(\kappa, \lambda, V, W)$, is the set of all pairs (p, q) , where p is a condition in $\text{Add}(\kappa, \lambda)^V$ and q is a function in W such that $\text{dom}(q)$ is a subset of open interval (κ, λ) of size at most κ and for every $\alpha \in \text{dom}(q)$, α is a successor cardinal and the following holds:*

$$(2.3) \quad 1_{\text{Add}(\kappa, \alpha)^V} \Vdash^W q(\alpha) \in \dot{\text{Add}}(\kappa^+, 1)^W,$$

⁷This is just a technical assumption which will be useful in analysis of Mitchell forcing. See paragraph below Remark 2.16.

where $\text{Add}(\kappa^+, 1)^W$ is $\text{Add}(\kappa, \alpha)^V$ -name for the Cohen forcing at κ^+ over the model W . The ordering is defined by $(p, q) \leq (p', q')$ if

- (i) $p \leq p'$,
- (ii) $\text{dom}(q) \supseteq \text{dom}(q')$ and for every $\alpha \in \text{dom}(q')$, $p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

Now we review the original forcing which iteration was used to force the tree property below \aleph_ω . For more details see [6] and [1].

Fact 2.14 *Let λ be a supercompact cardinal. Then there is a function F from λ to V_λ such that for all $\mu \geq \lambda$ and all $x \in H_{\mu^+}$ there is a supercompactness measure U on $\mathcal{P}_\lambda(\mu)$ such that $j_U(F)(\lambda) = x$. We call F a Laver function for λ .*

Let $F_\lambda : \lambda \rightarrow V_\lambda$ denote a Laver function from previous fact for a given supercompact cardinal λ .

Definition 2.15 *Let $V \subseteq W$ be two inner models of ZFC with the same cardinals, κ be a regular cardinal and $\lambda > \kappa$ supercompact in W . Suppose that $\text{Add}(\kappa, \lambda)^V$ is κ^+ -cc and κ -distributive in W . The forcing $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ is the set of all triples (p, q, f) such that (p, q) is in the Mitchell forcing $\mathbb{M}(\kappa, \lambda, V, W)$ and f is a function in W of size less than κ^+ such that $\text{dom}(f)$ is a subset of*

$$(2.4) \quad \{\alpha < \lambda \mid \alpha \text{ inaccessible and } 1_{\mathbb{R}|\alpha} \Vdash^W F_\lambda(\alpha) \text{ is an } \alpha\text{-directed closed forcing}\},$$

and if $\alpha \in \text{dom}(f)$ then $f(\alpha) \in W^{\mathbb{R}|\alpha}$ and $1_{\mathbb{R}|\alpha} \Vdash^W f(\alpha) \in F_\lambda(\alpha)$.

The ordering is defined by $(p, q, f) \leq (p', q', f')$ if

- (i) $(p, q) \leq (p', q')$,
- (ii) $\text{dom}(f) \supseteq \text{dom}(f')$ and for every $\alpha \in \text{dom}(f')$, $(p \restriction \alpha, q \restriction \alpha, f \restriction \alpha) \Vdash f(\alpha) \leq f'(\alpha)$.

Note that the previous definition should be formally defined by induction on λ , for more details see [6]. Also note that the definition is made in the model W and all what we are state in further is in sense of the model W .

Mitchell forcing $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ is λ -Knaster and κ -distributive. Moreover, it collapses the cardinals in the open interval (κ^+, λ) to κ^+ and forces $2^\kappa = \lambda = \kappa^{++}$. The preservation of κ^+ is shown by means of the product analysis due to Abraham [1].

Let \mathbb{T} be defined as follows:

$$(2.5) \quad \mathbb{T} = \{(\emptyset, q, f) \mid (\emptyset, q, f) \in \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)\}.$$

The ordering on \mathbb{T} is the one induced from $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$. It is clear that \mathbb{T} is κ^+ -directed closed in W . We will call \mathbb{T} the *term forcing* (of the associated Mitchell-style forcing).

It is easy to see that the function

$$(2.6) \quad \pi : \text{Add}(\kappa, \lambda)^V \times \mathbb{T} \rightarrow \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$$

which maps $(p, (\emptyset, q, f))$ to (p, q, f) is a projection. Since the product $\text{Add}(\kappa, \lambda)^V \times \mathbb{T}$ preserves κ^+ (under assumption $\kappa^{<\kappa} = \kappa$), so does the forcing $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$.

There are natural projections from Mitchell forcing of length λ to Mitchell forcings of shorter lengths and a projection to Cohen forcing $\text{Add}(\kappa, \lambda)^V$. For the first claim, define a function $\sigma^{\lambda, \alpha}$ from $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ to $\mathbb{R}(\kappa, \alpha, V, W, F_\lambda)$, where α is an ordinal between κ and λ , as follows: $\sigma^{\lambda, \alpha}((p, q, f)) = (p \upharpoonright \alpha, q \upharpoonright \alpha, f \upharpoonright \alpha)$. For the second claim, define a function ρ from $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ to $\text{Add}(\kappa, \lambda)^V$ by $\rho((p, q, f)) = p$. It is easy to see that $\sigma^{\lambda, \alpha}$ and ρ are projections.

By the projection $\rho : \mathbb{R}(\kappa, \lambda, V, W, F_\lambda) \rightarrow \text{Add}(\kappa, \lambda)^V$, $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ is forcing-equivalent to $\text{Add}(\kappa, \lambda)^V * \mathbb{D}$, for some \mathbb{D} . Moreover, by the product analysis (i.e. of the existence of the projection π), \mathbb{D} is a name for a forcing notion which is forced to be κ^+ -distributive and κ -closed.

Remark 2.16 Notice that the term forcing \mathbb{T} collapses the cardinals between κ^+ and λ : Suppose $\kappa^{<\kappa} = \kappa$ and λ is inaccessible. As \mathbb{T} is κ^+ -closed, Cohen forcing $\text{Add}(\kappa, \lambda)$ is still κ^+ -cc and κ -closed in $V[\mathbb{T}]$. In particular, it does not collapse cardinals over $V[\mathbb{T}]$ (so it must be \mathbb{T} which collapses the cardinals).

The term forcing analysis carries over to quotients given by the projections $\sigma^{\lambda, \alpha}$ whenever α is an inaccessible cardinal between κ and λ . First note that if α is inaccessible then $\mathbb{R}(\kappa, \alpha + 1, V, W, F_\lambda)$ is equivalent to $\mathbb{R}(\kappa, \alpha, V, W, F_\lambda) * F(\alpha)$. This holds because at limit cardinals the first coordinates are not defined.

Let $G_{\alpha+1}$ be an $\mathbb{R}(\kappa, \alpha + 1, V, W, F_\lambda)$ -generic filter and define in $V[G_{\alpha+1}]$ the quotient $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)/G_{\alpha+1}$ as follows:

$$(2.7) \quad \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)/G_{\alpha+1} = \{(p, q, f) \in \mathbb{R}(\kappa, \lambda, V, W, F_\lambda) \mid (p \upharpoonright \alpha, q \upharpoonright \alpha, f \upharpoonright \alpha + 1) \in G_{\alpha+1}\}.$$

Regarding this quotient, we can now analogously define the term forcing \mathbb{T}^* in $V[G_{\alpha+1}]$

$$(2.8) \quad \mathbb{T}^* = \{(\emptyset, q, f) \mid (\emptyset, q, f) \in \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)/G_{\alpha+1}\}.$$

and a projection π^* from $\text{Add}(\kappa, \lambda - \alpha) \times \mathbb{T}^*$ to $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)/G_{\alpha+1}$ by setting $\pi^*((p, (\emptyset, q, f))) = (p, q, f)$.

Fact 2.17 *Let α be inaccessible and $G_{\alpha+1}$ an $\mathbb{R}(\kappa, \alpha + 1, V, W, F_\lambda)$ -generic filter. Then in $V[G_{\alpha+1}]$ the following hold:*

- (i) π^* is a projection from $\text{Add}(\kappa, \lambda - \alpha) \times \mathbb{T}^*$ to $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)/G_{\alpha+1}$.
- (ii) \mathbb{T}^* is κ^+ -closed in $V[G_{\alpha+1}]$.

At the end of the analysis, consider the quotient of $\text{Add}(\kappa, \lambda) \times \mathbb{T}$ after the forcing $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$. Let G be $\mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ -generic. We define

$$(2.9) \quad \mathbb{S} = (\text{Add}(\kappa, \lambda) \times \mathbb{T})/G = \{(p, (\emptyset, q, f)) \in \text{Add}(\kappa, \lambda) \times \mathbb{T} \mid (p, q, f) \in G\}.$$

Fact 2.18 \mathbb{S} is κ -closed, κ^+ -distributive and λ -cc over $V[\mathbb{R}]$.

The following lemma summarises properties which are preserved after forcing with a product of a Mitchell-style forcing and another forcing.

Lemma 2.19 *Let $V \subseteq W$ be two inner models of ZFC with the same cardinals, κ be a regular cardinal and $\lambda > \kappa$ supercompact in W . Suppose that $\text{Add}(\kappa, \lambda)^V$ is κ^+ -Knaster and κ -distributive in W . Assume P is κ^+ -Knaster, R is κ^+ -cc and Q and S are κ^+ -closed in W . Then the following hold:*

- (i) $R \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ forces that Q is κ^+ -distributive.
- (ii) $Q \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ forces that R is κ^+ -cc.
- (iii) $P \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ forces that R is κ^+ -cc.
- (iv) $Q \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)$ forces that S is κ^+ -distributive.

PROOF. (i). It is easy to check that the projection π in (2.6) extends to the projection π' ,

$$(2.10) \quad \pi' : R \times Q \times \text{Add}(\kappa, \lambda) \times \mathbb{T} \rightarrow R \times Q \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda),$$

which sends $(r_1, r_2, p, (\emptyset, q, f))$ to $(r_1, r_2, (p, q, f))$. It follows that $R \times Q \times \text{Add}(\kappa, \lambda)^V \times \mathbb{T}$ is forcing equivalent to

$$(2.11) \quad [R \times Q \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda)] * \dot{S}$$

for some quotient forcing \dot{S} .

Let $G \times g \times F$ be an arbitrary $R \times \mathbb{R}(\kappa, \lambda, V, W, F_\lambda) \times Q$ -generic filter over W . We will show that every sequence x of ordinals of length less than κ^+ which is in $V[G \times g \times F]$ is in $V[G \times g]$ which shows that Q is forced to be κ^+ -distributive as required. Let x as above be fixed. Let h be any \dot{S} -generic filter over $V[G \times g \times F]$. It follows by (2.11) that $V[G \times g \times F][h]$ can be written as $V[G \times g_0 \times g_1 \times F]$ where $g_0 \times g_1$ is $\text{Add}(\kappa, \lambda)^V \times \mathbb{T}$ -generic, and the following hold:

- (i) $V[G \times g \times F] \subseteq V[G \times g_0 \times g_1 \times F]$,
- (ii) $V[G \times g_0] \subseteq V[G \times g]$,

where (ii) holds because g_0 is the Cohen part of g . In particular x is in $V[G \times g_0 \times g_1 \times F]$.

By Easton's lemma, $R \times \text{Add}(\kappa, \lambda)^V$ (which is κ^+ -cc) forces that $\mathbb{T} \times Q$ (which is κ^+ -closed) is κ^+ -distributive. It follows that x is already in $V[G \times g_0]$, and hence in $V[G \times g]$ as desired.

(ii) – (iv). It suffices to argue similarly as in (i) that the forcing notion under consideration has the required property in the generic extension by $Q \times \text{Add}(\kappa, \lambda)^V \times \mathbb{T}$ for (ii) and (iv), and $P \times \text{Add}(\kappa, \lambda)^V \times \mathbb{T}$ for (iii). This is easy to show using the Easton's lemma (Lemma 2.6). \square

2.4 THE CUMMINGS-FOREMAN MODEL

Let $\kappa_2 < \kappa_3 < \dots$ be an ω -sequence of supercompact cardinals with limit λ and let κ_0 denote \aleph_0 and κ_1 denote \aleph_1 . And let F_n denote corresponding Laver function for κ_n for $n > 1$. Now we define Cummings-Foreman forcing used in [6] to force the tree property below \aleph_ω . We also state some basic facts about this forcing which can be found in [6].

Definition 2.20 *The iteration $\mathbb{R}_\omega = \langle \mathbb{R}_n * \dot{Q}_n \mid n < \omega \rangle$ of length ω is defined by induction as follows:*

- (i) *The first stage $\mathbb{Q}_0 = \mathbb{R}(\kappa_0, \kappa_2, V, V, F_2)$, let us denote $\mathbb{R}_1 = \mathbb{Q}_0$ and \mathbb{R}_0 be the trivial forcing.*
- (ii) *Suppose that we have defined the iteration up to stage $n > 0$. Let $\mathbb{R}_n = \mathbb{Q}_0 * \dots * \dot{Q}_{n-1}$. First define an \mathbb{R}_n -name \dot{F}_{n+2} by $\dot{F}_{n+2}(\alpha) = F_{n+2}(\alpha)$, if $F_{n+2}(\alpha)$ is an \mathbb{R}_n -name, and $\dot{F}_{n+2}(\alpha) = 0$ otherwise. Then define \dot{Q}_n to be a name for $\mathbb{R}(\kappa_n, \kappa_{n+2}, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_{n+2}^*)$, where F_{n+2}^* is the interpretation of \dot{F}_{n+2} in $V[\mathbb{R}_n]$.*

Let \mathbb{R}_ω denote the inverse limit of $\langle \mathbb{R}_n \mid n < \omega \rangle$.

Let us for $n < \omega$ fix the following notation corresponding to the analysis in the previous section. Let \mathbb{T}_n , \mathbb{D}_n and \mathbb{S}_n be the relevant partial orders and π_n , ρ_n and $\sigma^{\kappa_{n+2}, \alpha}$ the projections, where α is an ordinal between κ_n and κ_{n+2} .

For the proofs of the following facts see corresponding lemmas in [6] (Lemma 4.2, Lemma 4.3 and Lemma 4.4).

Fact 2.21 *Let \mathbb{P}_0 denote $\text{Add}(\kappa_0, \kappa_2)$ and \mathbb{T}_0 denote the term forcing of by $\mathbb{Q}_0 = \mathbb{R}(\kappa_0, \kappa_2, V, V, F_2)$. Then the following hold:*

- (i) *The size of \mathbb{Q}_0 is κ_2 and \mathbb{Q}_0 is κ_2 -Knaster.*
- (ii) *π_0 is a projection from $\mathbb{P}_0 \times \mathbb{T}_0$ to \mathbb{Q}_0 and ρ_0 is a projection from \mathbb{Q}_0 to \mathbb{P}_0 .*
- (iii) *\mathbb{Q}_0 forces $2^{\aleph_0} = \kappa_2 = \aleph_2$.*
- (iv) *$\text{Add}(\kappa_1, \xi)^V$ is κ_1 -distributive and κ_2 -Knaster after forcing with \mathbb{Q}_0 for a suitable ordinal $\xi > 0$.*
- (v) *$\dot{\mathbb{D}}_0$, given by the projection ρ_0 , is a \mathbb{P}_0 -name for κ_1 -distributive and κ_2 -cc forcing.*
- (vi) *$\dot{\mathbb{S}}_0$, given by the projection π_0 is a \mathbb{Q}_0 -name for κ_1 -distributive and κ_2 -cc forcing.*

Fact 2.22 *Let $n > 0$ and let us denote by $\mathbb{P}_n = \text{Add}(\kappa_n, \kappa_{n+2})^{V[\mathbb{R}_{n-1}]}$ and \mathbb{T}_n the term forcing of $\mathbb{Q}_n = (\kappa_n, \kappa_{n+2}, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_{n+2}^*)$. Then in $V[\mathbb{R}_n]$ the following hold:*

- (i) *$2^{\kappa_i} = \kappa_{i+2}$ for $i < n$ and $\kappa_i = \aleph_i$ for $i < n + 2$.*
- (ii) *The size of \mathbb{Q}_n is κ_{n+2} and \mathbb{Q}_n is κ_{n-1} -closed, κ_n -distributive and κ_{n+2} -Knaster.*
- (iii) *\mathbb{Q}_n is a projection of $\mathbb{P}_n \times \mathbb{T}_n$ and there is also projection from \mathbb{Q}_n to \mathbb{P}_n .*
- (iv) *\mathbb{Q}_n forces $2^{\kappa_n} = \kappa_{n+2} = \aleph_{n+2}$.*
- (v) *$\text{Add}(\kappa_{n+1}, \xi)^{V[\mathbb{R}_n]}$ is κ_{n+1} -distributive and κ_{n+2} -Knaster after forcing with \mathbb{Q}_n a suitable ordinal $\xi > 0$.*
- (vi) *$\dot{\mathbb{D}}_n$, given by the projection ρ_n , is a \mathbb{P}_n -name for κ_n -closed, κ_{n+1} -distributive and κ_{n+2} -cc forcing.*

(vii) \dot{S}_n , given by the projection π_n is a \mathbb{Q}_n -name for κ_n -closed, κ_{n+1} -distributive and κ_{n+2} -cc forcing.

Fact 2.23 *Let $n \geq 0$. Any κ_n -sequence of ordinals in $V[\mathbb{R}_\omega]$ is already added by $\mathbb{R}_n * \dot{\mathbb{P}}_n$.*

Theorem 2.24 (Cummings-Foreman) *In the generic extension by \mathbb{R}_ω the following hold:*

- (i) $2^{\kappa_n} = \kappa_{n+2}$ and $\kappa_n = \aleph_n$, for $n < \omega$,
- (ii) the tree property at κ_n , for $1 < n < \omega$.

3 MAIN THEOREM

Let $\kappa_2 < \kappa_3 < \dots$ be an ω -sequence of supercompact cardinals with limit λ and let κ_0 denote \aleph_0 and κ_1 denote \aleph_1 . In Theorem 3.1, we control the continuum function below $\aleph_\omega = \lambda$, while having the tree property at all \aleph_n , $n > 1$.

Let A denote the set $\{\kappa_i \mid i < \omega\}$, and let $e : A \rightarrow A$ be a function which satisfies for all α, β in A :

- (i) $i < j < \omega \rightarrow e(\kappa_i) \leq e(\kappa_j)$.
- (ii) $e(\kappa_i) \geq \kappa_{i+2}$ for all $i < \omega$.

We say that e is an Easton function on A which respects the κ_i 's (condition (ii)).

Theorem 3.1 *Assume GCH and let $\langle \kappa_i \mid i < \omega \rangle$, λ , and A be as above. Let e be an Easton function on A which respects the κ_i 's. Then there is a forcing notion \mathbb{Z} such that if G is a \mathbb{Z} -generic filter, then in $V[G]$:*

- (i) *Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_n$,*
- (ii) *The continuum function on $A = \{\aleph_n \mid n < \omega\}$ is controlled by e , i.e. $\forall n < \omega, 2^{\aleph_n} = e(\aleph_n)$.*
- (iii) *The tree property holds at every \aleph_n , $2 \leq n < \omega$.*

For obtaining the model we are using the Cummings-Foreman iteration from [6] followed by the Easton product of Cohen forcings which live in suitable inner models.

3.1 THE FORCING

Let e be an Easton function on A which respect the κ_n 's and let \mathbb{R}_ω be the forcing from Cummings and Foreman. Our forcing \mathbb{Z} is defined as follows:

$$(3.12) \quad \mathbb{Z} = \mathbb{R}_\omega * \prod_{n < \omega} \text{Add}(\kappa_n, e(\kappa_n))^{V[\mathbb{R}_{n-1}]},$$

where we identify $V[\mathbb{R}_{-1}]$ (for $n = 0$) with V .

Let us denote this product by \mathbb{E} and let $\dot{\mathbb{E}}$ be a canonical \mathbb{R}_ω -name for it. We can therefore write

$$(3.13) \quad \mathbb{Z} = \mathbb{R}_\omega * \dot{\mathbb{E}}.$$

Now we need to verify that the tree property holds in this model below \aleph_ω and that the continuum function is represented by e .

3.2 THE RIGHT CONTINUUM FUNCTION

In this section, we show that \mathbb{Z} forces the right continuum function:

Theorem 3.2 $\mathbb{R}_\omega * \dot{\mathbb{E}}$ forces that for all $n < \omega$, $\kappa_n = \aleph_n$ and $2^{\kappa_n} = e(\kappa_n)$.

We prove the theorem in a series of lemmas. Before we begin with the analysis of the forcing $\mathbb{R}_\omega * \dot{\mathbb{E}}$, let us fix some notation. For $n < \omega$ let $\dot{\mathbb{R}}_{[n,\omega]}$ denote the canonical \mathbb{R}_n -name for the tail $\mathbb{R}_{[n,\omega]}$ of the iteration \mathbb{R}_ω . If $i < n$ let us also denote $\dot{\mathbb{R}}_{[i,n]}$ the canonical \mathbb{R}_i -name for the iteration between i and n , $\mathbb{R}_{[i,n]}$.

In $V[\mathbb{R}_\omega]$, let us denote by $\mathbb{P}_n^{\mathbb{E}}$ the Cohen forcing $\text{Add}(\kappa_n, e(\kappa_n))^{V[\mathbb{R}_{n-1}]}$ in the product \mathbb{E} , $n < \omega$. Moreover, let us denote by \mathbb{E}_n the product of first n -many Cohen forcings in \mathbb{E} , i.e. $\mathbb{E}_n = \prod_{i < n} \mathbb{P}_i^{\mathbb{E}}$ and analogously let $\mathbb{E}_{[n,\omega]}$ denote the product of the rest of the forcing, i.e. $\mathbb{E}_{[n,\omega]} = \prod_{i \geq n} \text{Add}(\kappa_i, e(\kappa_i))^{V[\mathbb{R}_{i-1}]}$; we have $\mathbb{E} \cong \mathbb{E}_n \times \mathbb{E}_{[n,\omega]}$. Let us further define $\mathbb{E}_{(j,n)} = \prod_{j < i < n} \text{Add}(\kappa_i, e(\kappa_i))^{V[\mathbb{R}_{i-1}]}$ for $j \leq n$ and let $\dot{\mathbb{E}}_n$, $\dot{\mathbb{E}}_{[n,\omega]}$ and $\dot{\mathbb{E}}_{(j,n)}$ denote the canonical \mathbb{R}_ω -name for \mathbb{E}_n , $\mathbb{E}_{[n,\omega]}$ and $\mathbb{E}_{(j,n)}$ -name, respectively.

It is easy to see that for all $n < \omega$, $\dot{\mathbb{E}}_{n+2}$ can be identified with an \mathbb{R}_n -name as all Cohen forcings in $\dot{\mathbb{E}}_{n+2}$ live in $V[\mathbb{R}_n]$. Therefore we can factor the iteration as $\mathbb{R}_\omega * \dot{\mathbb{E}} = \mathbb{R}_n * (\dot{\mathbb{E}}_{n+2} \times \dot{\mathbb{R}}_{[n,\omega]}) * \dot{\mathbb{E}}_{[n+2,\omega]}$ for each $n < \omega$.

Lemma 3.3 Let $n > 0$. Then in $V[\mathbb{R}_n * \dot{\mathbb{E}}_{n+1}]$, the following hold:

- (i) $\aleph_i = \kappa_i$ for $i < n + 2$;
- (ii) $2^{\kappa_i} = e(\kappa_i)$ for $i < n + 1$.

PROOF. (i). Let $n > 0$ be given. First recall Cummings-Foreman result that for all $1 < i < n + 2$, $\kappa_i = \aleph_i$ in $V[\mathbb{R}_n]$, $2^{\kappa_{i-2}} = \kappa_i$ and GCH holds everywhere else.

We will show by induction starting with $i = n$ and descending to 0 that for each $0 \leq i \leq n$, the forcing $\mathbb{E}_{[i,n+1]}$ behaves well over the model $V[\mathbb{R}_n]$ in the sense that it does not unintentionally collapse cardinals and forces the right continuum function. The assumptions for the induction are as follows:

- (a) $\mathbb{E}_{[i,n+1]} = \mathbb{P}_i^{\mathbb{E}} \times \mathbb{E}_{(i,n+1)}$ is κ_{i-1} -closed in $V[\mathbb{R}_n]$,
- (b) $\mathbb{P}_i^{\mathbb{E}}$ is κ_i -distributive in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$,
- (c) $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$.

Notice that if we verify (a)–(c) for each $0 \leq i \leq n$, then the result follows because by stage $i = 0$ we have dealt with the whole forcing $\mathbb{E}_{[0,n+1]} = \mathbb{E}_{n+1}$ (items (b) and (c) imply that for each i , $\mathbb{P}_i^{\mathbb{E}}$ preserves cardinals over the model $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$, with (a) being a useful assumption which keeps the induction running).

The base case is $i = n$, which means that $\mathbb{P}_n^{\mathbb{E}}$ should satisfy points (a)–(c) in $V[\mathbb{R}_n]$. This is true by Lemma 2.5(ii), Lemma 2.19(i) (with a trivial forcing R) and Lemma 2.4(i), respectively.

For the induction step, let us assume that (a)–(c) hold for $0 < i + 1 \leq n$, and we will verify (a)–(c) for i .

- (a) It suffices to show that $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i-1} -closed in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$ because by the induction assumption (a), $\mathbb{E}_{(i,n+1)} = \mathbb{E}_{[i+1,n+1]}$ is κ_i -closed in $V[\mathbb{R}_n]$.

The forcing \mathbb{R}_n is equal to $\mathbb{R}_{i-1} * \dot{\mathbb{R}}_{[i-1,n]}$ and $\dot{\mathbb{R}}_{[i-1,n]}$ is forced to be κ_{i-1} -distributive by Fact 2.22(ii). Therefore $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i-1} -closed in $V[\mathbb{R}_n]$ by Lemma 2.5(ii).

- (b) We wish to show that $\mathbb{P}_i^{\mathbb{E}}$ is κ_i -distributive in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$.

\mathbb{R}_n can be written as

$$(3.14) \quad \mathbb{R}_{i-1} * \dot{\mathbb{Q}}_{i-1} * \dot{\mathbb{Q}}_i * \dot{\mathbb{R}}_{[i+1,n]}.$$

Working in $V[\mathbb{R}_{i-1}]$, $\mathbb{Q}_{i-1} * \dot{\mathbb{Q}}_i$ is short for $\mathbb{R}(\kappa_{i-1}, \kappa_{i+1}, V[\mathbb{R}_{i-2}], V[\mathbb{R}_{i-1}], F_{i+1}^*) * \mathbb{R}(\kappa_i, \kappa_{i+2}, V[\mathbb{R}_{i-1}], V[\mathbb{R}_i], F_{i+2}^*)$ and this forcing is forcing equivalent to

$$(3.15) \quad (\mathbb{R}(\kappa_{i-1}, \kappa_{i+1}, V[\mathbb{R}_{i-2}], V[\mathbb{R}_{i-1}], F_{i+1}^*) \times \mathbb{P}_i) * \dot{\mathbb{D}}_i$$

where $\dot{\mathbb{D}}_i$ is forced to be κ_i -closed after $\mathbb{R}(\kappa_{i-1}, \kappa_{i+1}, V[\mathbb{R}_{i-2}], V[\mathbb{R}_{i-1}], F_{i+1}^*) \times \mathbb{P}_i$ by Fact 2.22(vi). But $\mathbb{P}_i^{\mathbb{E}}$ is κ_i -distributive after the forcing

$$\mathbb{R}(\kappa_{i-1}, \kappa_{i+1}, V[\mathbb{R}_{i-2}], V[\mathbb{R}_{i-1}], F_{i+1}^*) \times \mathbb{P}_i$$

by Lemma 2.19(iv), therefore we can apply Lemma 2.5(i) to $\dot{\mathbb{D}}_i$ and $\mathbb{P}_i^{\mathbb{E}}$ and conclude that $\mathbb{P}_i^{\mathbb{E}}$ is κ_i -distributive in $V[\mathbb{R}_{i-1}][\mathbb{R}(\kappa_{i-1}, \kappa_{i+1}, V[\mathbb{R}_{i-2}], V[\mathbb{R}_{i-1}], F_{i+1}^*) \times \mathbb{P}_i) * \dot{\mathbb{D}}_i]$. The rest of the proof again follows by Lemma 2.5(i) from Fact 2.22(ii) that $\mathbb{R}_{[i+1,n]}$ is κ_i -closed and from the induction hypothesis that $\mathbb{E}_{(i,n+1)}$ is κ_i -closed in $V[\mathbb{R}_n]$.

- (c) We wish to show that $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$.

The forcing $\mathbb{R}_n * \dot{\mathbb{E}}_{(i,n+1)}$ is forcing equivalent to

$$(3.16) \quad \mathbb{R}_{i-1} * \dot{\mathbb{Q}}_{i-1} * (\dot{\mathbb{Q}}_i \times \dot{\mathbb{P}}_{i+1}^{\mathbb{E}}) * \dot{\mathbb{Q}}_{i+1} * \dot{\mathbb{R}}_{(i+1,n)} * \dot{\mathbb{E}}_{(i+1,n+1)}.$$

As both $\mathbb{P}_i^{\mathbb{E}}$ and \mathbb{Q}_{i-1} are κ_{i+1} -Knaster in $V[\mathbb{R}_{i-1}]$, \mathbb{Q}_{i-1} forces that $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc and thus $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc in $V[\mathbb{R}_i]$. Now, in $V[\mathbb{R}_i]$, $(\mathbb{Q}_i \times \mathbb{P}_{i+1}^{\mathbb{E}}) * \dot{\mathbb{Q}}_{i+1}$ is forcing equivalent to

$$(3.17) \quad (\mathbb{Q}_i \times \mathbb{P}_{i+1}^{\mathbb{E}} \times \mathbb{P}_{i+1}) * \dot{\mathbb{D}}_{i+1},$$

where \mathbb{D}_{i+1} is a $\mathbb{Q}_i \times \mathbb{P}_{i+1}$ -name for a forcing notion which is κ_{i+1} -closed. As $\mathbb{P}_{i+1}^{\mathbb{E}}$ stays κ_{i+1} -distributive after $\mathbb{Q}_i \times \mathbb{P}_{i+1}$ by Lemma 2.19(iv), \mathbb{D}_{i+1} is still forced to be κ_{i+1} -closed after forcing with $\mathbb{P}_{i+1}^{\mathbb{E}}$ by Lemma 2.5(ii).

Our forcing $\mathbb{P}_i^{\mathbb{E}}$ is still κ_{i+1} -cc after $\mathbb{Q}_i \times \mathbb{P}_{i+1}^{\mathbb{E}} \times \mathbb{P}_{i+1}$ by Lemma 2.19(ii). By the previous paragraph and Lemma 2.6(ii) it is still κ_{i+1} -cc after the forcing (3.17), which is forcing equivalent to $(\mathbb{Q}_i * \mathbb{Q}_{i+1}) \times \mathbb{P}_{i+1}^{\mathbb{E}}$.

In $V[\mathbb{R}_{i+2}]$, $\mathbb{P}_{i+1}^{\mathbb{E}}$ is κ_{i+1} -distributive and $\mathbb{R}_{(i+1,n)}$ is κ_{i+1} -closed by Fact 2.22(ii) and thus $\mathbb{R}_{(i+1,n)}$ is still κ_{i+1} -closed in $V[\mathbb{R}_{i+2}][\mathbb{P}_{i+1}^{\mathbb{E}}]$ by Lemma 2.5(ii). Therefore our forcing $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc in

$$(3.18) \quad V[\mathbb{R}_{i+2}][\mathbb{P}_{i+1}^{\mathbb{E}}][\mathbb{R}_{(i+1,n)}] = V[\mathbb{R}_n][\mathbb{P}_{i+1}^{\mathbb{E}}]$$

by Lemma 2.6(ii)

Now it is enough to realize that by the induction hypothesis $\mathbb{E}_{(i+1,n+1)}$ is κ_{i+1} -closed in $V[\mathbb{R}_n]$ and $\mathbb{P}_{i+1}^{\mathbb{E}}$ is κ_{i+1} -distributive and thus $\mathbb{E}_{(i+1,n+1)}$ is κ_{i+1} -closed in the model (3.18) by Lemma 2.5(ii). Therefore we can apply Lemma 2.6(ii) to $\mathbb{P}_i^{\mathbb{E}}$ and $\mathbb{E}_{(i+1,n+1)}$ over the model (3.18), hence $\mathbb{P}_i^{\mathbb{E}}$ is κ_{i+1} -cc in $V[\mathbb{R}_n][\mathbb{E}_{(i,n+1)}]$.

(ii). Easily follows from (i). □

Corollary 3.4 *Let $n < \omega$ be given. In $V[\mathbb{R}_n]$ the following hold:*

- (i) *For $i < n$, $\mathbb{E}_{(i,n+1)}$ forces \mathbb{E}_{i+1} is κ_{i+1} -cc.*
- (ii) *For $i < n + 1$, \mathbb{E}_{i+1} is κ_{i+1} -cc, in particular \mathbb{E}_{n+1} is κ_{n+1} -cc.*

PROOF. This is immediate from proof of (c) of the previous lemma using Lemma 2.2 and fact that chain condition is upward closed. □

Lemma 3.5 *In $V[\mathbb{R}_\omega]$, $\mathbb{E}_{[n,\omega]}$ is κ_{n-1} -closed for each $n > 0$.*

PROOF. Let $n > 0$ be given. As the product of κ_{n-1} -closed forcings is κ_{n-1} -closed, it suffices to show that for each $i \geq n$, $\mathbb{P}_i^{\mathbb{E}} = \text{Add}(\kappa_i, e(\kappa_i))^{V[\mathbb{R}_{i-1}]}$ is κ_{n-1} -closed. $\mathbb{P}_i^{\mathbb{E}}$ is defined in $V[\mathbb{R}_{i-1}]$ and it is even κ_i -closed there, but $\mathbb{R}_{[i-1,\omega]}$, the tail of the iteration \mathbb{R}_ω , is just κ_{i-1} -distributive in $V[\mathbb{R}_{i-1}]$ ⁸, and therefore \mathbb{P}_i remains κ_{i-1} -closed in $V[\mathbb{R}_\omega]$ and thus at least κ_{n-1} -closed. □

Lemma 3.6 *For each $0 \leq n < \omega$, any κ_n -sequence of ordinals in $V[\mathbb{R}_\omega][\mathbb{E}]$ is already added by $\mathbb{R}_n * (\mathbb{P}_n \times \mathbb{E}_{n+1})$.*

⁸To see that $\mathbb{R}_{[i-1,\omega]}$ is κ_{i-1} -distributive, note that $\mathbb{R}_{[i-1,\omega]} = \mathbb{Q}_{i-1} * \mathbb{R}_{[i,\omega]}$ and \mathbb{Q}_{i-1} is κ_{i-1} -distributive and forces that $\mathbb{R}_{[i,\omega]}$ is κ_{i-1} -closed by Fact 2.22(ii).

PROOF. Let $n \geq 0$ be given. First note that by Fact 2.22(ii), $\mathbb{R}_{[n+2, \omega]}$ is κ_{n+1} -closed in $V[\mathbb{R}_{n+2}]$ and $\mathbb{E}_{[n+2, \omega]}$ is κ_{n+1} -closed in $V[\mathbb{R}_\omega]$, therefore $\mathbb{R}_{[n+2, \omega]} * \dot{\mathbb{E}}_{[n+2, \omega]}$ is κ_{n+1} -closed in $V[\mathbb{R}_{n+2}]$ and thus also in $V[\mathbb{R}_{n+2}][\mathbb{P}_{n+1}^{\mathbb{E}}]$ by Lemma 2.5(ii) as $\mathbb{P}_{n+1}^{\mathbb{E}}$ is κ_{n+1} -distributive in $V[\mathbb{R}_{n+2}]$. By Corollary 3.4(i), \mathbb{E}_{n+1} is κ_{n+1} -cc in $V[\mathbb{R}_{n+2}][\mathbb{P}_{n+1}^{\mathbb{E}}]$, therefore by Lemma 2.6(i), $\mathbb{R}_{[n+2, \omega]} * \dot{\mathbb{E}}_{[n+2, \omega]}$ is κ_{n+1} -distributive in $V[\mathbb{R}_{n+2}][\mathbb{P}_{n+1}^{\mathbb{E}}][\mathbb{E}_{n+1}] = V[\mathbb{R}_{n+2}][\mathbb{E}_{n+2}]$. Hence any κ_n -sequence of ordinals is already added by $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+2}$.

Now, work in $V[\mathbb{R}_n]$. The forcing $\mathbb{Q}_n * \dot{\mathbb{Q}}_{n+1}$ is forcing equivalent to $(\mathbb{Q}_n \times \mathbb{P}_{n+1}) * \dot{\mathbb{D}}_{n+1}$, where $\dot{\mathbb{D}}_{n+1}$ is forced to be κ_{n+1} -closed and stays κ_{n+1} -closed after forcing with $\mathbb{P}_{n+1}^{\mathbb{E}}$ by Lemma 2.19(iv) and Lemma 2.5(ii). Now we can apply Lemma 2.6(i) over $V[\mathbb{R}_n][\mathbb{Q}_n \times \mathbb{P}_{n+1} \times \mathbb{P}_{n+1}^{\mathbb{E}}]$ to \mathbb{E}_{n+1} ⁹ and \mathbb{D}_{n+1} to show that \mathbb{D}_{n+1} is κ_{n+1} -distributive in $V[\mathbb{R}_n][\mathbb{Q}_n \times \mathbb{P}_{n+1} \times \mathbb{P}_{n+1}^{\mathbb{E}}][\mathbb{E}_{n+1}] = V[\mathbb{R}_{n+1}][\mathbb{P}_{n+1}][\mathbb{E}_{n+2}]$. Therefore any κ_n -sequence of ordinals is already added by $\mathbb{R}_{n+1} * \dot{\mathbb{P}}_{n+1} * \dot{\mathbb{E}}_{n+2}$.

Work again in $V[\mathbb{R}_n]$. \mathbb{E}_{n+1} is κ_{n+1} -cc and $\mathbb{P}_{n+1} \times \mathbb{P}_{n+1}^{\mathbb{E}}$ is κ_{n+1} -closed here, therefore by Lemma 2.19(i) $\mathbb{P}_{n+1} \times \mathbb{P}_{n+1}^{\mathbb{E}}$ is κ_{n+1} -distributive in $V[\mathbb{R}_n][\mathbb{Q}_n][\mathbb{E}_{n+1}] = V[\mathbb{R}_{n+1}][\mathbb{E}_{n+1}]$. Therefore any κ_n -sequence of ordinals is already in $V[\mathbb{R}_{n+1}][\mathbb{E}_{n+1}]$.

In $V[\mathbb{R}_n]$, \mathbb{Q}_n is a projection of $\mathbb{P}_n \times \mathbb{T}_n$, where \mathbb{T}_n is κ_{n+1} -closed and \mathbb{P}_n is κ_{n+1} -Knaster, therefore $\mathbb{E}_{n+1} \times \mathbb{P}_n$ is κ_{n+1} -cc and hence \mathbb{T}_n stays κ_{n+1} -distributive after forcing with $\mathbb{E}_{n+1} \times \mathbb{P}_n$ by Lemma 2.6(i). It follows that every κ_n -sequence is added by $\mathbb{R}_n * (\dot{\mathbb{P}}_n \times \dot{\mathbb{E}}_{n+1})$, as desired. \square

Now we can finish the proof of Theorem 3.2:

PROOF. (Proof of theorem 3.2.) The theorem follows from Lemma 3.6, Lemma 3.3 and the fact that $\mathbb{P}_n \times \mathbb{E}_{n+1}$ is isomorphic to \mathbb{E}_{n+1} over $V[\mathbb{R}_n]$. \square

3.3 THE TREE PROPERTY

In this section we finish the argument by showing:

Theorem 3.7 $\mathbb{R}_\omega * \dot{\mathbb{E}}$ forces that the tree property holds at κ_{n+2} , for every $n \geq 0$.

We prove the theorem in two subsections and several lemmas. Let us fix some $n \geq 0$, and let us denote κ_{n+2} by κ . We show the tree property at κ .

In $V[\mathbb{R}_{n+2}]$, let $\mathbb{E}_{n+3}|\kappa$ be the product $\prod_{i < n+3} \text{Add}(\kappa_i, \lambda_i)^{V[\mathbb{R}_{i-1}]}$, where $\lambda_i = \kappa$ for $e(\kappa_i) > \kappa$ and $\lambda_i = e(\kappa_i)$ otherwise.

Lemma 3.8 If $\mathbb{R}_\omega * \dot{\mathbb{E}}$ adds a κ -Aronszajn tree, so does $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+3}|\kappa$.

PROOF. Assume for contradiction that there is a κ -Aronszajn tree T in generic extension by $\mathbb{R}_\omega * \dot{\mathbb{E}}$. By Lemma 3.6, T has to be added by $\mathbb{R}_{n+2} * (\dot{\mathbb{P}}_{n+2} \times \dot{\mathbb{E}}_{n+3})$ and as this forcing is isomorphic to $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+3}$, T is in the generic extension by $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+3}$.

⁹Note that \mathbb{E}_{n+1} is κ_{n+1} -cc in $V[\mathbb{R}_n]$ by Corollary 3.4 and it remains κ_{n+1} -cc over the present model by Lemma 2.19(ii).

Now, work in $V[\mathbb{R}_{n+2}]$. In this model $\kappa^+ = \kappa_{n+3} = \aleph_{n+3}$ and by Lemma 3.3, \mathbb{E}_{n+3} is κ^+ -cc. Therefore there is a nice \mathbb{E}_{n+3} -name \dot{T} for T of size κ . Such a nice name contains at most κ -many conditions in \mathbb{E}_{n+3} , hence we can restrict each $\text{Add}(\kappa_i, e(\kappa_i))$ (if necessary) in the product \mathbb{E}_{n+3} to $\text{Add}(\kappa_i, A_i)$, where A_i has size at most κ and it is determined by the support of conditions in \dot{T} . The claim now follows as any bijection between A_i and κ gives an isomorphism between $\text{Add}(\kappa_i, A_i)$ and $\text{Add}(\kappa_i, \kappa)$. \square

Let us denote $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+3} | \kappa$ by $\mathbb{R}_{n+2} * \dot{\mathbb{E}}_{n+3}$ in the interest of brevity and let us keep in mind that all the Cohen forcings in \mathbb{E}_{n+3} have length less than or equal to κ .

Let us fix some notation now. Let G_i denote a \mathbb{Q}_i -generic over $V[G_0][\dots][G_{i-1}]$, for each $i < n+2$, and x_i a $\mathbb{P}_i^{\mathbb{E}}$ -generic over $V[G_0][\dots][G_{n+1}][x_0][\dots][x_{i-1}]$ for each $i < n+3$. Let us denote by V_{n-1} the model $V[G_0][\dots][G_{n-1}]$ and let us write for brevity $x_{<i}$ instead of $x_0 \times \dots \times x_{i-1}$ for $i \leq n+3$.

3.3.1 LIFTING AN EMBEDDING

We wish to lift an appropriate embedding to the model $V_{n-1}[G_n][G_{n+1}][x_{<n+3}]$ which contains the tree T .

In V , using the Laver function F_{n+2} , let us choose a supercompact embedding $j : V \rightarrow M$ such that:

- (i) $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$.¹⁰
- (ii) $j(F_{n+2})(\kappa)$ is the canonical \mathbb{R}_n -name for the canonical \mathbb{Q}_n -name for $\mathbb{T}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$.

We are going to lift j first to the model $V_{n-1}[G_n][G_{n+1}][x_{n+2}]$. The argument is essentially the same as in [6], except that we have the extra forcing $\mathbb{P}_{n+2}^{\mathbb{E}}$. Let us review the basic steps of the lifting.

- As $j(\mathbb{R}_n) = \mathbb{R}_n$, we can lift the embedding from V_{n-1} to $M_{n-1} = M[G_0][\dots][G_{n-1}]$.
- Since j is identity below $\kappa = \kappa_{n+2}$, $j(\mathbb{Q}_n) | \kappa = \mathbb{Q}_n$ and we can lift the embedding further from $V_{n-1}[G_n]$ to $M_{n-1}[G_n][h_n]$ in $V_{n-1}[G_n][h_n]$, where h_n is $j(\mathbb{Q}_n)/G_n$ -generic over $V_{n-1}[G_n]$.
- Now work in $V_{n-1}[G_n][h_n]$ and define:

$$(3.19) \quad G_{n+1}^1 \times x_{n+2} = \{f(\kappa)^{G_n} \mid \text{for some } p, q, (p, q, f) \in G_n * h_n\}.$$

By our choice of j , $G_{n+1}^1 \times x_{n+2}$ is $\mathbb{T}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$ -generic over $V_{n-1}[G_n]$.

By the projection $\sigma_\kappa^{j(\kappa)}$ (see the analysis below Remark 2.16), $V_{n-1}[G_n][h_n] = V_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$ for some $j(\mathbb{Q}_n)/(G_n * (G_{n+1}^1 \times x_{n+2}))$ -generic filter h_n^* .

The family of conditions $j''(G_{n+1}^1 \times x_{n+2})$ has a lower bound $t = ((\emptyset, p_m, q_m), t_m)$ in the product forcing $j(\mathbb{T}_{n+1}) \times j(\mathbb{P}_{n+2}^{\mathbb{E}})$ because $j(\mathbb{T}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}})$ is $j(\kappa)$ -directed

¹⁰Recall that λ is the limit of the sequence of the supercompact cardinals $\langle \kappa_n \mid n < \omega \rangle$.

closed and $j(\kappa) > \lambda > \kappa_{n+3}$. The condition t can be used as a master condition for j and $\mathbb{Q}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$: if $H_{n+1} \times y_{n+2}$ is $j(\mathbb{Q}_{n+1}) \times j(\mathbb{P}_{n+2}^{\mathbb{E}})$ -generic over $V_{n-1}[G_n][h_n]$ and H_{n+1} contains (\emptyset, p_m, q_m) and y_{n+2} contains t_m , then $j^{-1''}(H_{n+1} \times y_{n+2})$ generates a $\mathbb{Q}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$ -generic over $V_{n-1}[G_n]$. Let us denote by $G_{n+1} \times x_{n+2}$ the $\mathbb{Q}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$ -generic over $V_{n-1}[G_n]$ generated by $j^{-1''}(H_{n+1} \times y_{n+2})$.

$$(3.20) \quad G_{n+1} \times x_{n+2} = \pi_{n+1}''(\rho_{n+1}'' G_{n+1} \times G_{n+1}^1) \times x_{n+2}.$$

Therefore we can lift the embedding to

$$(3.21) \quad j : V_{n-1}[G_n][G_{n+1}][x_{n+2}] \rightarrow M_{n-1}[G_n][h_n][H_{n+1}][y_{n+2}].$$

Note that the model $M_{n-1}[G_n][h_n][H_{n+1}][y_{n+2}]$ is the same as $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*][H_{n+1}][y_{n+2}]$.

Now we need to lift j further to \mathbb{E}_{n+2} . Since j is identity below κ and $\mathbb{E}_{n+2} = \prod_{i < n+2} \mathbb{P}_i^{\mathbb{E}}$, j is the identity on conditions in \mathbb{E}_{n+2} . For each $i < n+2$, $j(\mathbb{P}_i^{\mathbb{E}}) = \mathbb{P}_i^{\mathbb{E}} \times j(\mathbb{P}_i^{\mathbb{E}})|[\kappa, j(\kappa)]$ ¹¹. Therefore we can lift the embedding further from the model $V_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{<n+2}]$ to $M_{n-1}[G_n][h_n][H_{n+1}][y_{n+2}][y_{<n+2}]$, where

- $y_{<n+2}$ denotes $y_0 \times \cdots \times y_{n+1}$ and
- for each $i < n+2$ there is x_i^* such that $y_i = x_i \times x_i^*$ and y_i is $j(\mathbb{P}_i^{\mathbb{E}})$ -generic over $V_{n-1}[G_n][h_n][H_{n+1}][y_{n+2}][y_{<i}]$.

Let us write the model $M_{n-1}[G_n][h_n][H_{n+1}][y_{n+2}][y_{<n+2}]$ equivalently as

$$(3.22) \quad M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*][H_{n+1}][y_{n+2}][y_{<n+2}].$$

We will rearrange the generics to be able to argue for the tree property in the next section.

H_{n+1} is $j(\mathbb{Q}_{n+1})$ -generic over the $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$ and by applying the projection $\rho_{n+1}^* : j(\mathbb{Q}_{n+1}) \rightarrow j(\mathbb{P}_{n+1})$ we get a $j(\mathbb{P}_{n+1})$ -generic; let us denote it by H_{n+1}^0 and let us also denote by H_{n+1}^1 a $j(\mathbb{D}_{n+1}) = j(\mathbb{Q}_{n+1})/H_{n+1}^0$ -generic over $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*][H_{n+1}^0]$ such that $H_{n+1} = H_{n+1}^0 * H_{n+1}^1$. Now the model (3.22) is equal to

$$(3.23) \quad M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*][H_{n+1}^0][H_{n+1}^1][y_{n+2}][y_{<n+2}].$$

The elementary embedding j is in particular a regular embedding from \mathbb{P}_{n+1} to $j(\mathbb{P}_{n+1})$ and therefore $j^{-1''}H_{n+1}^0$ yields a generic filter for \mathbb{P}_{n+1} over $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$. Let us denote this generic by G_{n+1}^0 and let h_{n+1}^0 be a generic filter such that $G_{n+1}^0 \times h_{n+1}^0 = H_{n+1}^0$. Therefore the model (3.23) can be decomposed further as

$$(3.24) \quad M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*][G_{n+1}^0][h_{n+1}^0][H_{n+1}^1][y_{n+2}][y_{<n+2}].$$

¹¹Note that $j(\mathbb{P}_i^{\mathbb{E}})|[\kappa, j(\kappa)]$ is isomorphic to $j(\mathbb{P}_i^{\mathbb{E}})$ therefore for simplification of the notation we will write $j(\mathbb{P}_i^{\mathbb{E}})$ instead of $j(\mathbb{P}_i^{\mathbb{E}})|[\kappa, j(\kappa)]$

Now note that \mathbb{P}_{n+1} lives already in M_{n-1} and as G_{n+1}^0 is generic over the model $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$, G_{n+1}^0 and h_n^* are mutually generic over $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}]$ and also G_{n+1}^0 , G_{n+1}^1 and x_{n+2} are mutually generic over $M_{n-1}[G_n]$. Therefore we can rearrange model (3.24) as

$$(3.25) \quad M_{n-1}[G_n][G_{n+1}^0 \times G_{n+1}^1 \times x_{n+2}][h_n^*][h_{n+1}^0][H_{n+1}^1][y_{n+2}][y_{<n+2}].$$

Recall that there is the projection $\pi_{n+1} : \mathbb{P}_{n+1} \times \mathbb{T}_{n+1} \rightarrow \mathbb{Q}_{n+1}$.¹² Therefore we can rewrite the model (3.25) as

$$(3.26) \quad M_{n-1}[G_n][G_{n+1}][G_{\mathbb{S}}][x_{n+2}][h_n^*][h_{n+1}^0][H_{n+1}^1][y_{n+2}][y_{<n+2}],$$

where $G_{\mathbb{S}}$ is \mathbb{S}_{n+1} -generic over $M_{n-1}[G_n][G_{n+1}]$ such that $G_{n+1}^0 \times G_{n+1}^1 = G_{n+1} * G_{\mathbb{S}}$. Recall that \mathbb{S}_{n+1} is the quotient forcing $\mathbb{P}_{n+1} \times \mathbb{T}_{n+1}/G_{n+1}$.

Finally, for each $i < n + 2$, $y_i = x_i \times x_i^*$, hence we can write the model (3.26) as follows:

$$(3.27) \quad M_{n-1}[G_n][G_{n+1}][G_{\mathbb{S}}][x_{n+2}][h_n^*][h_{n+1}^0][H_{n+1}^1][y_{n+2}][x_0 \times x_0^*][\dots][x_{n+1} \times x_{n+1}^*],$$

and again by mutual genericity we can rearrange the generic filters in (3.27) as follows:

$$(3.28) \quad M_{n-1}[G_n][G_{n+1}][x_{<n+3}][x_{<n+2}^*][h_{n+1}^0][G_{\mathbb{S}}][h_n^*][H_{n+1}^1][y_{n+2}].$$

3.3.2 THE TREE PROPERTY ARGUMENT

Recall that we assume that T is κ -Aronszajn tree in $V_{n-1}[G_n][G_{n+1}][x_{<n+3}]$. By the closure properties of the models, we can assume that T is also in $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$. As $j(T) \upharpoonright \kappa = T$, T has a cofinal branch in model (3.28). We will argue that the forcing from $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$ to the model (3.28) cannot add a cofinal branch to T over $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$. This will contradict the assumption that T is a κ -Aronszajn tree in $V_{n-1}[G_n][G_{n+1}][x_{<n+3}]$, and conclude the whole proof.

First we show that there are no cofinal branches in T in the smaller model:

$$(3.29) \quad M_{n-1}[G_n][G_{n+1}][x_{<n+3}][x_{<n+2}^*][h_{n+1}^0][G_{\mathbb{S}}][h_n^*].$$

Let us work for a while in $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}]$; h_n^* is $j(\mathbb{Q}_n)/(G_n * (G_{n+1}^1 \times x_{n+2}))$ -generic over this model and there is a projection $\pi_n^* : j(\mathbb{P}_n) \times \mathbb{T}_n^* \rightarrow j(\mathbb{Q}_n)/(G_n * (G_{n+1}^1 \times x_{n+2}))$. Therefore we can find $h_n^{*0} \times h_n^{*1}$ which is $j(\mathbb{P}_n) \times \mathbb{T}_n^*$ -generic over

$$M_{n-1}[G_n][G_{n+1}][x_{<n+3}][x_{<n+2}^*][G_{\mathbb{S}}][h_{n+1}^0]$$

such that $\pi_n^{*''}(h_n^{*0} \times h_n^{*1}) = h_n^*$.

¹² $\pi_{n+1}''(G_{n+1}^0 \times G_{n+1}^1) = G_{n+1}$

In order to argue that there are no cofinal branches through T in the model (3.29), it is enough to show that there are no such branches in the larger model:

$$(3.30) \quad M_{n-1}[G_n][G_{n+1}][x_{<n+3}][x_{<n+2}^*][h_n^{*0}][h_{n+1}^0][G_{\mathbb{S}}][h_n^{*1}].^{13}$$

We divide the proof of the proposition that T has no cofinal branch in (3.30) into two claims: First we use the κ -square-cc of the Cohen forcings which add the generic $x_{<n+2}^* \times h_n^{*0} \times h_{n+1}^0$ to show that they do not add cofinal branches to T , and then we use the closure property of forcings which add $G_{\mathbb{S}} * h_n^{*1}$ to show that they cannot add a cofinal branch to T either.

Claim 3.9 $j(\mathbb{E}_{n+2}) \times j(\mathbb{P}_n) \times j(\mathbb{P}_{n+1})$ is κ -square-cc in $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$.

PROOF. First note that the product $j(\mathbb{E}_{n+2}) \times j(\mathbb{P}_n) \times j(\mathbb{P}_{n+1})$ is isomorphic to $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ as $\mathbb{P}_n^{\mathbb{E}} \times \mathbb{P}_n$ is isomorphic to $\mathbb{P}_n^{\mathbb{E}}$, and $\mathbb{P}_{n+1}^{\mathbb{E}} \times \mathbb{P}_{n+1}$ is isomorphic to \mathbb{P}_{n+1} ¹⁴. Also note that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is isomorphic to its square. Hence to show that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1}) \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc, it suffices to show that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc.

In $M_{n-1}[G_n][G_{n+1}][x_{n+2}]$, $\mathbb{E}_{n+2} \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is isomorphic to $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$; if we show that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc in this model, we conclude that that $\mathbb{E}_{n+2} \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc, i.e. \mathbb{E}_{n+2} forces that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc, which implies $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc in $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$.

To show that $j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_{n+1})$ is κ -cc in $M_{n-1}[G_n][G_{n+1}][x_{n+2}]$, we proceed as in the proof of Lemma 3.3(c). \square

Since $j(\mathbb{E}_{n+2}) \times j(\mathbb{P}_n) \times j(\mathbb{P}_{n+1})$ is κ -square-cc in $M_{n-1}[G_n][G_{n+1}][x_{<n+3}]$, there are no cofinal branches through T in

$$(3.31) \quad M_{n-1}[G_n][G_{n+1}][x_{<n+3}][x_{<n+2}^*][h_n^{*0}][h_{n+1}^0],$$

by Fact 2.9

Claim 3.10 In the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0]$ the following hold:

- (i) $\mathbb{S}_{n+1} * \mathbb{T}_n^*$ is κ_{n+1} -closed.
- (ii) $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_n)$ is κ_{n+1} -cc.

PROOF. (i) The forcing \mathbb{S}_{n+1} lives in $M_{n-1}[G_n][G_{n+1}]$ and it is κ_{n+1} -closed there, but it is also κ_{n+1} -closed in $M_{n-1}[G_n][G_{n+1}][x_{n+2}]$ by Lemma 2.5(ii) as $\mathbb{P}_{n+2}^{\mathbb{E}}$ is κ_{n+1} -closed in $M_{n-1}[G_n][G_{n+1}]$ by Lemma 3.3(a).

Now, the term forcing \mathbb{T}_n^* lives in $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}]$ and it is κ_{n+1} -closed there. The model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][G_{\mathbb{S}}]$ is equal to $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2} \times G_{n+1}^0]$.

¹³Note that in contrast to h_n^{*1} , we can put h_n^{*0} before $G_{\mathbb{S}}$ as it is generic for the Cohen forcing $j(\mathbb{P}_n)$ and it already lives in V_{n-2} .

¹⁴Note that \mathbb{P}_{n+1} has length κ_{n+3} hence $\mathbb{P}_{n+1}^{\mathbb{E}} \times \mathbb{P}_{n+1}$ is not isomorphic to $\mathbb{P}_{n+1}^{\mathbb{E}}$ as this has length less or equal κ

Therefore to show that \mathbb{T}_n^* is κ_{n+1} -closed here it is enough to show that it stay closed after forcing with \mathbb{P}_{n+1} , but this holds by Lemma 2.5(ii) as \mathbb{P}_{n+1} is κ_{n+1} -distributive in $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}]$.¹⁵

By the previous two paragraphs, $\mathbb{S}_{n+1} * \mathbb{T}_n^*$ is κ_{n+1} -closed in $M_{n-1}[G_n][G_{n+1}][x_{n+2}]$. Now, the product of Cohen forcings which add the generic filter $y_{n+1} \times h_{n+1}^0$ is isomorphic to $j(\mathbb{P}_{n+1})$. This forcing $j(\mathbb{P}_{n+1})$ is κ_{n+1} -distributive in $M_{n-1}[G_n][G_{n+1}][x_{n+2}]$ by Lemma 3.3(b); therefore the forcing $\mathbb{S}_{n+1} * \mathbb{T}_n^*$ remains κ_{n+1} -closed in the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0]$ by Lemma 2.5(ii) as required.

(ii) As before, the product $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_n)$ is isomorphic to $j(\mathbb{E}_{n+1})$ and the proof that this forcing is κ_{n+1} -cc is as in the proof of Lemma 3.3(c). \square

Now we can apply Fact 2.10 to $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1}) \times j(\mathbb{P}_n)$ as P and $\mathbb{S}_{n+1} * \mathbb{T}_n^*$ as Q over the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0]$. Therefore there are no cofinal branches in T in the model (3.30) and hence neither in the model (3.29).

To finish the proof of the tree property at κ it is enough to show that $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ cannot add a cofinal branch to T over the model (3.29).

Claim 3.11 *In the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0][G_S][h_n^*]$ the following hold:*

- (i) $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ is κ_{n+1} -closed.
- (ii) $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1})$ is κ_{n+1} -cc.

PROOF. (i) First, the forcing $j(\mathbb{D}_{n+1})$ lives in $M_{n-1}[G_n][G_{n+1}][x_{n+2}][h_{n+1}^0][G_S][h_n^*]$ and it is κ_{n+1} -closed there.

Second, $j(\mathbb{P}_{n+2})$ lives in $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$ and it is κ_{n+1} -closed there. To get from model $M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$ to $M_{n-1}[G_n][G_{n+1}^0 \times G_{n+1}^1 \times x_{n+2}][h_{n+1}^0][h_n^*] = M_{n-1}[G_n][G_{n+1}][x_{n+2}][G_S][h_n^*][h_{n+1}^0]$ it suffices to force with $j(\mathbb{P}_{n+1})$, which adds a generic filter for $G_{n+1}^0 \times h_{n+1}^0$. This forcing lives in M_{n-1} and it is κ_{n+1} -distributive in $M_{n-1}[G_n][h_n] = M_{n-1}[G_n][G_{n+1}^1 \times x_{n+2}][h_n^*]$ by Fact 2.22(v) or by Lemma 2.19. Therefore $j(\mathbb{P}_{n+2})$ remains κ_{n+1} -closed in $M_{n-1}[G_n][G_{n+1}][x_{n+2}][G_S][h_n^*][h_{n+1}^0]$ by Lemma 2.5(ii).

As both forcings are κ_{n+1} -closed in

$$(3.32) \quad M_{n-1}[G_n][G_{n+1}][x_{n+2}][h_{n+1}^0][G_S][h_n^*],$$

their product $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ is κ_{n+1} -closed as well. The difference between the model (3.32) and the model

$$(3.33) \quad M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0][G_S][h_n^*]$$

– where we want to show that $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ is κ_{n+1} -closed – is just the forcing $j(\mathbb{P}_{n+1}^{\mathbb{E}})$ which adds the generic filter y_{n+1} . Therefore to finish the proof of the claim

¹⁵ \mathbb{P}_{n+1} is κ_{n+1} -distributive in $M_{n-1}[G_n]$ by Fact 2.22(v) and it stay κ_{n+1} -distributive by Lemma 2.5(i) after forcing with $\mathbb{T}_{n+1} \times \mathbb{P}_{n+2}^{\mathbb{E}}$ as this forcing is κ_{n+1} -closed in $M_{n-1}[G_n]$.

it suffices to show that $j(\mathbb{P}_{n+1})$ is κ_{n+1} -distributive in model (3.32). The model (3.32) is actually equal to

$$(3.34) \quad M_{n-1}[G_n][h_n][G_{n+1}^0][h_{n+1}^0].$$

By Lemma 2.19(iv), $j(\mathbb{P}_{n+1}) \times j(\mathbb{P}_{n+1}^{\mathbb{E}})$ is κ_{n+1} -distributive in $M_{n-1}[G_n][h_n][G_{n+1}^0]$. Therefore $j(\mathbb{P}_{n+1})$ forces that $j(\mathbb{P}_{n+1}^{\mathbb{E}})$ is κ_{n+1} -distributive and so $j(\mathbb{P}_{n+1})$ is κ_{n+1} -distributive in the model (3.32). Now we can apply Lemma 2.5(ii) to $j(\mathbb{P}_{n+1}^{\mathbb{E}})$ and $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ over the model (3.32) and conclude that $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ is κ_{n+1} -closed in (3.33).

(ii) Recall that the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0][G_{\mathbb{S}}][h_n^*]$ is equal to

$$(3.35) \quad M_{n-1}[G_n][h_n][G_{n+1}^0][h_{n+1}^0][y_{n+1}].$$

The proof that $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1})$ – which is isomorphic to $j(\mathbb{E}_{n+1})$ – is κ_{n+1} -cc in this model proceeds exactly as in the proof of Lemma 3.3(c). \square

By the previous claim, we can apply Fact 2.10 to $\mathbb{E}_{n+1} \times j(\mathbb{E}_{n+1})$ as P and $j(\mathbb{D}_{n+1}) \times j(\mathbb{P}_{n+2})$ as Q over the model $M_{n-1}[G_n][G_{n+1}][x_{n+2}][y_{n+1}][h_{n+1}^0][G_{\mathbb{S}}][h_n^*]$ and conclude that there are no cofinal branches in T in the model (3.28). This is a contradiction which finishes the proof of Theorem 3.7.

4 OPEN QUESTIONS

For the first question below, let us assume $e : \omega \rightarrow \omega$ satisfies $n < m \rightarrow e(n) \leq e(m)$ and $e(n) > n + 1$ for all $n, m < \omega$.

Question 4.1 Is it possible to have the tree property at every \aleph_n , $1 < n < \omega$, with $2^{\aleph_n} = \aleph_{e(n)}$, $n < \omega$, and $2^{\aleph_\omega} = \aleph_{\omega+m}$ for a prescribed $1 < m < \omega$? (Note that in our model we have $2^{\aleph_\omega} = \aleph_{\omega+1}$.)

A partial answer to this question was given by Honzik and Friedman in [11], who showed that that $2^{\aleph_\omega} = \aleph_{\omega+2}$ is consistent with the tree property at every even cardinal below \aleph_ω . However, this method does not seem to be appropriate for manipulating the continuum function as they used an iteration of the Sacks forcing, instead of the Mitchell forcing which allows greater flexibility. Unger [24] extended this result using the Cummings-Foreman method to show that $2^{\aleph_\omega} = \aleph_{\omega+2}$ is consistent with the tree property at every cardinal \aleph_n below \aleph_ω , for $n > 1$, with $2^{\aleph_n} = \aleph_{n+2}$ for each $n < \omega$.

Question 4.2 In our final model, can we in addition have the tree property at $\aleph_{\omega+2}$?

Note that this question is still open even with the trivial continuum function; i.e. with $2^{\aleph_n} = \aleph_{n+2}$ for $n < \omega$.

Question 4.3 Can we control generalized cardinal invariants together with the tree property? For instance, is it possible to combine the results of Cummings and Shelah in [7] for \mathfrak{d}_κ and \mathfrak{b}_κ with the tree property at relevant cardinals?

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