A STRONGLY NON-SATURATED ARONSZAJN TREE WITHOUT WEAK KUREPA TREES

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ABSTRACT. Assuming the negation of Chang's conjecture, there is a c.c.c. forcing which adds a strongly non-saturated Aronszajn tree. Using a Mahlo cardinal, we construct a model in which there exists a strongly non-saturated Aronszajn tree and the negation of the Kurepa hypothesis is c.c.c. indestructible. For any inaccessible cardinal κ, there exists a forcing poset which is Y-proper and κ-c.c., collapses κ to become ω_2, and adds a strongly non-saturated Aronszajn tree. The quotients of this forcing in intermediate extensions are indestructibly Y-proper on a stationary set with respect to any Y-proper forcing extension. As a consequence, we prove from an inaccessible cardinal that the existence of a strongly non-saturated Aronszajn tree is consistent with the non-existence of a weak Kurepa tree. Finally, we prove from a supercompact cardinal that the existence of a strongly non-saturated Aronszajn tree is consistent with two-cardinal tree properties such as the indestructible guessing model principle.

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1. INTRODUCTION

An Aronszajn tree is *saturated* if every family of uncountable downwards closed subtrees of it which is almost disjoint has cardinality less than ω_2. By almost disjoint we mean that any

Date: June 26, 2025.
2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E40.
Key words and phrases. Strongly non-saturated Aronszajn tree, weak Kurepa tree, side conditions, Y-proper.

two members of the family have countable intersection. This idea was introduced by König, Larson, Moore, and Veličković in the context of a study of the consistency strength of Moore's five element basis theorem ([KLMV08], [Moo06]). A classic example of an Aronszajn tree which is non-saturated is due to Todorčević, who showed that if U is a special Aronszajn tree and T is a Kurepa tree, then the tree product $U \otimes T$ is a special Aronszajn tree which has no base of subtrees of cardinality less than ω_2 ([Bau85]). Moore asked whether the existence of a non-saturated Aronszajn tree implies the existence of a Kurepa tree ([Moo08]). This problem was solved by the authors by showing that if κ is an inaccessible cardinal and T is a free Suslin tree, then there is a generic extension in which $\kappa = \omega_2$, the square $T \otimes T$ is a non-saturated Aronszajn tree, and there does not exist a Kurepa tree ([KS24]).

Martinez Mendoza and the author introduced a stronger form of non-saturation for an Aronszajn tree T. Given subtrees U and W of T, we say that U and W are *strongly almost disjoint* if $U \cap W$ is a finite union of countable chains (equivalently, $U \cap W$ does not contain an infinite antichain). A stronger property that we will use is that $U \cap W$ is contained in the downward closure of a finite subset of T, which we refer to as $U \cap W$ being *finitely generated*. The tree T is *strongly non-saturated* if there exists a family of ω_2-many strongly almost disjoint uncountable downwards closed subtrees of T ([KMM24]). Note that if there exists such a tree, then CH is false.

The idea of a strongly non-saturated Aronszajn tree is a relative of that of a collection of uncountable subsets of ω_1 with size ω_2 such that the intersection of any two members of the collection is finite. The existence of such a family was proven to be consistent by Baumgartner [Bau76]. Indeed, if there is a strongly non-saturated Aronszajn tree, then there is such a collection.[1] The consistency of a strongly non-saturated Aronszajn tree was proven by Martinez Mendoza and the author by showing, using the ρ-function of Todorčević [Tod87] under the assumption of $\square_{\omega_1}$, that there exists a c.c.c. forcing which adds an almost Kurepa Suslin tree whose square is strongly non-saturated ([KMM24])

With this stronger version of non-saturation for Aronszajn trees at hand, it is natural to ask whether the existence of a strongly non-saturated Aronszajn tree is consistent with the non-existence of a Kurepa tree, or even with the non-existence of a weak Kurepa tree. The latter question was asked in [KMM24]. The connection with the weak Kurepa hypothesis is that the existence of a weak Kurepa tree follows from CH and thus holds in the model of [KS24]. Both questions can be thought of as more ambitious versions of Moore's problem stated above which are suitable in the context of the negation of CH.

In order to solve these problems, we introduce two new forcing posets for adding a strongly non-saturated Aronszajn tree, both with finite conditions. The first forcing poset is c.c.c. assuming the negation of Chang's conjecture. For this forcing, the main idea is to use a weak kind of ρ-function to bound the intersection of the subtrees appearing in a condition. The existence of such a function follows from the negation of Chang's conjecture ([Tod91]). Using

[1] Without loss of generality, assume that the strongly non-saturated Aronszajn tree T is Hausdorff. Let $\{U_\alpha : \alpha < \omega_2\}$ be a strongly almost disjoint family of uncountable downwards closed subtrees of T. For each $\alpha < \omega_2$, define W_α to be the collection of all sets $\{x, y, z\} \subseteq U_\alpha$ such that y and z are distinct immediate successors of x in U_α. Then for all $\alpha < \beta < \omega_2$, the fact that $U_\alpha \cap U_\beta$ contains no infinite antichain implies by a straightforward argument that $W_\alpha \cap W_\beta$ is finite. So $\{W_\alpha : \alpha < \omega_2\}$ is a strongly almost disjoint family of uncountable subsets of $[T]^3$.

this forcing poset in combination with a result of Jensen and Schlechta [JS90], we have the following theorem which provides a strong answer to the first question.

Theorem 1. *Assume that κ is a Mahlo cardinal. Then there is a generic extension of L in which there exists a strongly non-saturated Aronszajn tree and the negation of the Kurepa hypothesis is c.c.c. indestructible.*

For the second problem, for any inaccessible cardinal κ we prove that there exists a forcing poset with finite conditions which adds a strongly non-saturated Aronszajn tree, collapses κ to become ω_2, and has quotients in intermediate extensions which do not add new cofinal branches to trees of height ω_1. The conditions in this forcing consist of two parts: a working part which is a finite approximation of the generic tree and its subtrees, and a side condition which is a finite set of countable models. The interaction between the working part and the side condition is that whenever the indices of two of the subtrees from the working part are members of a model in the side condition, then the intersection of the subtrees is a subset of the model as well. Using this forcing together with a standard Silver factor analysis ([Sil71]), we have the following theorem.

Theorem 2. *Assume that κ is an inaccessible cardinal. Then there exists a forcing poset which is proper, collapses κ to become ω_2, adds a strongly non-saturated Aronszajn tree, and forces the non-existence of a weak Kurepa tree.*

In order to prove that the quotient forcings do not add new cofinal branches, we use the concept of Y-properness which was recently introduced by Chodounský and Zapletal [CZ15]. This property implies not only not adding new cofinal branches to trees of height ω_1, but also the stronger ω_1-approximation property. As a result, we are able to prove much more than the negation of the weak Kurepa hypothesis. In fact, not only are the quotients of our forcing poset Y-proper, but they remain Y-proper on a stationary set after any further Y-proper forcing. This enables us to do additional forcing after the main forcing while still allowing for the usual factor analysis. In particular, we have the following theorem related to two-cardinal tree principles.

Theorem 3. *Assume that κ is a supercompact cardinal. Then there is a generic extension in which $\kappa = \omega_2$, there exists a strongly non-saturated Aronszajn tree, and the indestructible guessing model principle* IGMP *holds.*

We note that IGMP follows from PFA, whereas PFA implies that all Aronszajn trees are saturated ([CK17], [KLMV08]).

We assume that the reader has a background in ω_1-trees and forcing. We refer the reader to [KS24, Section 1] for basic definitions and terminology concerning trees. An *ω_1-tree* is a tree with height ω_1 and countable levels. An *Aronszajn tree* is an ω_1-tree with no cofinal branch. A *Kurepa tree* is an ω_1-tree with at least ω_2-many cofinal branches, and a *weak Kurepa tree* is a tree with height and size ω_1 which has at least ω_2-many cofinal branches. The *Kurepa hypothesis* is the statement that there exists a Kurepa tree, and the *weak Kurepa hypothesis* is the statement that there exists a weak Kurepa tree.

We make use of the following fact about Aronszajn trees, which follows from a classic theorem of Baumgartner-Malitz-Reinhardt [BMR70] together with the Dushnik-Miller theorem $\omega_1 \to (\omega_1, \omega)^2$. Suppose that T is a tree with no uncountable chains and $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a sequence of disjoint finite subsets of T. Then there exists a countably infinite set $Y \subseteq \omega_1$ such that for all distinct α and β in Y, for all $x \in x_\alpha$ and for all $y \in x_\beta$, x and y are incomparable in T.

2. Finite Trees and Subtrees

For the remainder of the article, fix a regular cardinal $\kappa \geq \omega_2$. In Section 4, we assume that κ is equal to ω_2, and in Section 5 and for the rest of the article we assume that κ is an inaccessible cardinal. In Sections 3 and 6 we define two forcing posets for adding a strongly non-saturated Aronszajn tree. The forcing introduced in Section 3 is c.c.c., and the forcing introduced in Section 6 is proper and involves countable models as side conditions. These two forcing posets will have a part in common which adds the Aronszajn tree. In this section, we work out the details of this common part.

Define $h : \omega_1 \to \omega_1$ by letting $h(\alpha)$ be the unique ordinal γ such that $\omega \cdot \gamma \leq \alpha < \omega \cdot (\gamma + 1)$. Let C_h be the club set of ordinals in ω_1 which are closed under the function which maps any $\gamma < \omega_1$ to $\omega \cdot (\gamma + 1)$. Note that if $\delta \in C_h$, then $\alpha < \delta$ iff $h(\alpha) < \delta$. The function h will coincide with the height function of the generic trees of Sections 3 and 6.

Definition 2.1. *A* standard finite tree *is an ordered pair* $(T, <_T)$ *satisfying:*

1. T *is a finite subset of* $\{0\} \cup (\omega_1 \setminus \omega)$*;*
2. $<_T$ *is a strict partial ordering of* T *such that for any* $x \in T$*, the set* $\{y \in T : y <_T x\}$ *is linearly ordered by* $<_T$*;*
3. *if* $x <_T y$*, then* $h(x) < h(y)$*;*
4. *if* T *is non-empty, then* $0 \in T$ *and* $0 <_T x$ *for all non-zero* $x \in T$.

If $(T, <_T)$ *and* $(U, <_U)$ *are standard finite trees,* $(U, <_U)$ *is an* end-extension *of* $(T, <_T)$ *if* $T \subseteq U$ *and* $<_U \cap T^2 = <_T$.

We will abbreviate a standard finite tree $(T, <_T)$ as just T. If T is a standard finite tree and $\delta < \omega_1$, let $T \upharpoonright \delta$ be equal to $(T \cap \delta, <_T \cap \delta^2)$. Note that $T \upharpoonright \delta$ is also a standard finite tree. Observe that if T is a standard finite tree, then for any incomparable elements x and y of T, there exists a $<_T$-largest $z \in T$ such that $z <_T x, y$. We use the notation $h[T]$ for $\{h(x) : x \in T\}$.

Definition 2.2. *A standard finite tree* T *is* downwards closed *if whenever* $x \in T$ *and* $\alpha \in h[T] \cap h(x)$*, then there exists some* $z \in T$ *with* $z <_T x$ *and* $h(z) = \alpha$.

Definition 2.3. *A standard finite tree* T *has* minimal splits *if whenever* x *and* y *are incomparable elements of* T *and* z *is the largest element of* T *below both* x *and* y*, then there exist distinct* x_0 *and* y_0 *such that* $z <_T x_0 \leq_T x$*,* $z <_T y_0 \leq_T y$*, and* $h(x_0) = h(y_0) = h(z) + 1$.

Definition 2.4. *Let* T *be a standard finite tree. A* subtree *of* T *is an ordered pair* $(W, <_W)$*, where* $W \subseteq T$ *and* $<_W = <_T \cap W^2$.

Any subset W of T can be considered as a subtree of T with the induced tree order $<_W = <_T \cap W^2$. We will abbreviate a subtree $(W, <_W)$ as just W.

Definition 2.5. *Let* T *be a standard finite tree and let* W *be a subtree of* T*. We say that* W *is* downwards closed in T *if whenever* $x \in W$ *and* $y <_T x$*, then* $y \in W$*. If* W *is a subtree of* T*, then the* downward closure of W in T *is the set of* $y \in T$ *such that for some* $x \in W$*,* $y \leq_T x$.

Definition 2.6. *Let T be a standard finite tree. A* subtree function on T *is a function W whose domain is a finite subset of κ such that for all $\eta \in \mathrm{dom}(W)$, $W(\eta)$ is a downwards closed subtree of T.*

If W is a subtree function on T and $\eta \in \kappa$, we will occasionally write $W(\eta)$ even if we do not know whether or not $\eta \in \mathrm{dom}(W)$; in the case that it is not, $W(\eta)$ should be taken to mean the empty-set.

Definition 2.7. *Let $1 < d < \omega$ and suppose that $T_0, \ldots, T_{d-1}$ are standard finite trees. Define $T_0 \oplus \cdots \oplus T_{d-1}$ to be the ordered pair*

$$(T_0 \cup \cdots \cup T_{d-1}, <_{T_0} \cup \cdots \cup <_{T_{d-1}}).$$

Definition 2.8. *Let $1 < d < \omega$. Suppose that for all $i < d$, T_i is a standard finite tree and W_i is a subtree function on T_i. Define $W_0 \oplus \cdots \oplus W_{d-1}$ to be the function with domain equal to $\mathrm{dom}(W_0) \cup \cdots \cup \mathrm{dom}(W_{d-1})$ such that for all $\eta \in \mathrm{dom}(W_0 \oplus \cdots \oplus W_{d-1})$,*

$$(W_0 \oplus \cdots \oplus W_{d-1})(\eta) = W_0(\eta) \cup \cdots \cup W_{d-1}(\eta).$$

In general, $T_0 \oplus \cdots \oplus T_{d-1}$ is not necessarily a standard finite tree, and even if it is, then $W_0 \oplus \cdots \oplus W_{d-1}$ is not necessarily a subtree function on it.

We define an auxiliary forcing $\mathbb{P}^*$ which will assist with our main forcings $\mathbb{P}'$ and $\mathbb{P}$ introduced in Sections 3 and 6. We will never force with $\mathbb{P}^*$ itself.

Definition 2.9. *Let $\mathbb{P}^*$ be the forcing poset consisting of conditions which are triples (T, W, D) satisfying:*

(1) *T is a standard finite tree;*
(2) *W is subtree function on T;*
(3) *$D \subseteq [\mathrm{dom}(W)]^2$.*

Let $(U, Y, E) \leq (T, W, D)$ if:

(a) *U end-extends T;*
(b) *$\mathrm{dom}(W) \subseteq \mathrm{dom}(Y)$ and for all $\eta \in \mathrm{dom}(W)$, $W(\eta) \subseteq Y(\eta)$;*
(c) *$D \subseteq E$;*
(d) *if $\{\eta, \xi\} \in D$ and x is in $Y(\eta) \cap Y(\xi)$, then there exists some $z \in W(\eta) \cap W(\xi)$ such that $x \leq_U z$.*

Notation 2.10. *For any $p \in \mathbb{P}^*$, we write (T_p, W_p, D_p) for p, and we write $<_p$ for $<_{T_p}$*

Lemma 2.11. *For any $p \in \mathbb{P}^*$, there exists $q \leq p$ in $\mathbb{P}^*$ such that T_q is downwards closed and has minimal splits, $D_q = D_p$, $\mathrm{dom}(W_q) = \mathrm{dom}(W_p)$, and for all $\eta \in \mathrm{dom}(W_q)$, $W_q(\eta)$ is the downward closure of $W_p(\eta)$ in T_q. Moreover, for all distinct $\eta, \xi \in \mathrm{dom}(W_p)$, if $x \in W_q(\eta) \cap W_q(\xi)$, then there exists some $z \in W_p(\eta) \cap W_p(\xi)$ such that $x \leq_q z$.*

Proof. The proof is straightforward and we leave it as an exercise for the interested reader. □

Definition 2.12. *Let $1 < d < \omega$. Let $p_0, \ldots, p_{d-1}$ be in $\mathbb{P}^*$. Define $p_0 \oplus \cdots \oplus p_{d-1}$ to be the triple (T, W, D) satisfying:*

(1) *$T = T_0 \oplus \cdots \oplus T_{d-1}$;*
(2) *$W = W_0 \oplus \cdots \oplus W_{d-1}$;*
(3) *$D = D_0 \cup \cdots \cup D_{d-1}$.*

In general, $p_0 \oplus \cdots \oplus p_{d-1}$ is not necessarily a condition in $\mathbb{P}^*$, and even if it is, it is not necessarily an extension of $p_0, \ldots, p_{d-1}$.

Definition 2.13. *Let p and q be in $\mathbb{P}^*$ and let $\delta_p < \delta_q$ be in C_h. We say that the ordered pair (p, q) is (δ_p, δ_q)-*split *if the following are satisfied:*

(1) $T_p \restriction \delta_p = T_q \restriction \delta_q$;
(2) $T_p \subseteq \delta_q$;
(3) *for all* $\eta \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, $W_p(\eta) \cap \delta_p = W_q(\eta) \cap \delta_q$;
(4) *for all distinct* $\eta, \xi \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, $W_p(\eta) \cap W_p(\xi) \subseteq \delta_p$ *and* $W_q(\eta) \cap W_q(\xi) \subseteq \delta_q$.

Lemma 2.14. *Suppose that p and q are (δ_p, δ_q)-split, where $p, q \in \mathbb{P}^*$ and $\delta_p < \delta_q$ are in C_h. Then:*

(a) $T_p \cap T_q \subseteq \delta_p$;
(b) *if* $\eta \in \mathrm{dom}(W_p)$ *and* $\xi \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, *then* $W_p(\eta) \cap W_q(\xi) \subseteq W_p(\xi)$;
(c) *if* $\eta \in \mathrm{dom}(W_q)$ *and* $\xi \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, *then* $W_q(\eta) \cap W_p(\xi) \subseteq W_q(\xi)$.

Proof. (a) follows from Definition 2.13(1,2). For (b), assume that $x \in W_p(\eta) \cap W_q(\xi)$, and we show that $x \in W_p(\xi)$. Since $\xi \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, by Definition 2.13(3) we have that $W_p(\xi) \cap \delta_p = W_q(\xi) \cap \delta_q$. But $x \in W_p(\eta) \cap W_q(\xi)$ implies that $x \in T_p \cap T_q \subseteq \delta_q$. So $x \in W_q(\xi) \cap \delta_q \subseteq W_p(\xi)$. For (c), assume that $x \in W_q(\eta) \cap W_p(\xi)$, and we show that $x \in W_q(\xi)$. Since $\xi \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$, by Definition 2.13(3) we have that $W_p(\xi) \cap \delta_p = W_q(\xi) \cap \delta_q$. But $x \in W_q(\eta) \cap W_p(\xi)$ implies that $x \in T_p \cap T_q \subseteq \delta_p$. So $x \in W_p(\xi) \cap \delta_p \subseteq W_q(\xi)$. □

Lemma 2.15. *Let $1 < d < \omega$. Suppose that $p_0, \ldots, p_{d-1}$ are in $\mathbb{P}^*$ and $\delta_0 < \cdots < \delta_{d-1}$ are in C_h. Assume:*

(1) *for all* $i < j < d$, (p_i, p_j) *is* (δ_i, δ_j)*-split;*
(2) $\{\mathrm{dom}(W_i) : i < d\}$ *is a* Δ*-system.*

Then $p_0 \oplus \cdots \oplus p_{d-1}$ is a condition in $\mathbb{P}^*$ which extends $p_0, \ldots, p_{d-1}$.

Proof. For each $i < d$, write $p_i = (T_i, W_i, D_i)$, and write $p_0 \oplus \cdots \oplus p_{d-1} = (T, W, D)$. Let r be the root of the Δ-system of (2). We claim that T is a standard finite tree. For transitivity, suppose that $a <_T b <_T c$. If for some $i < d$, $a <_{T_i} b <_{T_i} c$, then we are done since p_i is a condition. Otherwise, for some distinct $i, j < d$, $a <_{T_i} b <_{T_j} < c$. Let $k = \min\{i, j\}$. Then $b \in T_i \cap T_j \subseteq \delta_k$. As $T_i \restriction \delta_k = T_j \restriction \delta_k$, $a <_{T_j} b$, so $a <_{T_j} c$ and we are done.

Now let $c \in T$ and assume that $a <_T c$ and $b <_T c$, where a and b are different. We will show that a and b are comparable in T. If for some $i < d$, $a <_i c$ and $b <_i c$, then we are done since p_i is a condition. Otherwise, for some distinct $i, j < d$, $a <_i c$ and $b <_j c$. Let $k = \min\{i, j\}$. Then $c \in T_i \cap T_j \subseteq \delta_k$, and since $T_i \restriction \delta_k = T_j \restriction \delta_k$, it follows that $a <_{T_i} c$ and $b <_{T_i} c$. Thus, a and b are comparable in T_i and therefore also in T. The other properties of T being a standard finite tree are obvious.

Since $\mathrm{dom}(W_i) \subseteq \mathrm{dom}(W)$ are for all $i < d$, obviously $D \subseteq [\mathrm{dom}(W)]^2$. We claim that for all $\eta \in \mathrm{dom}(W)$, $W(\eta)$ is a downwards closed subtree of T. Let $y \in W(\eta)$ and suppose that $x <_T y$. Fix $i, j < d$ such that $y \in W_i(\eta)$ and $x <_{T_j} y$. If $i = j$, then we are done since p_i is a condition. Assume that $i \neq j$. Let $k = \min\{i, j\}$. Then $y \in T_i \cap T_j \subseteq \delta_k$. As $T_i \restriction \delta_k = T_j \restriction \delta_k$, $x <_{T_i} y$. Since $y \in W_i(\eta)$ and $W_i(\eta)$ is downwards closed in T_i, $x \in W_i(\eta)$.

This completes the proof that (T, W, D) is in $\mathbb{P}^*$. It remains to show that for all $i < d$, (T, W, D) is an extension of (T_i, W_i, D_i). Clearly, $T_i \subseteq T$ and $<_{T_i} \subseteq <_T \cap T_i^2$. To show that T is an end-extension of T_i, suppose that $a, b \in T_i$ and $a <_T b$. If $a <_i b$, then we are done. Otherwise, for some $j < d$ different from i, $a <_j b$. Let $k = \min\{i, j\}$. Then $a, b \in T_i \cap T_j \subseteq \delta_k$, and since $T_i \restriction \delta_k = T_j \restriction \delta_k$, $a <_i b$. This proves that T end-extends T_i. It is clear that $\mathrm{dom}(W_i) \subseteq \mathrm{dom}(W)$ and for all $\eta \in \mathrm{dom}(W_i)$, $W_i(\eta) \subseteq W(\eta)$. And obviously $D_i \subseteq D$.

Finally, let $\{\eta, \xi\} \in D_i$ and let $x \in W(\eta) \cap W(\xi)$. We find some $z \in W_i(\eta) \cap W_i(\xi)$ such that $x \leq_T z$. In fact, we prove that $x \in W_i(\eta) \cap W_i(\xi)$, so $x = z$ works. Note that $\eta, \xi \in \mathrm{dom}(W_i)$. Fix $l, m < d$ such that $x \in W_l(\eta) \cap W_m(\xi)$. Assume first that one of l or m is equal to i. Without loss of generality, assume that $i = l$. Then $x \in W_i(\eta)$. If $i = m$, then $x \in W_i(\eta) \cap W_i(\xi)$ and we are done, so assume that $i \neq m$. Then $\eta \in \mathrm{dom}(W_i)$, $\xi \in \mathrm{dom}(W_i) \cap \mathrm{dom}(W_m)$, and $x \in W_i(\eta) \cap W_m(\xi)$. It follows by Lemma 2.14(b,c) that $x \in W_i(\xi)$ and we are done.

Now assume that both l and m are not equal to i. Then $\xi \in \mathrm{dom}(W_i) \cap \mathrm{dom}(W_m) = r \subseteq \mathrm{dom}(W_l)$. So $\eta \in \mathrm{dom}(W_l)$, $\xi \in \mathrm{dom}(W_l) \cap \mathrm{dom}(W_m)$, and $x \in W_l(\eta) \cap W_m(\xi)$. By Lemma 2.13(b,c), $x \in W_l(\xi)$. So in fact $x \in W_l(\eta) \cap W_l(\xi)$. We have that η and ξ are both in $\mathrm{dom}(W_i) \cap \mathrm{dom}(W_l)$. So by Definition 2.13(4), $W_l(\eta) \cap W_l(\xi) \subseteq \delta_l$. By Definition 2.13(3), $W_l(\eta) \cap \delta_l = W_i(\eta) \cap \delta_i$ and $W_l(\xi) \cap \delta_l = W_i(\xi) \cap \delta_i$. So $x \in W_i(\eta) \cap W_i(\xi)$ and we are done. □

3. The First Forcing

In this section, we define a c.c.c. forcing for adding a strongly non-saturated Aronszajn tree assuming the existence of a function from κ^2 into ω_1 with a special property. This forcing is a suborder of $\mathbb{P}^*$ consisting of conditions which satisfy a restriction on the intersection of their subtrees related to this function.

For the remainder of this section, fix a function $e : \kappa^2 \to \omega_1$.

Definition 3.1 (e-Separation)**.** *Let T be a standard finite tree and let W be a subtree function on T. We say that W is* e-separated *if for all distinct $\eta, \xi \in \mathrm{dom}(W)$, if $x \in W(\eta) \cap W(\xi)$, then $e(\eta, \xi) \geq h(x)$.*

Definition 3.2. *Let $\mathbb{P}'$ be the suborder of $\mathbb{P}^*$ consisting of triples $(T, W, D) \in \mathbb{P}^*$ such that W is e-separated.*

We begin by analyzing the generic object which is added by $\mathbb{P}'$.

Definition 3.3. *For any generic filter G on $\mathbb{P}'$, define $(T_G, <_G)$ by:*
- *$x \in T_G$ if there exists some $p \in G$ such that $x \in T_p$;*
- *$x <_G y$ if there exists some $p \in G$ such that $x <_p y$.*

We abbreviate $(T_G, <_G)$ by T_G. We occasionally write $\dot{G}$ for the canonical $\mathbb{P}'$-name for a generic filter on $\mathbb{P}'$. Let $T_{\dot{G}}$ be a $\mathbb{P}'$-name for the above object.

The following is easy to verify.

Lemma 3.4. *If G is a generic filter on $\mathbb{P}'$, then T_G is a tree with a root.*

Lemma 3.5. *For any $p \in \mathbb{P}'$, there exists $q \leq p$ such that T_q is downwards closed and has minimal splits.*

Proof. Fix $q \leq p$ in $\mathbb{P}^*$ satisfying the properties described in Lemma 2.11. In particular, T_q is downwards closed and has minimal splits. To show that q is in $\mathbb{P}'$, we prove that W_q is e-separated. So let η and ξ be distinct elements of $\mathrm{dom}(W_q)$, and assume that $x \in W_q(\eta) \cap W_q(\xi)$. By Lemma 2.11, there exists some $z \in W_p(\eta) \cap W_p(\xi)$ such that $x \leq_q z$. Since W_p is e-separated, $e(\eta, \xi) \geq h(z) \geq h(x)$. □

The next lemma is easy using Lemma 3.5.

Lemma 3.6. *Let $p \in \mathbb{P}'$.*

(1) *If $x \in T_p$ and $\alpha < h(x)$, then there exists $q \leq p$ and $y \in T_q$ such that $h(y) = \alpha$ and $y <_q x$.*
(2) *If $x \in T_p$ and $h(x) < \beta < \omega_1$, then there exists $q \leq p$ and $y \in T_q \setminus T_p$ such that $h(y) = \beta$ and $x <_U y$.*

Lemma 3.7. *Let G be a generic filter on $\mathbb{P}'$. Then the height function of T_G coincides with h. So T_G has height ω_1^V.*

Proof. By Lemma 3.6(1). □

Definition 3.8. *For any generic filter G on $\mathbb{P}'$ and for any $\eta < \kappa$, define*

$$W_G(\eta) = \bigcup \{W_p(\eta) : p \in G,\ \eta \in \mathrm{dom}(W_p)\}.$$

The next two lemmas are easy to check.

Lemma 3.9. *For any $p \in \mathbb{P}'$, for any $\eta < \kappa$, and for any $\alpha < \omega_1$, there exists $q \leq p$ such that $\eta \in \mathrm{dom}(W_q)$ and there exists some $x \in W_q(\eta)$ with $h(x) = \alpha$.*

Lemma 3.10. *For any generic filter G on $\mathbb{P}'$ and for any $\eta < \kappa$, $W_G(\eta)$ is an uncountable downwards closed subtree of T_G.*

Lemma 3.11. *Let G be a generic filter on $\mathbb{P}'$ and let $\eta < \xi < \kappa$. Then $W_G(\eta) \cap W_G(\xi)$ is finitely generated, and hence $W_G(\eta)$ and $W_G(\xi)$ are strongly almost disjoint.*

Proof. It is easy to prove that for all $p \in \mathbb{P}'$, there exists $q \leq p$ such that $\{\eta, \xi\} \in D_q$. It follows that there exists some $q \in G$ such that $\{\eta, \xi\} \in D_q$. We claim that any member of $W_G(\eta) \cap W_G(\xi)$ is less than or equal to some member of the finite set $W_q(\eta) \cap W_q(\xi)$ in T_G. So let $x \in W_G(\eta) \cap W_G(\xi)$. Then there exists some $r \leq q$ in G such that $x \in W_r(\eta) \cap W_r(\xi)$. Since $\{\eta, \xi\} \in D_q$, by the definition of $\mathbb{P}'$ there exists some $z \in W_q(\eta) \cap W_q(\xi)$ such that $x \leq_r z$. Then $x \leq_G z$. □

We now describe a special property of e and prove that it implies that $\mathbb{P}'$ is c.c.c.

Definition 3.12. *A function $f : \kappa^2 \to \omega_1$ is a* weak ρ-function *if whenever $\langle F_i : i < \omega_1 \rangle$ is a pairwise disjoint sequence of finite subsets of κ, then for any $\gamma < \omega_1$ there exist $i < j < \omega_1$ such that for all $\eta \in F_i$ and for all $\xi \in F_j$, $f(\eta, \xi) \geq \gamma$.*

Theorem 3.13. *Suppose that e is a weak ρ-function. Assume that $\langle p_\alpha : \alpha < \omega_1 \rangle$ is a sequence of conditions in $\mathbb{P}'$. Then for some $\alpha < \beta$ in C_h, (p_α, p_β) is (α, β)-split and $p_\alpha \oplus p_\beta$ is a condition in $\mathbb{P}'$ which extends p_α and p_β.*

Proof. Write $p_\alpha = (T^\alpha, W^\alpha, D^\alpha)$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, enumerate $\mathrm{dom}(W^\alpha)$ in increasing order as $\langle \eta_0^\alpha, \ldots, \eta_{n_\alpha - 1}^\alpha \rangle$.

By a standard pressing down argument, we can find a stationary set $Z_0 \subseteq C_h \cap \mathrm{cof}(>\omega)$, a standard finite tree T, $n < \omega$, and downwards closed subtrees $w_0, \ldots, w_{n-1}$ of T such that for all $\alpha \in Z_0$:

- $T^\alpha \restriction \alpha = T$;
- $n_\alpha = n$;
- for all $k < n$, $W^\alpha(\eta_k^\alpha) \cap \alpha = w_k$;

and moreover, for all $\alpha < \beta$ in Z_0, $T^\alpha \subseteq \beta$.

Applying the Δ-system lemma, fix an uncountable set $Z_1 \subseteq Z_0$ and a finite set $r \subseteq \kappa$ such that for all $\alpha < \beta$ in Z_1, $\mathrm{dom}(W^\alpha) \cap \mathrm{dom}(W^\beta) = r$. Now find an uncountable set $Z \subseteq Z_1$, an ordinal $\zeta \in C_h$, and a set $x \subseteq n$ such that for all $\alpha \in Z$:

- $\{k < n : \eta_k^\alpha \in r\} = x$
- $T \subseteq \zeta$;
- $\{e(\eta, \xi) : \eta, \xi \in r\} \subseteq \zeta$.

Note that for all $\alpha < \beta$ in Z and for all $k \in x$, $\eta_k^\alpha = \eta_k^\beta$.

Claim: For all $\alpha < \beta$ in Z, (p_α, p_β) is (α, β)-split.

Proof: (1) and (2) of Definition 2.13 are immediate by the choice of T and Z. For (3), let $\eta \in \mathrm{dom}(W^\alpha) \cap \mathrm{dom}(W^\beta) = r$. Then for some $k \in x$, $\eta = \eta_k^\alpha = \eta_k^\beta$. Hence, $W^\alpha(\eta) \cap \alpha = W^\alpha(\eta_k^\alpha) \cap \alpha = w_k = W^\beta(\eta_k^\beta) \cap \beta = W^\beta(\eta) \cap \beta$. For (4), let η and ξ be distinct elements of $\mathrm{dom}(W^\alpha) \cap \mathrm{dom}(W^\beta)$. Then $\eta, \xi \in r$. So $e(\eta, \xi) < \zeta < \alpha$. Let $x \in W^\alpha(\eta) \cap W^\alpha(\xi)$ and we show that $x < \alpha$. Since W^α is e-separated, $h(x) \leq e(\eta, \xi) < \alpha$, so $x < \alpha$. A similar argument show that $W^\beta(\eta) \cap W^\beta(\xi) \subseteq \beta$. This completes the proof of the claim.

By Lemma 2.15, it follows that for all $\alpha < \beta$ in Z, $p_\alpha \oplus p_\beta$ is in $\mathbb{P}^*$ and is an extension of p_α and p_β. Applying the assumption that e is a weak ρ-function to $\langle \mathrm{dom}(W^\alpha) \setminus r : \alpha \in Z \rangle$, fix $\alpha < \beta$ in Z such that for all $\eta \in \mathrm{dom}(W^\alpha) \setminus r$ and for all $\xi \in \mathrm{dom}(W^\beta) \setminus r$, $e(\eta, \xi) \geq \zeta$.

We claim that p_α and p_β are as required. We already know that (p_α, p_β) is (α, β)-split and $p_\alpha \oplus p_\beta$ is in $\mathbb{P}^*$ and is an extension of p_α and p_β. So it suffices to show that $W^\alpha \oplus W^\beta$ is e-separated. Consider distinct η and ξ in $\mathrm{dom}(W^\alpha \oplus W^\beta)$ and assume that $x \in (W^\alpha \oplus W^\beta)(\eta) \cap (W^\alpha \oplus W^\beta)(\xi)$. We show that $e(\eta, \xi) \geq h(x)$.

Case 1: $x \in T^\alpha \setminus \alpha$. Then $x \notin T^\beta$. By the definition of $W^\alpha \oplus W^\beta$, we must have that $x \in W^\alpha(\eta) \cap W^\alpha(\xi)$, for the other other possibilities imply that $x \in T^\beta$. Since W^α is e-separated, it follows that $e(\eta, \xi) \geq h(x)$.

Case 2: $x \in T^\beta \setminus \beta$. Then $x \notin T^\alpha$. So as in Case 1, $x \in W^\beta(\eta) \cap W^\beta(\xi)$. Since W^β is e-separated, it follows that $e(\eta, \xi) \geq h(x)$.

Case 3: $x \in T$. Then $h(x) < \zeta$. If one of η or ξ is in $\mathrm{dom}(W^\alpha) \setminus r$ and the other is in $\mathrm{dom}(W^\beta) \setminus r$, then by the choice of α and β we have that $e(\eta, \xi) \geq \zeta > h(x)$ and we are done. Otherwise, one of η or ξ is in $\mathrm{dom}(W^\alpha) \cap \mathrm{dom}(W^\beta)$. Without loss of generality, assume that $\xi \in \mathrm{dom}(W^\alpha) \cap \mathrm{dom}(W^\beta)$. If x is either in $W^\alpha(\eta) \cap W^\alpha(\xi)$ or in $W^\beta(\eta) \cap W^\beta(\xi)$, then $e(\eta, \xi) \geq h(x)$ by the fact that W^α and W^β are e-separated. So assume not. Then either $x \in W^\alpha(\eta) \cap W^\beta(\xi)$ or $x \in W^\beta(\eta) \cap W^\alpha(\xi)$. Since (p_α, p_β) is (α, β)-split, by Lemma 2.14 we have that $x \in W^\alpha(\xi)$ in the first case and $x \in W^\beta(\xi)$ in the second case. So

$x \in W^\alpha(\eta) \cap W^\alpha(\xi)$ in the first case and $x \in W^\beta(\eta) \cap W^\beta(\xi)$ in the second case, both of which contradict our current assumptions. □

Corollary 3.14. *If e is a weak ρ-function, then $\mathbb{P}'$ is c.c.c.*

Corollary 3.15. *If e is a weak ρ-function, then $\mathbb{P}'$ forces that $T_{\dot{G}}$ has no uncountable chain.*

Proof. Suppose for a contradiction that $p \in \mathbb{P}'$ forces that $\dot{b}$ is an uncountable chain of $T_{\dot{G}}$. For each $\alpha < \omega_1$, fix a condition $p_\alpha \leq p$ and some $x_\alpha \in T_{p_\alpha}$ such that p_α forces that $x_\alpha \in \dot{b} \setminus \alpha$. Applying Theorem 3.13, find $\alpha < \beta$ in C_h such that (p_α, p_β) is (α, β)-split and $p_\alpha \oplus p_\beta$ is a condition in $\mathbb{P}'$ which extends both p_α and p_β. Write $T_{p_\alpha} = T^\alpha$ and $T_{p_\beta} = T^\beta$.

Since (p_α, p_β) is (α, β)-split, $T^\alpha \upharpoonright \alpha = T^\beta \upharpoonright \beta$ and $T^\alpha \subseteq \beta$. As $x_\alpha \geq \alpha$, $x_\alpha \in T^\alpha \setminus T^\beta$, and since $x_\beta \geq \beta$, $x_\beta \in T^\beta \setminus T^\alpha$. By the definition of $T^\alpha \oplus T^\beta$, x_α and x_β are incomparable in $T^\alpha \oplus T^\beta$. Now for any $r \leq p_\alpha \oplus p_\beta$, T_r is an end-extension of $T^\alpha \oplus T^\beta$, and therefore x_α and x_β are incomparable in T_r. Consequently, $p_\alpha \oplus p_\beta$ forces that x_α and x_β are incomparable in $T_{\dot{G}}$, which contradicts that $p_\alpha \oplus p_\beta$ forces that x_α and x_β are both in the chain $\dot{b}$. □

Theorem 3.16. *Suppose that e is a weak ρ-function. Let G be a generic filter on $\mathbb{P}'$. Then T_G is a normal infinitely splitting Aronszajn tree and $\{W_G(\eta) : \eta < \kappa\}$ is a pairwise strongly almost disjoint family of uncountable downwards closed subtrees of T_G witnessing that T_G is strongly non-saturated.*

Proof. Lemma 3.7 and Corollary 3.15 imply that T_G is an Aronszajn tree. Lemma 3.5 implies that T_G is Hausdorff, and Lemma 3.6(2) implies that T_G is normal and infinitely splitting. By Lemmas 3.10 and 3.11 we are done. □

4. The Main Theorems: Part 1

Our first main theorem is proven by combining the results of the previous section with work of Jensen-Schlechta and Todorčević. Recall that the *generic Kurepa hypothesis* (GKH) is the statement that there exists a Kurepa tree in some c.c.c. forcing extension ([JS90]). So ¬GKH is equivalent to the statement that the negation of Kurepa's hypothesis is c.c.c. indestructible.

Theorem 4.1 ([Tod91, Lemma 4])**.** *The negation of Chang's conjecture is equivalent to the existence of a weak ρ-function $e : (\omega_2)^2 \to \omega_1$.*

Theorem 4.2 ([JS90, Proposition 1.4])**.** *Suppose that κ is a Mahlo cardinal. Then the Lévy collapse* $\mathrm{Col}(\omega_1, <\kappa)$ *forces ¬GKH.*

Finally, we use the fact that Chang's conjecture implies the existence of $0^\#$ and therefore fails in any generic extension of L.

Theorem 4.3. *Suppose that there exists a Mahlo cardinal. Then there is a generic extension of L in which there exists a strongly non-saturated Aronszajn tree and ¬GKH holds.*

Proof. Let κ be a Mahlo cardinal. Then κ is a Mahlo cardinal in L. Let K be an L-generic filter on the Lévy collapse $\mathrm{Col}(\omega_1, <\kappa)^L$. Working in $L[K]$, $0^\#$ does not exist and hence Chang's conjecture fails. By Theorem 4.1, in $L[K]$ we can fix a weak ρ-function $e : (\omega_2)^2 \to \omega_1$. Define $\mathbb{P}'$ in $L[K]$ using $\kappa = \omega_2$ and the function e.

Let G be an $L[K]$-generic filter on $\mathbb{P}'$. By Theorem 3.16, in $L[K][G]$ we have that T_G is a normal infinitely splitting Aronszajn tree which is strongly non-saturated. Consider any c.c.c.

forcing poset $\mathbb{Q}$ in $L[K][G]$. In $L[K]$ fix a $\mathbb{P}'$-name $\dot{\mathbb{Q}}$ for a c.c.c. forcing such that $\dot{\mathbb{Q}}^G = \mathbb{Q}$. Then the two-step forcing iteration $\mathbb{P}' * \dot{\mathbb{Q}}$ is c.c.c. in $L[K]$, and hence by Theorem 4.2, $\mathbb{P}' * \dot{\mathbb{Q}}$ forces over $L[K]$ that there does not exist a Kurepa tree. So in $L[K][G]$, $\mathbb{Q}$ forces that there does not exist a Kurepa tree. So $\neg\mathsf{GKH}$ holds in $L[K][G]$. □

We note that in contrast to the model of [KS24], in the model of the above theorem $\neg\mathsf{GKH}$ implies that there does not exist an almost Kurepa Suslin tree.

5. Adequate Sets

We now turn to developing our second forcing poset for adding a strongly non-saturated Aronszajn tree. For the remainder of the article, assume that κ is an inaccessible cardinal. In this section we review the type of side conditions which are used in this forcing. We refer the reader to [Kru17] for the proofs of Proposition 5.8 and Theorems 5.11 and 5.15 below, as well as for a general discussion of this style of side conditions and its history. Other than these three black boxes, we include the remaining proofs for completeness, all of which are easy. We do note that in [Kru17] the context is a bit different, since the inaccessible cardinal κ is replaced with ω_2. But everything works almost identically in both cases, with the difference being that in our current situation κ will be collapsed to become ω_2.

Fix a bijection $\psi : \kappa \to H(\kappa)$. Define a well-ordering $\lhd$ of $H(\kappa)$ by $a \lhd b$ if $\psi^{-1}(a) < \psi^{-1}(b)$. Let $\mathcal{A}$ denote the structure $(H(\kappa), \in, \psi)$. Since $\lhd$ is a well-ordering of $H(\kappa)$ which is definable in $\mathcal{A}$, the structure $\mathcal{A}$ has definable Skolem functions. For any set $x \subseteq H(\kappa)$, let $\mathrm{Sk}(x)$ denote the closure of x under these definable Skolem functions. And let $\mathrm{cl}(x)$ denote the set consisting of the elements of x together with the limit points of x. Define Λ_0 to be the club of all $\beta < \kappa$ such that $\mathrm{Sk}(\beta) \cap \kappa = \beta$.

Definition 5.1. *Define Λ to be the set of all $\beta < \kappa$ with uncountable cofinality which are limit points of Λ_0 and satisfy that $[\beta]^\omega \subseteq \mathrm{Sk}(\beta)$.*

Since κ is inaccessible, Λ is the intersection of some club subset of κ with $\kappa \cap \mathrm{cof}(>\omega)$.

In [Kru17] we also fix a thin stationary subset of $[\omega_2]^\omega$ which is only needed in the case that CH is false. In this article, this set will just be $[\kappa]^\omega$ and will not be mentioned explicitly.

Definition 5.2. *Define $\mathcal{X}$ to be the set of all $N \in [\kappa]^\omega$ such that $\mathrm{Sk}(N) \cap \kappa = N$ and for all $\gamma \in N$, $\sup(\gamma \cap \Lambda_0) \in N$.*

The main point of this definition for us is the property that $\mathrm{Sk}(N) \cap \kappa = N$. The second requirement is of minor technical importance and can be ignored for this article.

Note that $\mathcal{X}$ is a club subset of $[\kappa]^\omega$. It is easy to check that $\mathcal{X}$ is closed under intersections and if $M \in \mathcal{X}$ and $\beta \in \Lambda$, then $M \cap \beta \in \mathcal{X}$. Observe that if $\beta \in \Lambda$ then $\mathcal{X} \cap \mathcal{P}(\beta) = \mathcal{X} \cap \mathrm{Sk}(\beta)$.

Definition 5.3. *For all $M, N \in \mathcal{X}$, define*

$$\beta_{M,N} = \min(\Lambda \setminus \sup(cl(M) \cap cl(N))).$$

The ordinal $\beta_{M,N}$ is called the comparison point *of M and N.*

Definition 5.4. *A set A is* adequate *if A is a finite subset of $\mathcal{X}$ and for all M and N in A, one of the following holds:*

(1) *($M < N$) $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$;*
(2) *($N < M$) $N \cap \beta_{M,N} \in \mathrm{Sk}(M)$;*

(3) *($M \sim N$)* $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.

If A is adequate and $M, N \in A$, then $M < N$ is equivalent to $M \cap \omega_1 < N \cap \omega_1$, and $M \sim N$ is equivalent to $M \cap \omega_1 = N \cap \omega_1$. We also write $M \leq N$ to mean that either $M < N$ or $M \sim N$.

The next lemma follows easily from the definitions.

Lemma 5.5. *Assume that A is adequate, $M, N \in A$, and $M < N$. Then $M \cap N = M \cap \beta_{M,N}$, and hence $M \cap N \in \mathrm{Sk}(N)$.*

Lemma 5.6. *Suppose that A is adequate, $N \in \mathcal{X}$, and $A \subseteq \mathrm{Sk}(N)$. Then $A \cup \{N\}$ is adequate.*

Proof. If $M, N \in \mathcal{X}$ and $M \in \mathrm{Sk}(N)$, then easily $\beta_{M,N} > \sup(M)$, and hence $M \cap \beta_{M,N} = M \in \mathrm{Sk}(N)$. □

Lemma 5.7. *Suppose that $\chi \geq \kappa$ is regular and M is a countable elementary substructure of $(H(\chi), \in, \psi)$ such that $N = M \cap \kappa \in \mathcal{X}$. Then $M \cap H(\kappa) = \mathrm{Sk}(N)$.*

Proof. Since $M \cap H(\kappa)$ is easily an elementary substructure of $(H(\kappa), \in, \psi)$, $\mathrm{Sk}(N) \subseteq M \cap H(\kappa)$. But since $\psi : \kappa \to H(\kappa)$ is a bijection, by elementarity $M \cap H(\kappa) = \psi[N] \subseteq \mathrm{Sk}(N)$. □

Proposition 5.8 ([Kru17, Proposition 3.4])**.** *Suppose that $A, C \subseteq \mathcal{X}$ are finite, A is adequate, $A \subseteq C$, and for all $K \in C \setminus A$, there exists some $M \in A$ and some $\beta \in \Lambda$ such that $K = M \cap \beta$. Then C is adequate.*

Definition 5.9. *Let A be adequate and let $N \in A$. We say that A is N-closed if for all $M \in A$, if $M < N$ then $M \cap N \in A$.*

Lemma 5.10. *Suppose that A is adequate and $N \in A$. Then*

$$A \cup \{M \cap N : M \in A,\ M < N\}$$

is adequate and N-closed.

Proof. This follows immediately from Lemma 5.5 and Proposition 5.8. □

Theorem 5.11 ([Kru17, Proposition 3.9])**.** *Let A be adequate, let $N \in A$, and suppose that A is N-closed. Assume that B is adequate and*

$$A \cap \mathrm{Sk}(N) \subseteq B \subseteq \mathrm{Sk}(N).$$

Then $A \cup B$ is adequate.

In this article, we need an extension of Theorem 5.11 to finitely many adequate sets.

Corollary 5.12. *Let $1 < d < \omega$. Suppose:*

(1) *$A_0, \ldots, A_{d-1}$ are adequate;*
(2) *for all $0 < i < d$, $N_i \in A_i$ and A_i is N_i-closed;*
(3) *for all $0 < i < d$,*

$$A_i \cap \mathrm{Sk}(N_i) \subseteq A_{i-1} \subseteq \mathrm{Sk}(N_i).$$

Then $A_0 \cup \cdots \cup A_{d-1}$ is adequate.

Proof. By induction on d using Theorem 5.11. □

Definition 5.13. *Let A be adequate and let $\beta \in \Lambda$. We say that A is β-*closed *if for all $M \in A$, $M \cap \beta \in A$.*

Lemma 5.14. *Suppose that A is adequate and $\beta \in \Lambda$. Then*

$$C = A \cup \{M \cap \beta : M \in A\}$$

is adequate and β-closed. Moreover, if $N \in A$ and A is N-closed, then C is also N-closed.

Proof. The set C is adequate by Proposition 5.8, and it is easily β-closed. Consider $M \in A$ and we show that $(M \cap \beta) \cap N \in C$. But $(M \cap \beta) \cap N = (M \cap N) \cap \beta$, and since A is N-closed, $(M \cap N) \in A$. Hence, $(M \cap N) \cap \beta \in C$. □

Theorem 5.15 ([Kru17, Proposition 3.11])**.** *Let A be adequate, let $\beta \in \Lambda$, and assume that A is β-closed. Suppose that B is adequate and*

$$A \cap \mathrm{Sk}(\beta) \subseteq B \subseteq \mathrm{Sk}(\beta).$$

Then $A \cup B$ is adequate.

Lemma 5.16. *Suppose that $\beta \in \Lambda$, $A \subseteq \mathrm{Sk}(\beta)$ is adequate, $N \in \mathcal{X}$, and $N \cap \beta \in A$. Then $A \cup \{N\}$ is adequate.*

Proof. Note that for all $M \in \mathcal{X} \cap \mathrm{Sk}(\beta)$, $\mathrm{cl}(M) \cap \mathrm{cl}(N) = \mathrm{cl}(M) \cap \mathrm{cl}(N \cap \beta)$, and therefore by definition $\beta_{M,N} = \beta_{M,N\cap\beta}$. Since $\beta \in \Lambda$, $\beta_{M,N} \leq \beta$. Hence, $N \cap \beta_{M,N} = (N \cap \beta) \cap \beta_{M,N}$. Let $M \in A$. If $N \cap \omega_1 < M \cap \omega_1$, then $(N \cap \beta) < M$, so $N \cap \beta_{M,N} = (N \cap \beta) \cap \beta_{M,N\cap\beta}$ is in $\mathrm{Sk}(M)$. If $N \cap \omega_1 = M \cap \omega_1$, then $(N \cap \beta) \sim M$, so $N \cap \beta_{M,N} = (N \cap \beta) \cap \beta_{M,N\cap\beta} = M \cap \beta_{M,N\cap\beta} = M \cap \beta_{M,N}$. If $M \cap \omega_1 < N \cap \omega_1$, then $M < (N \cap \beta)$, so $M \cap \beta_{M,N} = M \cap \beta_{M,N\cap\beta} \in \mathrm{Sk}(N \cap \beta) \subseteq \mathrm{Sk}(N)$. □

6. The Second Forcing

In this section, we introduce the second main forcing poset $\mathbb{P}$ of the article. A condition in $\mathbb{P}$ will consist of a working part, which is a member of $\mathbb{P}^*$, together with an adequate set as a side condition. The next definition describes the required interaction between the working part and the side condition.

Definition 6.1 (A-Separation)**.** *Let T be a standard finite tree, let W be a subtree function on T, and let A be adequate. We say that W is A-*separated *if whenever $M \in A$, η and ξ are distinct elements of $M \cap \mathrm{dom}(W)$, and $x \in W(\xi) \cap W(\eta)$, then $x \in M$.*

Definition 6.2. *Let $\mathbb{P}$ be the forcing poset consisting of all quadruples (T, W, D, A) such that:*

(1) *$(T, W, D) \in \mathbb{P}^*$;*
(2) *A is adequate;*
(3) *W is A-separated.*

Let $(U, Y, E, B) \leq (T, W, D, A)$ in $\mathbb{P}$ if $(U, Y, E) \leq (T, W, D)$ in $\mathbb{P}^*$ and $A \subseteq B$.

Notation 6.3. *For any $p \in \mathbb{P}$, we write (T_p, W_p, D_p, A_p) for p.*

Note that $\mathbb{P} \subseteq H(\kappa)$.

Lemma 6.4. *For any $p \in \mathbb{P}$, there exists $q \leq p$ such that T_q is downwards closed and has minimal splits.*

Proof. Fix $(T, W, D) \leq (T_p, W_p, D_p)$ in $\mathbb{P}^*$ satisfying the properties described in Lemma 2.11. We claim that W is A_p-separated, which easily implies that $q = (T, W, D, A_p)$ is as required. So let $M \in A_p$, let η and ξ be distinct elements of $M \cap \mathrm{dom}(W)$, and let $x \in W(\eta) \cap W(\xi)$. We claim that $x \in M$. By Lemma 2.11, fix $z \in W_p(\eta) \cap W_p(\xi)$ such that $x \leq_T z$. Since W_p is A_p-separated, $z \in M \cap \omega_1$. As $x \leq z$, $x \in M \cap \omega_1$ as well. □

Definition 6.5. *For any $p \in \mathbb{P}$ and for any $N \in \mathcal{X}$, define $p + N = (T_p, W_p, D_p, A_p \cup \{N\})$.*

Lemma 6.6. *For any $p \in \mathbb{P}$ and for any $N \in \mathcal{X}$ with $p \in \mathrm{Sk}(N)$, $p + N$ is in $\mathbb{P}$ and is an extension of p.*

Proof. The proof is easy using Lemmas 5.6. □

Lemma 6.7. *Suppose that $p \in \mathbb{P}$ and $N \in A_p$. Define $C = A \cup \{M \cap N : M \in A,\ M < N\}$. Then C is N-closed and (T_p, W_p, D_p, C) is in $\mathbb{P}$ and extends p.*

Proof. By Lemma 5.10, C is adequate and N-closed. It suffices to prove that W_p is C-separated. Let $M \in C$, let η and ξ be distinct elements of $\mathrm{dom}(W_p) \cap M$, and let $x \in W_p(\eta) \cap W_p(\xi)$. If $M \in A_p$, then since W_p is A_p-separated, $x \in M$. Otherwise, for some $K \in A_p$ with $K < N$, $M = K \cap N$. But then η and ξ are in K, so $x \in K \cap \omega_1 = M \cap \omega_1$. □

Lemma 6.8. *Suppose that $p \in \mathbb{P}$ and $\beta \in \Lambda$. Define $C = A_p \cup \{M \cap \beta : M \in A\}$. Then C is β-closed and (T_p, W_p, A_p, C) is in $\mathbb{P}$ and extends p. Moreover, if $N \in A_p$ and A_p is N-closed, then C is N-closed.*

Proof. Similar to the proof of Lemma 6.7 using Lemma 5.14. □

Definition 6.9. *Let $1 < d < \omega$. Let $p_0, \ldots, p_{d-1}$ be in $\mathbb{P}$. Define $p_0 \oplus \cdots \oplus p_{d-1}$ to be the quadruple (T, W, D, A) satisfying:*

(1) *$(T, W, D) = (T_{p_0}, W_{p_0}, D_{p_0}) \oplus \cdots \oplus (T_{p_{d-1}}, W_{p_{d-1}}, D_{p_{d-1}})$;*
(2) *$A = A_{p_0} \cup \cdots \cup A_{p_{d-1}}$.*

The question of when $p_0 \oplus \cdots \oplus p_{d-1}$ is a condition extending each of $p_0, \ldots, p_{d-1}$, in $\mathbb{P}$ or in quotients of $\mathbb{P}$, is one of the central issues we deal with in this article.

The next lemma is critical for analyzing quotients of $\mathbb{P}$ in later sections.

Lemma 6.10. *Let $1 < d < \omega$. Let $p_0, \ldots, p_{d-1}$ be in $\mathbb{P}$. Suppose that $p_0 \oplus \cdots \oplus p_{d-1}$ is a condition in $\mathbb{P}$ which extends each of $p_0, \ldots, p_{d-1}$. Assume that:*

(1) *r is in $\mathbb{P}$;*
(2) *$r \leq p_0, \ldots, p_{d-1}$;*
(3) *for all $i < j < d$, for all $x \in T_{p_i} \setminus T_{p_j}$ and for all $y \in T_{p_j} \setminus T_{p_i}$, x and y are incomparable in T_r.*

Then $r \leq p_0 \oplus \cdots \oplus p_{d-1}$.

Proof. For each $i < d$, write $p_i = (T_i, W_i, D_i, A_i)$, and write $p_0 \oplus \cdots \oplus p_{d-1} = (T, W, D, A)$. By (2), $A_0, \ldots, A_{d-1}$ are subsets of A_r. Hence, $A \subseteq A_r$. We claim that $(T_r, W_r, D_r) \leq (T, W, D)$ in $\mathbb{P}^*$. By (2), $D \subseteq D_r$, $T \subseteq T_r$, and $<_T \subseteq <_r$.

To see that T_r is an end-extension of T, suppose that $x <_r y$ where $x, y \in T$, and we prove that $x <_T y$. Fix $i, j < d$ such that $x \in T_i$ and $y \in T_j$. If $i = j$, then we done since $r \leq p_i$. Assume that $i \neq j$. By (3) and the fact that $x <_r y$, it cannot be the case that both $x \in T_i \setminus T_j$

and $y \in T_j \setminus T_i$. Without loss of generality, assume that $x \in T_i \cap T_j$. Then x and y are in T_j, and since $r \leq p_j$, it follows that $x <_{T_j} y$ and hence $x <_T y$.

By (2), $\mathrm{dom}(W) \subseteq \mathrm{dom}(W_r)$. Consider $\eta \in \mathrm{dom}(W)$. Then $W(\eta) = W_0(\eta) \cup \cdots \cup W_{d-1}(\eta)$. Since $r \leq p_0, \ldots, p_{d-1}$, for all $i < d$, $W_i(\eta)$ is a subset of $W(\eta)$. So $W(\eta) \subseteq W_r(\eta)$. Now assume that $\{\eta, \xi\} \in D$ and $x \in W_r(\eta) \cap W_r(\xi)$. We will find some $z \in W(\eta) \cap W(\xi)$ such that $x \leq_r z$. Fix $i < d$ such that $\{\eta, \xi\} \in D_i$. Since $r \leq p_i$, there exists some $z \in W_i(\eta) \cap W_i(\xi)$ such that $x \leq_r z$. Then $z \in W(\eta) \cap W(\xi)$ and we are done. □

7. Properness and Collapsing

In this section, we prove that the forcing poset $\mathbb{P}$ is proper and collapses cardinals larger than ω_1 and less than κ. Lemma 7.1 describes properties which are sufficient for amalgamating conditions over countable elementary substructures. The case that $d = 2$ is used in Theorem 7.2 to prove that $\mathbb{P}$ is proper, and the general case that $d \geq 2$ is used in Section 7 to prove that $\mathbb{P}$ is Y-proper. Lemma 7.1 is also used implicitly in proving that quotients of $\mathbb{P}$ are Y-proper, by way of Lemma 8.6.

Lemma 7.1. *Let $1 < d < \omega$. Let $p_0, \ldots, p_{d-1}$ be in $\mathbb{P}$. Write $p_i = (T_i, W_i, D_i, A_i)$ for all $i < d$. Assume that for all $i < d$, $N_i \in A_i$, A_i is N_i-closed, and for all $i < j < d$, $p_i \in \mathrm{Sk}(N_j)$. Let $\delta_i = N_i \cap \omega_1$ for all $i < d$.*

Assume that there exist commutative families of functions $\{f_{j,i} : i < j < d\}$ and $\{g_{j,i} : i < j < d\}$ such that for each $i < j < d$, $f_{j,i} : \mathrm{dom}(W_j) \to \mathrm{dom}(W_i)$ and $g_{j,i} : A_j \to A_i$ are bijective. Finally, assume that the following statements hold for all $i < j < d$:

1. *$T_i \upharpoonright \delta_i = T_j \upharpoonright \delta_j$;*
2. *$\mathrm{dom}(W_i) \cap N_i = \mathrm{dom}(W_j) \cap N_j$ and for all η in this set, $f_{j,i}(\eta) = \eta$;*
3. *for all $\eta \in \mathrm{dom}(W_j)$ and for all $M \in A_j$, $\eta \in M$ iff $f_{j,i}(\eta) \in g_{j,i}(M)$;*
4. *for all $\eta \in \mathrm{dom}(W_j)$, $W_i(f_{j,i}(\eta)) \cap \delta_i = W_j(\eta) \cap \delta_j$;*
5. *$A_i \cap \mathrm{Sk}(N_i) = A_j \cap \mathrm{Sk}(N_j)$ and for all M in this set, $g_{j,i}(M) = M$;*
6. *for all $M \in A_j$, if $M \cap \omega_1 < \delta_j$, then $g_{j,i}(M) \cap \omega_1 = M \cap \omega_1$ and $M \cap N_j \subseteq g_{j,i}(M)$.*

Then $p_0 \oplus \cdots \oplus p_{n-1}$ is a condition which extends $p_0, \ldots, p_{n-1}$.

Proof. Observe that by elementarity, $\delta_i \in C_h$ for all $i < d$. Note that for all $i < j < d$, by (1) and the fact that $p_i \in \mathrm{Sk}(N_j)$, $T_i \subseteq \delta_j$ and $T_i \cap T_j \subseteq \delta_i$.

Claim 1: For all $k < d$, $\{\mathrm{dom}(W_i) : i < d\}$ is a Δ-system with root $\mathrm{dom}(W_k) \cap N_k$. So for all $i < j < d$, $\mathrm{dom}(W_i) \cap \mathrm{dom}(W_j) \subseteq N_i \cap N_j$.

Proof: This follows easily from (2) together with the fact that for all $i < j < d$, $p_i \in \mathrm{Sk}(N_j)$.

Claim 2: For all $i < j < d$, (T_i, W_i, D_i) and (T_j, W_j, D_j) are (δ_i, δ_j)-split.

Proof: We have that $T_i \upharpoonright \delta_i = T_j \upharpoonright \delta_j$ by (1), and since $p_i \in \mathrm{Sk}(N_j)$, $T_i \subseteq \delta_j$. If $\eta \in \mathrm{dom}(W_i) \cap \mathrm{dom}(W_j)$, then by (2), $f_{j,i}(\eta) = \eta$. Hence by (4), $W_i(\eta) \cap \delta_i = W_j(\eta) \cap \delta_j$. Finally, consider distinct $\eta, \xi \in \mathrm{dom}(W_i) \cap \mathrm{dom}(W_j)$. Then η and ξ are in $N_i \cap N_j$. Since W_i is A_i-separated and $N_i \in A_i$, it follows that $W_i(\eta) \cap W_i(\xi) \subseteq N_i \cap \omega_1 = \delta_i$. And because W_j is A_j-separated and $N_j \in A_j$, $W_j(\eta) \cap W_j(\xi) \subseteq N_j \cap \omega_1 = \delta_j$. This completes the proof of the claim.

Define $r = p_0 \oplus \cdots \oplus p_{d-1}$, which we denote by (T, W, D, A). We prove that $r \in \mathbb{P}$ and r is an extension of p_i for all $i < d$. By claims 1 and 2 and Lemma 2.15, (T, W, D) is in $\mathbb{P}^*$ and is an extension of (T_i, W_i, D_i) for all $i < d$. Obviously, $A_i \subseteq A$ for all $i < d$.

Using (5), it is easy to check that the assumptions of Corollary 5.12 hold for $A_0, \dots, A_{d-1}$ and $N_1, \dots, N_{d-1}$, so $A = A_0 \cup \cdots \cup A_{d-1}$ is adequate.

Finally, we prove that W is A-separated. Let $M \in A$, let η and ξ be distinct elements of $M \cap \mathrm{dom}(W)$, and let $x \in W_r(\eta) \cap W_r(\xi)$. We show that $x \in M$. If there exists some $i < d$ such that $M \in A_i$ and $x \in W_i(\eta) \cap W_i(\xi)$, then we are done since p_i is a condition. So assume not. Fix $i, j, k < d$ such that $M \in A_i$, $x \in W_j(\eta)$, and $x \in W_k(\xi)$.

Case 1: $j = k$, $j < i$, and $M \in A_i \setminus \mathrm{Sk}(N_i)$. Then $p_j \in \mathrm{Sk}(N_i)$, so $x \in T_j \subseteq \delta_i$ and η and ξ are in N_i. And $x \in W_j(\eta) \cap W_j(\xi)$. If $N_i \leq M$, then $x \in \delta_i \subseteq M$ and we are done. Otherwise, $M \cap \omega_1 < \delta_i$. By (6), $M \cap \omega_1 = g_{i,j}(M) \cap \omega_1$ and η and ξ are in $M \cap N_i \subseteq g_{i,j}(M)$. Since W_j is A_j-separated and $g_{i,j}(M) \in A_j$, $x \in g_{i,j}(M) \cap \omega_1 \subseteq M$.

Case 2: $j = k$, $j < i$, and $M \in A_i \cap \mathrm{Sk}(N_i)$. By (5), $M \in A_j$. So we are done since W_j is A_j-separated.

Case 3: $j = k$ and $i < j$. Then $p_i \in \mathrm{Sk}(N_j)$, so $M \in \mathrm{Sk}(N_j)$. As $\eta, \xi \in M$, it follows that $\eta, \xi \in N_j$. So by (2), $\eta, \xi \in \mathrm{dom}(W_j) \cap N_j \subseteq \mathrm{dom}(W_i)$. By Definition 2.13(4), $W_j(\eta) \cap W_j(\xi) \subseteq \delta_j$, and hence $x < \delta_j$. By Definition 2.13(3), $W_i(\eta) \cap \delta_i = W_j(\eta) \cap \delta_j$ and $W_i(\xi) \cap \delta_i = W_j(\xi) \cap \delta_j$. So $x \in W_i(\eta) \cap W_i(\xi)$. Since W_i is A_i-separated and $M \in A_i$, $x \in M$.

In the remaining cases, $j \neq k$. Without loss of generality assume that $j < k$.

Case 4: $j \neq k$ and either $\eta \in \mathrm{dom}(W_k)$ or $\xi \in \mathrm{dom}(W_j)$. Assume that $\eta \in \mathrm{dom}(W_k)$. Then $\xi \in \mathrm{dom}(W_k)$ and $\eta \in \mathrm{dom}(W_j) \cap \mathrm{dom}(W_k)$. By Lemma 2.14(c), $W_k(\xi) \cap W_j(\eta) \subseteq W_k(\eta)$. Hence, $x \in W_k(\eta) \cap W_k(\xi)$. So we are back to the situation of Cases 1, 2, and 3, and we are done. The case that $\xi \in \mathrm{dom}(W_j)$ is similar using Lemma 2.14(b).

For the remaining cases, we may assume that $\eta \in \mathrm{dom}(W_j) \setminus \mathrm{dom}(W_k)$ and $\xi \in \mathrm{dom}(W_k) \setminus \mathrm{dom}(W_j)$. Note that $x \in W_j(\eta) \cap W_k(\xi) \subseteq T_j \cap T_k \subseteq \delta_j$. So $x < \delta_j$. By (4), $W_j(f_{k,j}(\xi)) \cap \delta_j = W_k(\xi) \cap \delta_k$. So $x \in W_j(f_{k,j}(\xi))$. If $\delta_j \leq M \cap \omega_1$, then we are done. So assume that $M \cap \omega_1 < \delta_j$.

Case 5: $k < i$. Then $M \cap \omega_1 < \delta_i$. And $p_k \in \mathrm{Sk}(N_i)$ implies that $\xi \in M \cap N_i \subseteq g_{i,k}(M)$ by (6). So by (3) and commutativity, $f_{k,j}(\xi) \in g_{k,j}(g_{i,k}(M)) = g_{i,j}(M)$. And $p_j \in \mathrm{Sk}(N_i)$ and $M \cap \omega_1 < \delta_i$ implies by (6) that $\eta \in M \cap N_i \subseteq g_{i,j}(M)$. So $x \in W_j(\eta) \cap W_j(f_{k,j}(\xi))$, η and $f_{k,j}(\xi)$ are in $g_{i,j}(M)$, and $g_{i,j}(M) \in A_j$. As W_j is A_j-separated, it follows that $x \in g_{i,j}(M) \cap \omega_1 \subseteq M$.

Case 6: $k = i$. By (3), $f_{k,j}(\xi) \in g_{k,j}(M) = g_{i,j}(M)$. Also, $p_j \in \mathrm{Sk}(N_i)$. By (6), $\eta \in \mathrm{dom}(W_j) \cap M \subseteq N_i \cap M \subseteq g_{i,j}(M)$. So $x \in W_j(\eta) \cap W_j(f_{k,j}(\xi))$, and η and $f_{k,j}(\xi)$ are in $g_{i,j}(M)$. Since W_j is A_j-separated and $g_{i,j}(M) \in A_j$, $x \in g_{i,j}(M) \cap \omega_1 \subseteq M$.

Case 7: $i < k$. Then $M \in A_i \subseteq \mathrm{Sk}(N_k)$. So $\xi \in M \subseteq N_k$. By (2), $\xi \in \mathrm{dom}(W_k) \cap N_k = \mathrm{dom}(W_j) \cap N_j$, which contradicts our assumption that $\xi \notin \mathrm{dom}(W_j)$. □

Theorem 7.2. *Let $\chi > \kappa$ be regular. Let M be a countable elementary substructure of $\mathcal{B} = (H(\chi), \in, \psi, \mathbb{P})$ such that $N = M \cap \kappa \in \mathcal{X}$. Then for any $u \in M \cap \mathbb{P}$, $u + N$ is in $\mathbb{P}$, $u + N \leq u$, and $u + N$ is $(M, \mathbb{P})$-generic. In fact, for any dense open set $\mathcal{D} \subseteq \mathbb{P}$ in M, for any $q \leq u + N$ in $\mathcal{D}$ such that A_q is N-closed, there exists some $\bar{q} \in M \cap \mathcal{D}$ such that $\bar{q} \oplus q$ is in $\mathbb{P}$ and extends $\bar{q}$ and q.*

Proof. By Lemma 5.7, $M \cap H(\kappa) = \mathrm{Sk}(N)$. So $u \in \mathrm{Sk}(N)$. By Lemma 6.6, $u + N$ is in $\mathbb{P}$ and extends u. Fix a dense open set $\mathcal{D} \subseteq \mathbb{P}$ in M. Fix $q \leq u + N$ in $\mathcal{D}$ which is N-closed.

Let $N_q = N$ and let $\delta_q = N \cap \omega_1$. Enumerate $\mathrm{dom}(W_q) = \{\eta_0, \dots, \eta_{m-1}\}$ and $A_q = \{M_0, \dots, M_{n-1}\}$, where $m, n < \omega$ and $M_0 = N$. Define:

- $U_0 = \{k < m : \eta_k \in N\}$;
- $U_1 = \{l < n : M_l \in \mathrm{Sk}(N)\}$;
- $U_2 = \{l < n : M_l < N\}$;
- $U_3 = \{(k, l) \in m \times n : \eta_k \in M_l\}$.

Define a formula with free variables

$$\varphi = \varphi(\dot{q}, \dot{T}, \dot{W}, \dot{D}, \dot{A}, \dot{\delta}, \dot{\eta}_0, \ldots, \dot{\eta}_{m-1}, \dot{M}_0, \ldots, \dot{M}_{n-1})$$

to be conjunction of the following:

(1) $\dot{q} = (\dot{T}, \dot{W}, \dot{D}, \dot{A}) \in \mathcal{D}$;
(2) $\mathrm{dom}(\dot{W}) = \{\dot{\eta}_0, \ldots, \dot{\eta}_{m-1}\}$ and $\dot{A} = \{\dot{M}_0, \ldots, \dot{M}_{n-1}\}$;
(3) $\dot{\delta} = \dot{M}_0 \cap \omega_1$;
(4) $\dot{T} \restriction \dot{\delta} = T_q \restriction \delta_q$;
(5) for all $k \in U_0$, $\dot{\eta}_k = \eta_k$;
(6) for all $(k, l) \in m \times n$, $(k, l) \in U_3$ iff $\dot{\eta}_k \in \dot{M}_l$;
(7) $\mathrm{dom}(\dot{W}) \cap \dot{M}_0 = \mathrm{dom}(W_q) \cap N$;
(8) for all $k < m$, $\dot{W}(\dot{\eta}_k) \cap \dot{\delta} = W_q(\eta_k) \cap \delta_q$;
(9) $\dot{A} \cap \mathrm{Sk}(\dot{M}_0) = A_q \cap \mathrm{Sk}(N)$;
(10) for all $l \in U_1$, $\dot{M}_l = M_l$;
(11) for all $l \in U_2$, $\dot{M}_l \cap \omega_1 = M_l \cap \omega_1$ and $M_l \cap N \subseteq \dot{M}_l$;
(12) for all $l < n$, $l \in U_2$ iff $\dot{M}_l < \dot{M}_0$;
(13) for all $l \in U_2$, $\dot{M}_l \cap \dot{M}_0 \in \dot{A}$.

It is routine to check that all of the parameters appearing in φ are members of M, and that

$$\mathcal{B} \models \varphi[q, T_q, W_q, D_q, A_q, N \cap \omega_1, \eta_0, \ldots, \eta_{m-1}, M_0, \ldots, M_{n-1}].$$

By elementarity, we can find objects $\bar{q}$, $\bar{T}$, $\bar{W}$, $\bar{D}$, $\bar{A}$, $\bar{\delta}$, $\bar{\eta}_0, \ldots, \bar{\eta}_{m-1}$, and $\bar{M}_0, \ldots, \bar{M}_{n-1}$ in M which satisfy the same.

Now it is straightforward to check that the assumptions of Lemma 7.1 are satisfied, where $d = 2$, $\bar{q}$ and $\bar{M}_0$ serve the roles of p_0 and N_0, q and N serve the roles of p_1 and N_1, and the functions $f_{1,0}$ and $g_{1,0}$ are defined by $f_{1,0}(\eta_k) = \bar{\eta}_k$ for all $k < m$ and $g_{1,0}(M_l) = \bar{M}_l$ for all $l < n$. It follows that $\bar{q} \oplus q$ is a condition which extends $\bar{q}$ and q. □

Since $\mathcal{X}$ is a club subset of $[\kappa]^\omega$, we immediately have the following corollary.

Corollary 7.3. *The forcing poset $\mathbb{P}$ is proper.*

Since $\mathbb{P}$ is proper, it preserves ω_1. Let us check that $\mathbb{P}$ collapses every cardinal μ such that $\omega_1 < \mu < \kappa$.

Proposition 7.4. *Suppose that μ is a cardinal and $\omega_1 < \mu < \kappa$. Then $\mathbb{P}$ forces that μ is not a cardinal.*

Proof. Let G be a generic filter on $\mathbb{P}$. In $V[G]$, define

$$Z = \{M \in \mathcal{X} : \exists p \in G\ (M \in A_p\ \wedge\ \mu \in M)\}.$$

If M and N are in Z, then $\mu \in M \cap N \cap \kappa$, and hence $\beta_{M,N} > \mu$. It easily follows from the definition of $\mathbb{P}$ that for all $M, N \in Z$, $M \cap \omega_1 < N \cap \omega_1$ iff $M \cap \mu \subsetneq N \cap \mu$. So $Z_\mu = \{M \cap \mu : M \in Z\}$ is a well-ordered chain of countable sets with order-type at most ω_1. It easily follows from Lemma 6.6 that the union of this chain is equal to μ, and hence this chain has order type equal to ω_1 since ω_1 is preserved. So μ is the union of ω_1-many countable sets and therefore has size ω_1. □

We now briefly discuss the generic object which is added by $\mathbb{P}$.

Definition 7.5. *Let G be a generic filter on $\mathbb{P}$. Define $(T_G, <_G)$ by:*

- *$x \in T_G$ if there exists some $p \in G$ such that $x \in T_p$;*
- *$x <_G y$ if there exists some $p \in G$ such that $x <_p y$.*

For any $\eta < \kappa$, define $W_G(\eta) = \bigcup\{W_p(\eta) : p \in G,\ \eta \in \mathrm{dom}(W_p)\}$.

As usual, we abbreviate $(T_G, <_G)$ by T_G. We occasionally write $\dot{G}$ for the canonical $\mathbb{P}$-name for a generic filter on $\mathbb{P}$. Let $T_{\dot{G}}$ be a $\mathbb{P}$-name for the above object.

The following proposition has almost the same proof as the analogous fact about $\mathbb{P}'$ from Section 3.

Proposition 7.6. *Let G be a generic filter on $\mathbb{P}$. Then T_G is a normal infinitely splitting ω_1-tree and $\{W_G(\eta) : \eta < \kappa\}$ is a pairwise strongly almost disjoint family of uncountable downwards closed subtrees of T_G.*

Proposition 7.7. *The forcing poset $\mathbb{P}$ forces that $T_{\dot{G}}$ is Aronszajn.*

Proof. Suppose for a contradiction that $u \in \mathbb{P}$ forces that $\dot{b}$ is a cofinal branch of $T_{\dot{G}}$. Fix a regular cardinal $\chi > \kappa$ such that $\dot{b} \in H(\chi)$. Let M be a countable elementary substructure of $(H(\chi), \in, \psi, \mathbb{P})$ such that u and $\dot{b}$ are in M and $N = M \cap \kappa \in \mathcal{X}$. By Theorem 7.2, $u + N$ is a condition extending u which is $(M, \mathbb{P})$-generic.

Fix $q \le u + N$ and $x_q \ge N \cap \omega_1$ such that $x_q \in T_q$ and q forces that $x_q \in \dot{b}$. By extending q further if necessary, we may assume that A_q is N-closed. Fix $\delta < N \cap \omega_1$ such that $T_q \cap (N \cap \omega_1) \subseteq \delta$. Let $\mathcal{D}$ be the set of conditions $s \in \mathbb{P}$ such that for some $x_s \ge \delta$ in T_s, s forces that $x_s \in \dot{b}$. Note that $\mathcal{D}$ is dense open, $\mathcal{D} \in M$, and $q \in \mathcal{D}$.

By Theorem 7.2, fix $\bar{q}$ in $\mathcal{D} \cap M$ such that $r = \bar{q} \oplus q$ is in $\mathbb{P}$ and is an extension of $\bar{q}$ and q. As $\bar{q} \in \mathcal{D} \cap M$, fix $x_{\bar{q}} \ge \delta$ in $T_{\bar{q}} \cap M$ such that $\bar{q}$ forces that $x_{\bar{q}} \in \dot{b}$. Now $T_q \cap (N \cap \omega_1) \subseteq \delta$ and $\bar{q} \in M$ imply that $x_{\bar{q}} \in T_{\bar{q}} \setminus T_q$ and $x_q \in T_q \setminus T_{\bar{q}}$. By the definition of $T_{\bar{q}} \oplus T_q$, $x_{\bar{q}}$ and x_q are not comparable in T_r. For all $s \le r$, T_s is an end-extension of T_r and hence x and y are incomparable in T_s. So r forces that x and y are incomparable in $T_{\dot{G}}$, which contradicts that r forces that x and y are both in $\dot{b}$. □

8. Y-Properness

In our applications of the forcing poset $\mathbb{P}$, we need to know that quotients of $\mathbb{P}$ have the ω_1-approximation property, and in particular, that they do not add new cofinal branches of trees with height ω_1. The key to this fact is the property of Y-properness due to Chodounský and Zapletal ([CZ15]).

Definition 8.1. *A forcing poset $\mathbb{Q}$ is* Y-proper *if for all large enough regular cardinals χ with $\mathbb{Q} \in H(\chi)$, there are club many $M \in [H(\chi)]^\omega$ such that M is an elementary substructure of $(H(\chi), \in, \mathbb{Q})$, and for all $p \in M \cap \mathbb{Q}$ there exists $q \le p$ which is $(M, \mathbb{Q})$-generic and satisfies that for all $r \le q$, there exists a filter $\mathcal{F} \in M$ on the Boolean completion $\mathcal{B}(\mathbb{Q})$ such that $\{s \in M \cap \mathcal{B}(\mathbb{Q}) : r \le s\} \subseteq \mathcal{F}$. If the above holds for stationarily many (rather than club many) M in $[H(\chi)]^\omega$, then we say that $\mathbb{Q}$ is* Y-proper on a stationary set.

Note that Y-proper implies proper. Recall that a forcing poset $\mathbb{Q}$ has the ω_1-*approximation property* if whenever $X \in V$, $B \subseteq X$ is in $V^{\mathbb{Q}}$, and for all countable $a \subseteq X$ in V, $B \cap a \in V$, then $B \in V$. If $\mathbb{Q}$ has the ω_1-approximation property, then for any regular uncountable cardinal μ, $\mathbb{Q}$ does not add new cofinal branches to any tree with height μ. Our interest in Y-properness comes from the following consequence of it (see [CZ15, Corollary 4.1] and the proof of [CZ15, Theorem 2.8]).

Theorem 8.2. *Suppose that $\mathbb{Q}$ is a forcing poset which is Y-proper on a stationary set. Then $\mathbb{Q}$ has the ω_1-approximation property.*

In Section 12, we also make use of the Y-c.c. property of a forcing poset ([CZ15]). The only things which the reader needs to know about this property is that it implies Y-properness and it is preserved under finite support forcing iterations.

We now proceed towards proving that $\mathbb{P}$ is Y-proper.

Notation 8.3. *For any $p \in \mathbb{P}$ define:*

- $m_p = |\mathrm{dom}(W_p)|$ *and* $n_p = |A_p|$*;*
- $\langle \eta_i^p : i < m_p \rangle$ *is the unique enumeration of* $\mathrm{dom}(W_p)$ *such that* $\eta_i^p < \eta_j^p$ *for all* $i < j < m_p$*;*
- $\langle K_i^p : i < n_p \rangle$ *is the unique enumeration of* A_p *such that* $K_i^p \lhd K_j^p$ *for all* $i < j < n_p$.

Definition 8.4. *Define a function w as follows. The domain of w is the set of ordered pairs (p, N) such that $p \in \mathbb{P}$, $N \in A_p$, and A_p is N-closed. For any such (p, N), letting $\delta = N \cap \omega_1$, define*

$$w(p, N) = (t, a, b, m, n, w_0, \ldots, w_{m-1}, U_0, U_1, U_2, U_3, h_0, h_1),$$

where:

(a) $t = T \restriction \delta$;
(b) $a = \mathrm{dom}(W) \cap N$;
(c) $b = A \cap \mathrm{Sk}(N)$;
(d) $m = m_p$ *and* $n = n_p$;
(e) $w_k = W(\eta_k^p) \cap \delta$ *for all* $k < m$;
(f) $U_0 = \{k < m : \eta_k^p \in N\}$;
(g) $U_1 = \{l < n : K_l^p \in \mathrm{Sk}(N)\}$;
(h) $U_2 = \{l < n : K_l^p \cap \omega_1 < \delta\}$.
(i) $U_3 = \{(k, l) \in m \times n : \eta_k^p \in K_l^p\}$;
(j) $h_0 : U_2 \to \delta$ *is a function and for all* $l \in U_2$, $h_0(l) = K_l^p \cap \omega_1$;
(k) $h_1 : U_2 \to \mathrm{Sk}(N)$ *is a function and for all* $l \in U_2$, $h_1(l) = K_l^p \cap N$.

Lemma 8.5. *For all (p, N) in the domain of w, $w(p, N) \in \mathrm{Sk}(N)$.*

Proof. Every member of the tuple $w(p, N)$ is a finite subset of $\mathrm{Sk}(N)$. □

Lemma 8.6. *Let $1 < d < \omega$. Let $p_0, \ldots, p_{d-1}$ be in $\mathbb{P}$, let $N_0, \ldots, N_{d-1}$ be in $\mathcal{X}$, and suppose that for all $i < d$, $(p_i, N_i) \in \mathrm{dom}(w)$. Assume that for all $i < j < d$, $w(p_i, N_i) = w(p_j, N_j)$ and $p_i \in \mathrm{Sk}(N_j)$. Then $p_0 \oplus \cdots \oplus p_{d-1}$ is a condition which extends each of $p_0, \ldots, p_{d-1}$.*

Proof. Let $m = m_{p_i}$ and $n = n_{p_i}$ for some (any) $i < d$. For each $i < j < d$, define $f_{j,i}$: $\mathrm{dom}(W_j) \to \mathrm{dom}(W_i)$ by letting $f_{j,i}(\eta_k^{p_j}) = \eta_k^{p_i}$ for all $k < m$, and define $g_{j,i} : A_j \to A_i$ by letting $g_{j,i}(K_l^{p_j}) = K_l^{p_i}$ for all $l < n$. Clearly, this definition gives commutative families of bijections.

We verify properties (1)-(6) of Lemma 7.1 using (a)-(k) of Definition 8.4, with the other required properties being immediate. (1) follows from (a). (2) follows from (b) and (f). (3) follows from (i). (4) follows from (e). (5) follows from (c) and (g). The first part of (6) follows from (h) and (j). For the second part of (6), if $i < j < d$ and $l \in U_2$, then by (k), $K_l^{p_j} \cap N_j = h_1(l) = K_l^{p_i} \cap N_i \subseteq g_{j,i}(K_l^{p_j})$.

By Lemma 7.1, $p_0 \oplus \cdots \oplus p_{d-1}$ is in $\mathbb{P}$ and extends each of $p_0, \ldots, p_{d-1}$. □

Lemma 8.7. *Under the assumptions of Lemma 8.6, for all $k < d$:*

- $\{T_{p_i} : i < d\}$ *is a Δ-system with root $T_{p_k} \cap (N_k \cap \omega_1)$;*
- $\{\mathrm{dom}(W_{p_i}) : i < d\}$ *is a Δ-system with root* $\mathrm{dom}(W_{p_k}) \cap N_k$*;*
- $\{A_{p_i} : i < d\}$ *is a Δ-system with root $A_{p_k} \cap \mathrm{Sk}(N_k)$.*

Proof. Straightforward using (a), (b), and (c) of Definition 8.4 together with the fact that $p_i \in \mathrm{Sk}(N_j)$ whenever $i < j < d$. □

Definition 8.8. *Let $\vec{z}$ be in the range of w. A set $R \subseteq \mathbb{P}$ is said to be $\vec{z}$-*robust *if the set*

$$\{N \in \mathcal{X} : \exists p \in R\ (w(p, N) = \vec{z})\}$$

is stationary in $[\kappa]^\omega$.

Proposition 8.9. *For any $\vec{z}$ in the range of w, the collection $\{ \sum R : R \subseteq \mathbb{P}$ is $\vec{z}$-robust $\}$ is centered.*

Proof. Let $1 < d < \omega$ and let $R_0, \ldots, R_{d-1}$ be $\vec{z}$-robust sets. We prove that there exists some $r \in \mathbb{P}$ such that for all $i < d$, $r \leq \sum R_i$. By induction, we choose $p_0, \ldots, p_{d-1}$ and $N_0, \ldots, N_{d-1}$ as follows. Fix any $p_0 \in R_0$ and N_0 such that $w(p_0, N_0) = \vec{z}$. Now let $0 < i < d$ and assume that p_j and N_j are defined for all $j < i$. Since R_i is $\vec{z}$-robust, by stationarity we can find some $p_i \in R$ and N_i such that $w(p_i, N_i) = \vec{z}$ and for all $j < i$, $p_j \in \mathrm{Sk}(N_i)$. This completes the induction. By Lemma 8.6, $q = p_0 \oplus \cdots \oplus p_{d-1}$ is in $\mathbb{P}$ and extends each of $p_0, \ldots, p_{d-1}$. Hence, for all $i < d$, $q \leq p_i \leq \sum R_i$. □

Theorem 8.10. *The forcing poset $\mathbb{P}$ is Y-proper.*

Proof. Fix a regular cardinal $\chi > \kappa$. Let M be a countable elementary substructure of $(H(\chi), \in, \psi, \mathbb{P})$ such that $N = M \cap \kappa \in \mathcal{X}$. Note that there are club many such M in $[H(\chi)]^\omega$. Consider $u \in M \cap \mathbb{P}$. By Theorem 7.2, $u + N$ is a condition in $\mathbb{P}$ extending u which is $(M, \mathbb{P})$-generic. Consider any condition $q \leq u + N$. We will find a filter $\mathcal{F}$ on $\mathcal{B}(\mathbb{P})$ in M such that for every $s \in M \cap \mathcal{B}(\mathbb{P})$, if $q \leq s$ then $s \in \mathcal{F}$.

Using Lemma 6.7, extend q to r such that A_r is N-closed. Then (r, N) is in the domain of w. Let $\vec{z} = w(r, N)$. Then $\vec{z} \in \mathrm{Sk}(N) \subseteq M$. Define $\mathcal{F}_0 = \left\{\sum R : R \subseteq \mathbb{P} \text{ is } \vec{z}\text{-robust}\right\}$. By Proposition 8.9, $\mathcal{F}_0$ is centered, and by elementarity, $\mathcal{F}_0 \in M$. Define $\mathcal{F} = \{b \in \mathcal{B}(\mathbb{P}) : \exists c \in \mathcal{F}_0\ (c \leq b)\}$. Then $\mathcal{F}$ is a filter on $\mathcal{B}(\mathbb{P})$ and $\mathcal{F} \in M$.

Suppose that $q \leq s$ and $s \in M \cap \mathcal{B}(\mathbb{P})$. Define $R = \{t \in \mathbb{P} : t \leq s\}$. Clearly, $s = \sum R$, $R \in M$, and $r \in R$. We claim that R is $\vec{z}$-robust, and therefore $s = \sum R \in \mathcal{F}_0 \subseteq \mathcal{F}$. Let C be a club subset of $[\kappa]^\omega$ in M. Then $N \in C$. So $N \in C$, $r \in R$, and $w(r, N) = \vec{z}$. By elementarity, it follows that the set of all $K \in [\kappa]^\omega$ for which there exists some $t \in R$ such that $w(t, K) = \vec{z}$ is stationary. □

9. A Dense Set For Projecting

The main goal for the remainder of the article is to prove that certain quotients of the forcing $\mathbb{P}$ are Y-proper in an intermediate extension. The proof of this fact is complex and will be completed in several steps over the next few sections. In this section, we identify a dense subset of $\mathbb{P}$ which we use in the next section to define a natural projection mapping.

Definition 9.1. *Define Σ to be the set of $\theta \in \Lambda$ such that* $\mathrm{Sk}(\theta)$ *is an elementary substructure of* $(H(\kappa), \in, \psi, \mathcal{X}, \mathbb{P})$.

Recall that $\mathrm{Sk}(\theta)$ denotes the closure of θ under the definable Skolem functions for the structure $\mathcal{A} = (H(\kappa), \in, \psi)$. If $\theta \in \Lambda$, then $\theta = \mathrm{Sk}(\theta) \cap \kappa = \psi[\theta]$. The set Σ is equal to a club subset of κ intersected with $\kappa \cap \mathrm{cof}(>\omega)$.

Definition 9.2. *Let $\theta \in \Sigma$. Define D_θ to be the set of conditions $r \in \mathbb{P}$ satisfying that A_r is θ-closed and there exist functions $f : \mathrm{dom}(W_r) \setminus \theta \to \mathrm{dom}(W_r) \cap \theta$ and $g : A_r \setminus \mathrm{Sk}(\theta) \to A_r \cap \mathrm{Sk}(\theta)$ satisfying:*

(a) *for all $\eta \in \mathrm{dom}(W_r) \setminus \theta$, $W(\eta) = W(f(\eta))$;*
(b) *for all $M \in A_r \setminus \mathrm{Sk}(\theta)$, $M \cap \omega_1 = g(M) \cap \omega_1$ and $M \cap \theta \subseteq g(M)$;*
(c) *for all $\eta \in \mathrm{dom}(W_r) \setminus \theta$ and for all $M \in A_r \setminus \mathrm{Sk}(\theta)$, $\eta \in M$ iff $f(\eta) \in g(M)$;*
(d) *for all $\eta \in \mathrm{dom}(W_r) \setminus \theta$ and for all $\xi \in \mathrm{dom}(W_r) \cap \theta$, if $\{\eta, \xi\} \in D_r$ then $\{f(\eta), \xi\} \in D_r$;*
(e) *for all $\eta, \xi \in \mathrm{dom}(W_r) \setminus \theta$, if $\{\eta, \xi\} \in D_r$ then $\{f(\eta), f(\xi)\} \in D_r$;*
(f) *if $K, M \in A_r \setminus \mathrm{Sk}(\theta)$ and $K \subseteq M$, then $g(K) \subseteq g(M)$.*

Lemma 9.3. *Let $\theta \in \Sigma$. Suppose that $q = (T, W, D, A) \in \mathbb{P}$ and A is θ-closed. Assume that $\bar{q} = (\bar{T}, \bar{W}, \bar{D}, \bar{A}) \in \mathrm{Sk}(\theta) \cap \mathbb{P}$, where $\bar{T} = T$, and there exist bijections $f : \mathrm{dom}(W) \to \mathrm{dom}(\bar{W})$ and $g : A \to \bar{A}$ satisfying:*

(1) *for all $\eta \in \mathrm{dom}(W) \cap \theta$, $f(\eta) = \eta$;*
(2) *for all $M \in A \cap \mathrm{Sk}(\theta)$, $g(M) = M$;*
(3) *for all $\eta \in \mathrm{dom}(W)$, $W(\eta) = \bar{W}(f(\eta))$;*
(4) *for all $M \in A$, $M \cap \omega_1 = g(M) \cap \omega_1$ and $M \cap \theta \subseteq g(M)$;*
(5) *for all $\eta \in \mathrm{dom}(W)$ and for all $M \in A$, $\eta \in M$ iff $f(\eta) \in g(M)$.*

Then $\bar{q} \oplus q$ is a condition in $\mathbb{P}$ which extends $\bar{q}$ and q.

Proof. Write $\bar{q} \oplus q = (T, Y, E, C)$. Then $\mathrm{dom}(Y) = \mathrm{dom}(\bar{W}) \cup \mathrm{dom}(W)$. By (1) and (2), $\mathrm{dom}(W) \cap \theta \subseteq \mathrm{dom}(\bar{W})$ and $A \cap \mathrm{Sk}(\theta) \subseteq \bar{A}$. By (1) and (3), for all $\eta \in \mathrm{dom}(\bar{W}) \cap \mathrm{dom}(W)$, $\bar{W}(\eta) = W(\eta)$. If $\eta \in \mathrm{dom}(\bar{W})$, then $Y(\eta) = \bar{W}(\eta)$, and if $\eta \in \mathrm{dom}(W)$, then $Y(\eta) = W(\eta)$.

Choosing any $\bar{\delta} < \delta$ in C_h such that $\bar{\delta} > \max(T)$, it is simple to check that $(\bar{T}, \bar{W}, \bar{D})$ and (T, W, D) are $(\bar{\delta}, \delta)$-split. By Lemma 2.15, it follows that $(\bar{T}, \bar{W}, \bar{D}) \oplus (T, W, D)$ is in $\mathbb{P}^*$ and extends $(\bar{T}, \bar{W}, \bar{D})$ and (T, W, D). We know that $A \cap \mathrm{Sk}(\theta) \subseteq \bar{A} \subseteq \mathrm{Sk}(\theta)$. By Theorem 5.15, $C = \bar{A} \cup A$ is adequate. Also, obviously $\bar{A}$ and A are subsets of C.

It remains to prove that Y is C-separated. Let $M \in C$, let η and ξ be distinct elements of $M \cap \mathrm{dom}(Y)$, and let $x \in Y(\eta) \cap Y(\xi)$. We prove that $x \in M$.

Case 1: $M \in \bar{A}$ and η and ξ are both in $\mathrm{dom}(\bar{W})$. Then $Y(\eta) = \bar{W}(\eta)$ and $Y(\xi) = \bar{W}(\xi)$. So $x \in \bar{W}(\eta) \cap \bar{W}(\xi)$. Since $\bar{W}$ is $\bar{A}$-separated, $x \in M$.

Case 2: $M \in \bar{A}$ and at least one of η or ξ is not in $\mathrm{dom}(\bar{W})$. Without loss of generality, assume that $\eta \notin \mathrm{dom}(\bar{W})$. Since $\mathrm{dom}(W) \cap \theta \subseteq \mathrm{dom}(\bar{W})$, η is not in θ. But $\eta \in M \subseteq \theta$, which is a contradiction.

Case 3: $M \in A$ and η and ξ are both in $\mathrm{dom}(W)$. Then $Y(\eta) = W(\eta)$ and $Y(\xi) = W(\xi)$. So $x \in W(\eta) \cap W(\xi)$, and therefore $x \in M$ since W is A-separated.

Case 4: $M \in A$ and neither η nor ξ are in $\mathrm{dom}(W)$. Then η and ξ are in $\mathrm{dom}(\bar{W}) \subseteq \theta$. Since A is θ-closed and $\theta^\omega \subseteq \mathrm{Sk}(\theta)$, $M \cap \theta \in A \cap \mathrm{Sk}(\theta) \subseteq \bar{A}$. So η and ξ are in $(M \cap \theta) \cap \mathrm{dom}(\bar{W})$. By Case 1, $x \in M \cap \theta \subseteq M$.

Case 5: $M \in A$ and one of η or ξ is in $\mathrm{dom}(W)$ and the other is not in $\mathrm{dom}(W)$. Without loss of generality, assume that $\eta \in \mathrm{dom}(W)$ and $\xi \in \mathrm{dom}(\bar{W}) \setminus \mathrm{dom}(W)$. Then $\xi \in \theta$, $Y(\eta) = W(\eta) = \bar{W}(f(\eta))$ by (3), and $Y(\xi) = \bar{W}(\xi)$. By (4), $M \cap \omega_1 = g(M) \cap \omega_1$ and $M \cap \theta \subseteq g(M)$. So $\xi \in g(M)$. Also, $\eta \in M$ implies that $f(\eta) \in g(M)$ by (5). So $x \in \bar{W}(f(\eta)) \cap \bar{W}(\xi)$ and $g(M) \in \bar{A}$. As $\bar{A}$ is $\bar{W}$-separated, it follows that $x \in g(M) \cap \omega_1 \subseteq M$. □

Proposition 9.4. *Let $\theta \in \Sigma$. Then D_θ is dense in $\mathbb{P}$. In fact, if $q \in \mathbb{P}$ and $N \in A_q$, then there exists $r \leq q$ which is in D_θ and satisfies that A_r is N-closed.*

Proof. Let $\mathcal{B} = (H(\kappa), \in, \psi, \mathcal{X}, \mathbb{P})$. Consider $q \in \mathbb{P}$ and $N \in A_q$, and we find an extension $r \leq q$ which is in D_θ and satisfies that A_r is N-closed. Write $q = (T, W, D, A)$. By extending further if necessary using in succession Lemmas 6.7 and 6.8, we may assume that A is N-closed and θ-closed.

Enumerate $\mathrm{dom}(W) = \{\eta_0, \ldots, \eta_{m-1}\}$ and $A = \{K_0, \ldots, K_{n-1}\}$, where $m, n < \omega$. Define:

- $U_0 = \{k < m : \eta_k \in \theta\}$;
- $U_1 = \{l < n : M_l \in \mathrm{Sk}(\theta)\}$;
- $U_2 = \{(k, l) \in m \times n : \eta_k \in K_l\}$;
- $U_3 = \{(j, k) \in m \times m : \{\eta_j, \eta_k\} \in D\}$;
- $U_4 = \{(k, l) \in n \times n : K_k \subseteq K_l\}$.

Define a formula with free variables

$$\varphi = \varphi(\dot{q}, \dot{T}, \dot{W}, \dot{D}, \dot{A}, \dot{\eta}_0, \ldots, \dot{\eta}_{m-1}, \dot{K}_0, \ldots, \dot{K}_{n-1})$$

to be the conjunction of the following statements:

(1) $\dot{q} = (\dot{T}, \dot{W}, \dot{D}, \dot{A}) \in \mathbb{P}$;
(2) $\dot{T} = T$;
(3) $\mathrm{dom}(\dot{W}) = \{\dot{\eta}_0, \ldots, \dot{\eta}_{m-1}\}$;
(4) $\dot{A} = \{\dot{K}_0, \ldots, \dot{K}_{n-1}\}$;
(5) for all $k \in U_0$, $\dot{\eta}_k = \eta_k$;
(6) for all $l \in U_1$, $\dot{K}_l = K_l$;
(7) for all $k < m$, $\dot{W}(\dot{\eta}_k) = W(\eta_k)$;
(8) for all $l < n$, $\dot{K}_l \cap \omega_1 = K_l \cap \omega_1$ and $K_l \cap \theta \subseteq \dot{K}_l$;
(9) for all $(k, l) \in m \times n$, $(k, l) \in U_2$ iff $\dot{\eta}_k \in \dot{K}_l$;
(10) for all $(j, k) \in m \times m$, $(j, k) \in U_3$ iff $\{\dot{\eta}_j, \dot{\eta}_k\} \in \dot{D}$;
(11) for all $(k, l) \in n \times n$, $(k, l) \in U_4$ iff $\dot{K}_k \subseteq \dot{K}_l$;
(12) for all $l < n$, $\dot{K}_l \cap (N \cap \theta) \in \dot{A}$.

Note that all of the parameters appearing in φ are members of $\mathrm{Sk}(\theta)$, and that

$$\mathcal{B} \models \varphi[q, T, W, D, A, \eta_0, \ldots, \eta_{m-1}, K_0, \ldots, K_{n-1}].$$

By elementarity, we can find in $\mathrm{Sk}(\theta)$ objects $\bar{q}$, $\bar{T}$, $\bar{W}$, $\bar{D}$, $\bar{A}$, $\bar{\eta}_0, \ldots, \bar{\eta}_{m-1}$, and $\bar{K}_0, \ldots, \bar{K}_{n-1}$ which satisfy the same. By (2), $\bar{T} = T$. By (3) and (5), $\mathrm{dom}(W) \cap \theta \subseteq \mathrm{dom}(\bar{W})$, and by (4) and (6), $A \cap \mathrm{Sk}(\theta) \subseteq \bar{A}$. By (12), for all $K \in \bar{A}$, $K \cap N = K \cap (N \cap \theta) \in \bar{A}$. So $\bar{A}$ is N-closed.

Define functions $f : \mathrm{dom}(W) \to \mathrm{dom}(\bar{W})$ and $g : A \to \bar{A}$ by letting $f(\eta_k) = \bar{\eta}_k$ for all $k < m$ and $g(K_l) = \bar{K}_l$ for all $l < n$. Using the definition of φ, it is routine to check that q, $\bar{q}$, f, and g satisfy all of the assumptions of Lemma 9.3. Hence, $\bar{q} \oplus q$ is in $\mathbb{P}$ and extends $\bar{q}$ and q. Since $A_{\bar{q}}$ and A_q are both N-closed, so is $A_{\bar{q}} \cup A_q = A_{\bar{q} \oplus q}$.

We claim that $\bar{q} \oplus q$ is in D_θ, which completes the proof. Write $\bar{q} \oplus q = (T, Y, E, C)$. The set A is θ-closed, and if $K \in \bar{A}$, then $K \in \mathrm{Sk}(\theta)$ so $K \cap \theta = K \in \bar{A}$. Hence, $C = A \cup \bar{A}$ is θ-closed. For the functions described in Definition 9.2, we use $f_0 = f \restriction (\mathrm{dom}(W) \setminus \theta)$ and $g_0 = g \restriction (A \setminus \mathrm{Sk}(\theta))$.

(3a) Let $\eta \in \mathrm{dom}(Y) \setminus \theta = \mathrm{dom}(W) \setminus \theta$. Fix $k < m$ such that $\eta = \eta_k$. Then $Y(\eta_k) = W(\eta_k)$. By (7), $Y(f_0(\eta)) = \bar{W}(\bar{\eta}_k) = W(\eta_k) = Y(\eta_k)$.

(3b) Let $K \in C \setminus \mathrm{Sk}(\theta) = A \setminus \mathrm{Sk}(\theta)$. Fix $l < n$ such that $K = K_l$. By (8), $g_0(K_l) \cap \omega_1 = \bar{K}_l \cap \omega_1 = K_l \cap \omega_1$ and $K_l \cap \theta \subseteq \bar{K}_l = g_0(K_l)$.

(3c) Let $\eta \in \mathrm{dom}(Y) \setminus \theta = \mathrm{dom}(W) \setminus \theta$ and let $K \in C \setminus \mathrm{Sk}(\theta) = A \setminus \mathrm{Sk}(\theta)$. Fix $k < m$ and $l < n$ such that $\eta = \eta_k$ and $K = K_l$. Then by (9), $\eta_k \in K_l$ iff $(k, l) \in U_2$ iff $\bar{\eta}_k \in \bar{K}_l$ iff $f_0(\eta_k) \in g_0(K_l)$.

(3d) Let $\eta \in \mathrm{dom}(Y) \setminus \theta$ and let $\xi \in \mathrm{dom}(Y) \cap \theta$. Then $\eta \in \mathrm{dom}(W)$. Assume that $\{\eta, \xi\} \in E$. Since $E = \bar{D} \cup D$ and $\eta \notin \theta$, $\{\eta, \xi\} \in D$. So $\xi \in \mathrm{dom}(W)$. Fix $j, k < m$ such that $\eta = \eta_j$ and $\xi = \eta_k$. Then $(j, k) \in U_3$. By (10), $\{f_0(\eta), f_0(\xi)\} \in \bar{D}$. But $\xi \in \theta$ implies that $k \in U_0$. Hence by (5), $f_0(\xi) = \xi$. So $\{f_0(\eta), \xi\} \in \bar{D}$, and therefore $\{f_0(\eta), \xi\} \in E$.

(3e) Let $\eta, \xi \in \mathrm{dom}(Y) \setminus \theta$. Then $\eta, \xi \in \mathrm{dom}(W)$. Assume that $\{\eta, \xi\} \in E$. Since $E = \bar{D} \cup D$ and $\eta \notin \theta$, $\{\eta, \xi\} \in D$. Fix $j, k < m$ such that $\eta = \eta_j$ and $\xi = \eta_k$. Then $(j, k) \in U_3$. By (10), $\{f_0(\eta), f_0(\xi)\} \in \bar{D} \subseteq E$.

(3f) Let $K, M \in C \setminus \mathrm{Sk}(\theta)$ and assume that $K \subseteq M$. Fix $k, l < n$ such that $K = K_k$ and $M = K_l$. Then $(k, l) \in U_4$. By (11), $f_0(K_k) = \bar{K}_k \subseteq \bar{K}_l = f_0(K_l)$. □

10. Projection and Chain Condition

In this section, we prove that for all $\theta \in \Sigma$, a certain natural map of a dense subset of $\mathbb{P}$ into the suborder $\mathbb{P} \cap \mathrm{Sk}(\theta)$ is a projection mapping.

Definition 10.1. *For any $\theta \in \Sigma$, let $\mathbb{P}_\theta = \mathbb{P} \cap \mathrm{Sk}(\theta)$.*

Definition 10.2. *For any $\theta \in \Sigma$, define π_θ with domain $\mathbb{P}$ by letting*

$$\pi_\theta(p) = (T_p, W_p \restriction \theta, D_p \cap [\theta]^2, A_p \cap \mathrm{Sk}(\theta)).$$

Lemma 10.3. *Let $\theta \in \Sigma$.*

1. *For any $p \in \mathbb{P}$, $\pi_\theta(p) \in \mathbb{P}_\theta$ and $p \leq \pi_\theta(p)$.*
2. *If $q \leq p$, then $\pi_\theta(q) \leq \pi_\theta(p)$.*
3. *If $q \leq s$, where $q \in \mathbb{P}$ and $s \in \mathbb{P}_\theta$, then $\pi_\theta(q) \leq s$.*

The proof is straightforward.

Lemma 10.4. *Let $\theta \in \Sigma$. Let $1 < d < \omega$ and suppose that $p_0, \dots, p_{d-1}$ are in $\mathbb{P}$ and $p_0 \oplus \cdots \oplus p_{d-1} \in \mathbb{P}$. Then*

$$\pi_\theta(p_0 \oplus \cdots \oplus p_{d-1}) = \pi_\theta(p_0) \oplus \cdots \oplus \pi_\theta(p_{d-1}).$$

The proof is easy.

Definition 10.5. *Define a function w_θ as follows. The domain of w_θ is the set of ordered pairs (q, N) such that $q \in D_\theta$, $N \in A_q$, and A_q is N-closed. For any such ordered pair (q, N), define*

$$w_\theta(q, N) = w(q, N)^\frown \langle f \restriction N, g \restriction \mathrm{Sk}(N)\rangle,$$

where f and g are the $\lhd$-least witnesses to the fact that $q \in D_\theta$.

Lemma 10.6. *Let $\theta \in \Sigma$. Suppose that p and q are in D_θ as witnessed by functions f_p and g_p for p and f_q and g_q for q. Assume that $w_\theta(p, M) = w_\theta(q, N)$ and $p \in \mathrm{Sk}(N)$. Then:*

1. *For all $\eta \in \mathrm{dom}(W_q) \setminus \theta$ and for all $K \in A_p \setminus \mathrm{Sk}(\theta)$, $\eta \in K$ implies that $f_q(\eta) \in g_p(K)$.*
2. *For all $\eta \in \mathrm{dom}(W_p) \setminus \theta$ and for all $K \in A_q \setminus \mathrm{Sk}(\theta)$ such that $K < N$, $\eta \in K$ implies that $f_p(\eta) \in g_q(K)$.*

Proof. Since $w_\theta(p, M) = w_\theta(q, N)$, $f_p \restriction M = f_q \restriction N$ and $g_p \restriction \mathrm{Sk}(M) = g_q \restriction \mathrm{Sk}(N)$. By Lemma 8.7, for all $\eta \in (\mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)) \setminus \theta$, $\eta \in M \cap N$ and hence $f_p(\eta) = f_q(\eta)$, and for all $K \in ((A_p \cap A_q) \setminus \mathrm{Sk}(\theta))$, $K \in \mathrm{Sk}(M) \cap \mathrm{Sk}(N)$ and so $g_p(K) = g_q(K)$.

(1) Let $\eta \in \mathrm{dom}(W_q) \setminus \theta$ and $K \in A_p \setminus \mathrm{Sk}(\theta)$, and suppose that $\eta \in K$. First, assume that $\eta \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$. Then $f_p(\eta) = f_q(\eta)$. Since $\eta \in K$, by Definition 9.2(c), $f_q(\eta) = f_p(\eta) \in g_p(K)$. Now assume that $\eta \in \mathrm{dom}(W_q) \setminus \mathrm{dom}(W_p)$. Since $p \in \mathrm{Sk}(N)$, $K \in \mathrm{Sk}(N)$. As $\eta \in K$, $\eta \in \mathrm{dom}(W_q) \cap N = \mathrm{dom}(W_p) \cap M$, which contradicts that $\eta \notin \mathrm{dom}(W_p)$.

(2) Let $\eta \in \mathrm{dom}(W_p) \setminus \theta$ and $K \in A_q \setminus \mathrm{Sk}(\theta)$ with $K < N$, and suppose that $\eta \in K$. First, assume that $\eta \in \mathrm{dom}(W_p) \cap \mathrm{dom}(W_q)$. Then $f_p(\eta) = f_q(\eta)$. Since $\eta \in K$, by Definition 9.2(c), $f_p(\eta) = f_q(\eta) \in g_q(K)$. Secondly, assume that $K \in A_p \cap A_q$. Then $g_p(K) = g_q(K)$. Since $\eta \in K$, by Definition 9.2(c), $f_p(\eta) \in g_p(K) = g_q(K)$.

Now assume that $\eta \in \mathrm{dom}(W_p) \setminus \mathrm{dom}(W_q)$ and $K \in A_q \setminus A_p$. Since $p \in \mathrm{Sk}(N)$, $\eta \in N$. So $\eta \in K \cap N$. Since $(q, N) \in \mathrm{dom}(w)$, A_q is N-closed. As $K < N$, it follows that $K \cap N \in A_q \cap \mathrm{Sk}(N) = A_p \cap \mathrm{Sk}(M)$. Since $\eta \in (K \cap N) \setminus \theta$, $K \cap N \notin \mathrm{Sk}(\theta)$. So $g_p(K \cap N) = g_q(K \cap N)$. As $\eta \in K \cap N$, by Definition 9.2(c), $f_p(\eta) \in g_p(K \cap N) = g_q(K \cap N)$. By Definition 9.2(f), $g_q(K \cap N) \subseteq g_q(K)$, so $f_p(\eta) \in g_q(K)$. □

It follows from Proposition 10.9 below that π_θ restricted to the dense set D_θ is a projection mapping into $\mathbb{P}_\theta$. However, in order to prove that quotient forcings of $\mathbb{P}$ are Y-proper, this is not enough. In particular, Lemma 11.6 below needs π_θ to be a projection mapping on a larger set of conditions, which we introduce now.

Definition 10.7. *Let $\theta \in \Sigma$. Define E_θ to be the set of conditions $p \in \mathbb{P}$ such that either $p \in D_\theta$ and $A_p \neq \emptyset$, or else for some $1 < d < \omega$, $p_0, \dots, p_{d-1}$, and $N_0, \dots, N_{d-1}$:*

1. *$p_0, \dots, p_{d-1}$ are in D_θ;*
2. *$p = p_0 \otimes \cdots \otimes p_{d-1}$;*
3. *for all $i < d$, $(p_i, N_i) \in \mathrm{dom}(w_\theta)$;*
4. *for all $i < j < d$, $w_\theta(p_i, N_i) = w_\theta(p_j, N_j)$;*

(5) *for all* $i < j < d$, $p_i \in \mathrm{Sk}(N_j)$.

Note that by Lemma 8.6, in the above p is an extension of each of $p_0, \dots, p_{d-1}$.

Lemma 10.8. *For any* $\theta \in \Sigma$, E_θ *is dense in* $\mathbb{P}$.

Proof. Immediate by Lemma 6.6 and Proposition 9.4. □

Proposition 10.9. *Let* $\theta \in \Sigma$. *Assume that* $p \in E_\theta$ *and* $s \leq \pi_\theta(p)$ *in* $\mathbb{P}_\theta$. *Define* Y *with domain equal to* $\mathrm{dom}(W_s) \cup \mathrm{dom}(W_p)$ *so that for all* $\eta \in \mathrm{dom}(W_s)$, $Y(\eta) = W_s(\eta)$, *and for all* $\xi \in \mathrm{dom}(W_p) \setminus \mathrm{dom}(W_s)$, $Y(\xi)$ *is the downward closure of* $W_p(\xi)$ *in* T_s. *Then* $(T_s, Y, D_s \cup D_p, A_s \cup A_p)$ *is in* $\mathbb{P}$ *and is an extension of* s *and* p.

Proof. We begin by fixing some notation. If $p \notin D_\theta$, then fix $1 < d < \omega$, $p_0, \dots, p_{d-1}$, and $N_0, \dots, N_{d-1}$ witnessing that $p \in E_\theta \setminus D_\theta$ (and in particular, $p = p_0 \oplus \cdots \oplus p_{d-1}$), and for each $i < d$, fix functions f_i and g_i witnessing that $p_i \in D_\theta$. If $p \in D_\theta$, then let $p_0 = p$, let N_0 be any member of A_p, and let f_0 and g_0 witness that $p \in D_\theta$. By Lemma 10.3(2), for all $i < d$, $s \leq \pi_\theta(p_i)$. Since $T_{\pi_\theta(p)} = T_p$, T_s is an end-extension of T_p. For each $i < d$, write $p_i = (T_i, W_i, D_i, A_i)$.

Claim: If $\eta \in \mathrm{dom}(W_p) \setminus \theta$ and $x \in Y(\eta)$, then there exists $j < d$ such that $\eta \in \mathrm{dom}(W_j)$, x is in the downward closure of $W_j(f_j(\eta))$ in T_s, and $x \in W_s(f_j(\eta))$.

Proof: Since $Y(\eta)$ is the downward closure of $W_p(\eta)$ in T_s, we can fix $j < d$ such that x is in the downward closure of $W_j(\eta)$ in T_s. But $W_j(\eta) = W_j(f_j(\eta))$ by Definition 9.2(a). So x is in the downward closure of $W_j(f_j(\eta))$ in T_s. Since $s \leq \pi_\theta(p_j)$, $W_j(f_j(\eta)) \subseteq W_s(f_j(\eta))$, and as $W_s(f_j(\eta))$ is downward closed in T_s, $x \in W_s(f_j(\eta))$. This completes the proof of the claim.

It is straightforward to check that $(T_s, Y, D_s \cup D_p)$ is in $\mathbb{P}^*$ and extends (T_s, W_s, D_s) in $\mathbb{P}^*$. Concerning $(T_s, Y, D_s \cup D_p)$ being an extension of (T_p, W_p, D_p) in $\mathbb{P}^*$, (a-c) of Definition 2.9 follow easily from the fact that $s \leq \pi_\theta(p)$.

For (d) of Definition 2.9, suppose that $\{\eta, \xi\} \in D_p$ and $x \in Y(\eta) \cap Y(\xi)$. We show that there exists some $z \in W_p(\eta) \cap W_p(\xi)$ such that $x \leq_{T_s} z$. Fix $i < d$ such that $\{\eta, \xi\} \in D_i$.

Case A: η and ξ are both in θ. Since $s \leq \pi_\theta(p)$, easily η and ξ are both in $\mathrm{dom}(W_s)$. Hence, $Y(\eta) = W_s(\eta)$ and $Y(\xi) = W_s(\xi)$. So $x \in W_s(\eta) \cap W_s(\xi)$. Also, $\{\eta, \xi\} \in D_p \cap [\theta]^2 = D_{\pi_\theta(p)}$. Since $s \leq \pi_\theta(p)$, there exists $z \in W_{\pi_\theta(p)}(\eta) \cap W_{\pi_\theta(p)}(\xi) = W_p(\eta) \cap W_p(\xi)$ such that $x \leq_{T_s} z$.

Case B: One of η or ξ is in θ and the other is not. Without loss of generality, assume that $\eta \notin \theta$ and $\xi \in \theta$. Since $s \leq \pi_\theta(p)$, easily $\xi \in \mathrm{dom}(W_s)$. By Definition 9.2(d), $\{f_i(\eta), \xi\} \in D_i$. By the claim, fix $j < d$ such that $\eta \in \mathrm{dom}(W_j)$ and $x \in W_s(f_j(\eta))$. Now $\eta \in \mathrm{dom}(W_i) \cap \mathrm{dom}(W_j)$ implies that $f_i(\eta) = f_j(\eta)$. So $\{f_j(\eta), \xi\} \in D_i \cap [\theta]^2 \subseteq D_p \cap [\theta]^2 = D_{\pi_\theta(p)}$. As $s \leq \pi_\theta(p)$ and $x \in W_s(f_j(\eta)) \cap W_s(\xi)$, there exists $z \in W_{\pi_\theta(p)}(f_j(\eta)) \cap W_{\pi_\theta(p)}(\xi)$ such that $x \leq_{T_s} z$. Then $z \in W_p(f_i(\eta)) \cap W_p(\xi)$. Since $\{f_i(\eta), \xi\} \in D_i$ and $p \leq p_i$, there exists $c \in W_i(f_i(\eta)) \cap W_i(\xi)$ such that $z \leq_{T_p} c$. Then $c \in W_i(\eta)$ by Definition 9.2(a). So $c \in W_p(\eta) \cap W_p(\xi)$. As T_s end-extends T_p, $x \leq_{T_s} z \leq_{T_s} c$ and so $x \leq_{T_s} c$.

Case C: η and ξ are both in $\mathrm{dom}(W_p) \setminus \theta$. By the claim, fix $j, k < d$ such that $\eta \in \mathrm{dom}(W_j)$, $\xi \in \mathrm{dom}(W_k)$, and $x \in W_s(f_j(\eta)) \cap W_s(f_k(\xi))$. By Definition 9.2(e), $\{f_i(\eta), f_i(\xi)\} \in D_i$. Since $\eta \in \mathrm{dom}(f_i) \cap \mathrm{dom}(f_j)$ and $\xi \in \mathrm{dom}(f_i) \cap \mathrm{dom}(f_k)$, $f_i(\eta) = f_j(\eta)$ and $f_i(\xi) = f_k(\xi)$. So $\{f_j(\eta), f_k(\xi)\} \in D_i \cap [\theta]^2 \subseteq D_p \cap [\theta]^2 = D_{\pi_\theta(p)}$. Since $s \leq \pi_\theta(p)$ and $x \in W_s(f_j(\eta)) \cap W_s(f_k(\xi))$, there exists some $z \in W_{\pi_\theta(p)}(f_j(\eta)) \cap W_{\pi_\theta(p)}(f_k(\xi))$ such that $x \leq_{T_s} z$. Then $z \in W_p(f_i(\eta)) \cap W_p(f_i(\xi))$. As $p \leq p_i$ and $\{f_i(\eta), f_i(\xi)\} \in D_i$, there exists

$c \in W_i(f_i(\eta)) \cap W_i(f_i(\xi))$ such that $z \leq_{T_p} c$. Then $c \in W_i(\eta) \cap W_i(\xi)$ by Definition 9.2(a), and hence $c \in W_p(\eta) \cap W_p(\xi)$. As T_s end-extends T_p, $x \leq_{T_s} z \leq_{T_s} c$, so $x \leq_{T_s} c$. This completes the proof that $(T_s, Y, D_s \cup D_p)$ is an extension of (T_p, W_p, D_p) in $\mathbb{P}^*$.

Since A_i is θ-closed for all $i < d$, it easily follows that $A_p = A_0 \cup \cdots \cup A_{d-1}$ is θ-closed. As $s \leq \pi_\theta(p)$, $A_p \cap \mathrm{Sk}(\theta) = A_{\pi_\theta(p)} \subseteq A_s$. Also, $s \in \mathbb{P}_\theta$ implies $A_s \subseteq \mathrm{Sk}(\theta)$. By Theorem 5.15, $A_s \cup A_p$ is adequate.

It remains to prove that Y is $(A_s \cup A_p)$-separated. Let $K \in A_s \cup A_p$, let η and ξ be distinct elements of $K \cap \mathrm{dom}(Y)$, and let $x \in Y(\eta) \cap Y(\xi)$. We prove that $x \in K$.

Case 1: $K \in A_s$ and η and ξ are both in $\mathrm{dom}(W_s)$. Then by definition, $Y(\eta) = W_s(\eta)$ and $Y(\xi) = W_s(\xi)$. Hence, $x \in W_s(\eta) \cap W_s(\xi)$. As W_s is A_s-separated, it follows that $x \in K$.

Case 2: $K \in A_s$ and at least one of η or ξ is not in $\mathrm{dom}(W_s)$. Without loss of generality, assume that $\eta \notin \mathrm{dom}(W_s)$. Since $\mathrm{dom}(W_p) \cap \theta \subseteq \mathrm{dom}(W_s)$, η is not in θ. But $\eta \in K \subseteq \theta$, which is a contradiction.

Case 3: $K \in A_p \setminus A_s$ and η and ξ are both in $\mathrm{dom}(W_s)$. Then $K \notin \mathrm{Sk}(\theta)$. We have that $Y(\eta) = W_s(\eta)$ and $Y(\xi) = W_s(\xi)$, so $x \in W_s(\eta) \cap W_s(\xi)$. Fix $i < d$ such that $K \in A_i$. Then $\eta, \xi \in K \cap \theta \subseteq g_i(K)$ by Definition 9.2(b). Since W_s is A_s-separated and $g_i(K) \in A_s$, it follows that $x \in g_i(K) \cap \omega_1 \subseteq K$.

Case 4: $K \in A_p \setminus A_s$ and one of η or ξ is in $\mathrm{dom}(W_p) \setminus \theta$ and the other is in $\mathrm{dom}(W_s)$. Without loss of generality, assume that $\eta \in \mathrm{dom}(W_p) \setminus \theta$ and $\xi \in \mathrm{dom}(W_s)$. Then $Y(\xi) = W_s(\xi)$. By the claim, fix $j < d$ such that $\eta \in \mathrm{dom}(W_j)$, x is in the downward closure of $W_j(f_j(\eta))$ in T_s, and $x \in W_s(f_j(\eta))$. Fix $i < d$ such that $K \in A_i \setminus \mathrm{Sk}(\theta)$. By Definition 9.2(b), $\xi \in K \cap \theta \subseteq g_i(K)$.

First, consider the case that $j < i$ and $N_i \leq K$. Since x is in the downward closure of $W_j(f(\eta))$ in T_s, fix $z \in W_j(f(\eta))$ such that $x \leq_{T_s} z$. Then $j < i$ implies that $p_j \in \mathrm{Sk}(N_i)$, and hence $z \in N_i \cap \omega_1 \leq K \cap \omega_1$. So $z \in K \cap \omega_1$, and as $x \leq z$, $x \in K \cap \omega_1$ as well.

Secondly, assume that either $i \leq j$, or $j < i$ and $K < N_i$. We claim that $f_j(\eta) \in g_i(K)$. If $i = j$, then $\eta \in K$ implies by Definition 9.2(c) that $f_j(\eta) \in g_j(K) = g_i(K)$. If $i < j$, then $f_j(\eta) \in g_i(K)$ by Lemma 10.6(1). If $j < i$ and $K < N_i$, then $f_j(\eta) \in g_i(K)$ by Lemma 10.6(2). So indeed $f_j(\eta) \in g_i(K)$. But $x \in W_s(\xi) \cap W_s(f_j(\eta))$ and $g_i(K) \in A_i \cap \mathrm{Sk}(\theta) = A_{\pi_\theta(p_i)} \subseteq A_s$. Since ξ and $f_j(\eta)$ are in $g_i(K)$ and W_s is A_s-separated, it follows that $x \in g_i(K) \cap \omega_1 = K \cap \omega_1$.

Case 5: $K \in A_p \setminus A_s$ and η and ξ are both in $\mathrm{dom}(W_p) \setminus \theta$. By the claim, fix $j < d$ such that $\eta \in \mathrm{dom}(W_j)$, x is in the downward closure of $W_j(f_j(\eta))$ in T_s, and $x \in W_s(f_j(\eta))$. And fix $k < d$ such that $\xi \in \mathrm{dom}(W_k)$, x is in the downward closure of $W_k(f_k(\xi))$ in T_s, and $x \in W_s(f_k(\xi))$. Fix $i < d$ such that $K \in A_i$.

Without loss of generality, assume that $j \leq k$. If $j < i$ and $N_i \leq K$, then by the same argument as in the first subcase of Case 4, $x \in K$. The remaining cases are: $i < j$, $j = i = k$, $j = i < k$, $j < i < k$ and $K < N_i$, $j < k = i$ and $K < N_i$, and $k < i$ and $K < N_i$. Note that by Definition 9.2(c), if $i = j$ then $f_j(\eta) \in g_i(K)$, and if $i = k$, then $f_k(\xi) \in g_i(K)$. Combining this information with Lemma 10.6, it is routine to check that both $f_j(\eta)$ and $f_k(\xi)$ are in $g_i(K)$. But $g_i(K) \in A_s$ and $x \in W_s(f_j(\eta)) \cap W_s(f_k(\xi))$. Since W_s is A_s-separated, it follows that $x \in g_i(K) \cap \omega_1 = K \cap \omega_1$. □

Proposition 10.10. *Let $\theta \in \Sigma$. Then $\pi_\theta \restriction E_\theta : E_\theta \to \mathbb{P}_\theta$ is a projection mapping.*

Proof. Clearly, π_θ maps the maximum condition in D_θ to the maximum condition in $\mathbb{P}_\theta$. The map π_θ is order-preserving by Lemma 10.3(2).

Suppose that $p \in E_\theta$ and $s \leq \pi_\theta(p)$ in $\mathbb{P}_\theta$. We find $r \leq p$ in E_θ such that $\pi_\theta(r) \leq s$. By extending further if necessary, assume that A_s is non-empty. Define Y with domain equal to $\mathrm{dom}(W_s) \cup \mathrm{dom}(W_p)$ so that for all $\eta \in \mathrm{dom}(W_s)$, $Y(\eta) = W_s(\eta)$, and for all $\xi \in \mathrm{dom}(W_p) \setminus \mathrm{dom}(W_s)$, $Y(\xi)$ is the downward closure of $W_p(\xi)$ in T_s. By Proposition 10.9, $(T_s, Y, D_s \cup D_p, A_s \cup A_p)$ is in $\mathbb{P}$ and extends p and s. Since D_θ is dense, fix $r \leq (T_s, Y, D_s \cup D_p, A_s \cup A_p)$ in D_θ. As A_s is non-empty, so is A_r, so $r \in E_\theta$. Then $r \leq p$, and since $r \leq s$, $\pi_\theta(r) \leq s$ by Lemma 10.3(3). □

Proposition 10.11. *The forcing poset $\mathbb{P}$ is κ-c.c.*

Proof. Let A be a maximal antichain of $\mathbb{P}$, and we show that $|A| < \kappa$. Fix a regular cardinal $\chi > \kappa$ such that $A \in H(\chi)$. Fix an elementary substructure Q of $\mathcal{B} = (H(\chi), \in, \psi, \mathcal{X}, \mathbb{P})$ satisfying that $|Q| < \kappa$, $\theta = Q \cap \kappa \in \Sigma$, and $A \in Q$. Note that by elementarity, $Q \cap H(\kappa) = \psi[\theta] = \mathrm{Sk}(\theta)$.

Suppose for a contradiction that $|A| \geq \kappa$. Then in particular, A is not a subset of Q, so we can fix some $p \in A \setminus Q$. Fix $q \leq p$ in E_θ. Then $\pi_\theta(q) \in \mathrm{Sk}(\theta) \subseteq Q$. Since A is maximal, by the elementarity of Q we can fix some $u \in Q \cap \mathbb{P}$ which extends both $\pi_\theta(q)$ and some element s of $Q \cap A$. Then $u \in Q \cap H(\kappa) = \mathrm{Sk}(\theta)$, so $u \in \mathbb{P}_\theta$. Since $\pi_\theta \restriction E_\theta$ is a projection mapping, fix $r \leq q$ in E_θ such that $\pi_\theta(r) \leq u$. By Lemma 10.3(1), $r \leq \pi_\theta(r) \leq u \leq s$. So r is below both p and s. Since A is an antichain, $p = s$, which is false since $s \in Q$ and $p \notin Q$. □

Corollary 10.12. *The forcing poset $\mathbb{P}$ forces that $\omega_1^V = \omega_1$ and $\kappa = \omega_2$.*

Proof. By Corollary 7.3, Proposition 7.4, and Proposition 10.11. □

Combining Corollary 10.12 with Propositions 7.6 and 7.7, we have the following theorem, where $\dot{G}$ is the canonical $\mathbb{P}$-name for a generic filter on $\mathbb{P}$.

Theorem 10.13. *The forcing poset $\mathbb{P}$ forces that $T_{\dot{G}}$ is a normal infinitely splitting Aronszajn tree which is strongly non-saturated.*

11. THE QUOTIENT FORCING

With a projection mapping at hand, we are now in a position to analyze the quotient forcing in an intermediate extension. In this section, we provide some information about the quotient forcing which we use in the next section to show that it is Y-proper on a stationary set.

For the remainder of the section, fix $\theta \in \Sigma$. Recall that $\pi_\theta \restriction E_\theta$ is a projection mapping from E_θ into $\mathbb{P}_\theta = \mathbb{P} \cap \mathrm{Sk}(\theta)$. For any generic filter H on $\mathbb{P}_\theta$, define in $V[H]$ the quotient forcings $E_\theta/H = \{q \in E_\theta : \pi_\theta(q) \in H\}$ and

$$\mathbb{P}/H = \{p \in \mathbb{P} : \exists q \in E_\theta/H \ (q \leq p)\},$$

considered as suborders of $\mathbb{P}$.

Lemma 11.1. *Let H be a generic filter on $\mathbb{P}_\theta$.*

1. *If $q \in \mathbb{P}/H$, $p \in \mathbb{P}$, and $q \leq p$, then $p \in \mathbb{P}/H$.*
2. *$\mathbb{P}/H$ is the set of all $q \in \mathbb{P}$ which are compatible in $\mathbb{P}$ with every member of H.*
3. *If $q \in \mathbb{P}/H$ and $s \in H$, then there exists some $r \in \mathbb{P}/H$ which extends q and s.*
4. *If in V, $\mathcal{D}$ is a dense open subset of $\mathbb{P}$, then in $V[H]$, $\mathcal{D} \cap (\mathbb{P}/H)$ is a dense open subset of $\mathbb{P}/H$.*

Proof. (1) is immediate. (2), (3), and (4) have routine proofs using density arguments in V. □

The following lemma is standard.

Lemma 11.2. (1) *If G is a V-generic filter on $\mathbb{P}$, then $H = \pi_\theta[G] = G \cap \mathbb{P}_\theta$ is a V-generic filter on $\mathbb{P}_\theta$, G is a $V[H]$-generic filter on $\mathbb{P}/H$, and $V[G] = V[H][G]$.*

(2) *If H is a generic filter on $\mathbb{P}_\theta$ and G is a $V[H]$-generic filter on $\mathbb{P}/H$, then G is a V-generic filter on $\mathbb{P}$, $H = G \cap \mathbb{P}_\theta = \pi_\theta[G]$, and $V[G] = V[H][G]$.*

We occasionally write $\dot{H}$ for the canonical $\mathbb{P}_\theta$-name for a generic filter on $\mathbb{P}_\theta$ when working in V.

For the remainder of the section, fix a generic filter H on $\mathbb{P}_\theta$.

Lemma 11.3. *If $p \in \mathbb{P}/H$, then $\pi_\theta(p) \in H$. Consequently, $E_\theta/H = E_\theta \cap (\mathbb{P}/H)$.*

The proof is easy.

The converse of Lemma 11.3 is false in general. For example, if $\{M, N\} \subseteq \mathcal{X}$ is not adequate, $(\emptyset, \emptyset, \emptyset, \{M\}) \in H$, and $N \notin \mathrm{Sk}(\theta)$, then $\pi_\theta(\emptyset, \emptyset, \emptyset, \{N\}) = (\emptyset, \emptyset, \emptyset, \emptyset) \in H$, but $(\emptyset, \emptyset, \emptyset, \{M\})$ is incompatible with $(\emptyset, \emptyset, \emptyset, \{N\})$. Therefore, the condition $(\emptyset, \emptyset, \emptyset, \{N\})$ is not in $\mathbb{P}/H$ by Lemma 11.1(2).

Definition 11.4. *Define $(T_H, <_H)$ by:*

- *$x \in T_H$ if there exists some $p \in H$ such that $x \in T_p$;*
- *$x <_H y$ if there exists some $p \in H$ such that $x <_p y$.*

As usual, we abbreviate $(T_H, <_H)$ by T_H.

Lemma 11.5. *If G is a $V[H]$-generic filter on $\mathbb{P}/H$, then $T_G = T_H$.*

Proof. This follows easily from Lemma 11.2 and the fact that for any $p \in \mathbb{P}$, $T_p = T_{\pi_\theta(p)}$. □

Lemma 11.6. *Let $p \in E_\theta$. Assume that $p_0, \ldots, p_{d-1}$ are in $\mathbb{P}/H$, where $1 < d < \omega$, $p = p_0 \oplus \cdots \oplus p_{d-1}$, and p extends each of $p_0, \ldots, p_{d-1}$. Suppose that for all $i < j < d$, for all $x \in T_{p_i} \setminus T_{p_j}$ and for all $y \in T_{p_j} \setminus T_{p_i}$, x and y are incomparable in T_H. Then $p \in \mathbb{P}/H$.*

Proof. The condition $\pi_\theta(p_0 \oplus \cdots \oplus p_{d-1})$ extends each of $\pi_\theta(p_0), \ldots, \pi_\theta(p_{d-1})$. By Lemma 10.4, $\pi_\theta(p) = \pi_\theta(p_0 \oplus \cdots \oplus p_{d-1}) = \pi_\theta(p_0) \oplus \cdots \oplus \pi_\theta(p_{d-1})$. It suffices to show that $\pi_\theta(p_0) \oplus \cdots \oplus \pi_\theta(p_{d-1}) \in H$, for then $\pi_\theta(p) \in H$ so $p \in E_\theta/H \subseteq \mathbb{P}/H$. As $p_0, \ldots, p_{d-1}$ are all in $\mathbb{P}/H$, for all $i < d$, $\pi_\theta(p_i) \in H$. Fix $r \in H$ such that $r \leq \pi_\theta(p_i)$ for all $i < d$.

For all $i < d$, $T_{p_i} = T_{\pi_\theta(p_i)}$. So for all $i < j < d$, for all $x \in T_{\pi_\theta(p_i)} \setminus T_{\pi_\theta(p_j)}$ and for all $y \in T_{\pi_\theta(p_j)} \setminus T_{\pi_\theta(p_i)}$, x and y are incomparable in T_H. Since $r \in H$, all such x and y are incomparable in T_r as well. Applying Lemma 6.10 to the conditions $\pi_\theta(p_0), \ldots, \pi_\theta(p_{d-1})$ and r, we get that $r \leq \pi_\theta(p_0) \oplus \cdots \oplus \pi_\theta(p_{d-1})$. Since $r \in H$, $\pi_\theta(p_0) \oplus \cdots \oplus \pi_\theta(p_{d-1}) \in H$. □

Definition 11.7. *Define $\mathcal{X}(H)$ to be the set of all $N \in \mathcal{X}$ such that $(\emptyset, \emptyset, \emptyset, \{N \cap \theta\}) \in H$.*

Lemma 11.8. *In $V[H]$, $\mathcal{X}(H)$ is a stationary subset of $[\kappa]^\omega$.*

Proof. We give a density argument in V. Let $p \in \mathbb{P}_\theta$ and suppose that $\dot{F}$ is a $\mathbb{P}_\theta$-name for a function from $\kappa^{<\omega}$ to κ. We find $s \leq p$ in $\mathbb{P}_\theta$ and $N \in \mathcal{X}$ such that s forces that N is closed under $\dot{F}$ and $N \in \mathcal{X}(\dot{H})$.

Fix a regular cardinal $\chi > \kappa$ with $\dot{F} \in H(\chi)$. Let M be a countable elementary substructure of $(H(\chi), \in, \psi, \mathbb{P}, \theta, \dot{H}, \dot{F})$ such that $p \in M$ and $N = M \cap \kappa \in \mathcal{X}$. By Theorem 7.2, $p + N$ is in $\mathbb{P}$, $p + N \leq p$, and $p + N$ is $(M, \mathbb{P})$-generic. Since $p + N$ is $(M, \mathbb{P})$-generic, $p + N$ forces that M, and hence N, is closed under $\dot{F}$.

Fix $r \leq p + N$ in D_θ. Since $r \in D_\theta$ and $N \in A_r$, $N \cap \theta \in A_r$. As $\theta^\omega \subseteq \mathrm{Sk}(\theta)$, $N \cap \theta \in \mathrm{Sk}(\theta)$. So $N \cap \theta \in A_r \cap \theta = A_{\pi_\theta(r)}$. Therefore, $\pi_\theta(r) \leq (\emptyset, \emptyset, \emptyset, \{N \cap \theta\})$ in $\mathbb{P}_\theta$. So $\pi_\theta(r)$ forces that $N \in \mathcal{X}(\dot{H})$. Since $r \leq p$ and $p \in \mathbb{P}_\theta$, by Lemma 10.3(3), $\pi_\theta(r) \leq p$. As r forces in $\mathbb{P}$ that N is closed under $\dot{F}$, an easy argument using Lemma 11.2 shows that $\pi_\theta(r)$ forces in $\mathbb{P}_\theta$ that N is closed under $\dot{F}$. □

Lemma 11.9. *Suppose that $p \in D_\theta \cap (\mathbb{P}/H)$, A_p is non-empty, $N \in \mathcal{X}(H)$, p and θ are in* $\mathrm{Sk}(N)$*, and A_p is non-empty. Then $p + N$ is in $\mathbb{P}/H$ and is an extension of p.*

Proof. By Lemma 6.6, $p + N$ is in $\mathbb{P}$ and is an extension of p. We prove that $p + N \in \mathbb{P}/H$ by giving a density argument in V. Assume that $s \in \mathbb{P}_\theta$ and s forces that $N \in \mathcal{X}(\dot{H})$ and $p \in \mathbb{P}/\dot{H}$. We find an extension of s in $\mathbb{P}_\theta$ which forces that $p + N \in \mathbb{P}/\dot{H}$. By extending further if necessary using Lemma 11.3, we may assume that $s \leq \pi_\theta(p)$ and $N \cap \theta \in A_s$. To show that s forces that $p + N \in \mathbb{P}/\dot{H}$, by Lemma 11.1(2) it suffices to show that for all $t \leq s$ in $\mathbb{P}_\theta$, t and $p + N$ are compatible in $\mathbb{P}$.

Fix $t \leq s$ in $\mathbb{P}_\theta$. Define Y with domain equal to $\mathrm{dom}(W_t) \cup \mathrm{dom}(W_p)$ so that for all $\eta \in \mathrm{dom}(W_t)$, $Y(\eta) = W_t(\eta)$, and for all $\xi \in \mathrm{dom}(W_p) \setminus \mathrm{dom}(W_t)$, $Y(\xi)$ is the downward closure of $W_p(\xi)$ in T_t. Since $p \in E_\theta$ and $t \leq \pi_\theta(p)$, by Proposition 10.9 we have that $u = (T_t, Y, D_t \cup D_p, A_t \cup A_p)$ is in $\mathbb{P}$ and extends t and p. Define $v = u + N$.

We prove that $v \in \mathbb{P}$ and v is an extension of t and $p + N$, which completes the proof. Now u is in $\mathbb{P}$, u extends t and p, and $(T_u, W_u, D_u) = (T_v, W_v, D_v)$. It follows that (T_v, W_v, D_v) is in $\mathbb{P}^*$ and extends (T_t, W_t, D_t) and (T_p, W_p, D_p) in $\mathbb{P}^*$. Now A_t and A_p are subsets of A_u, and since $A_v = A_u \cup \{N\}$, clearly A_t and $A_{p+N} = A_p \cup \{N\}$ are subsets of A_v. We have that $A_v = A_u \cup \{N\} = A_t \cup A_p \cup \{N\}$, $A_u = A_t \cup A_p$ is adequate, and $A_{p+N} = A_p \cup \{N\}$ is adequate. So A_v is adequate provided that $A_t \cup \{N\}$ is adequate. Since $A_t \in \mathrm{Sk}(\theta)$ and $N \cap \theta \in A_t$, $A_t \cup \{N\}$ is adequate by Lemma 5.16.

Finally, we show that W_v is A_v-separated. We know that $W_v = Y$ and Y is A_u-separated. So it suffices to show that if η and ξ are distinct elements of $N \cap \mathrm{dom}(Y)$ and $x \in Y(\eta) \cap Y(\xi)$, then $x \in N$. First, assume that η and ξ are in θ. Then η and ξ are in $N \cap \theta$. Since $t \leq \pi_\theta(p)$, $\mathrm{dom}(W_p) \cap \theta \subseteq \mathrm{dom}(W_t)$, and hence $Y(\eta) = W_t(\eta)$ and $Y(\xi) = W_t(\xi)$. So $x \in W_t(\eta) \cap W_t(\xi)$ and $N \cap \theta \in A_t$, Since W_t is A_t-separated, $x \in N \cap \theta \subseteq N$.

For the remaining cases, fix the $\lhd$-least functions f and g witnessing that $p \in D_\theta$. Since p and θ are members of $\mathrm{Sk}(N)$, it is clear by elementarity that f and g are also in $\mathrm{Sk}(N)$.

Claim: If $\eta \in \mathrm{dom}(W_p) \setminus \theta$ and $x \in Y(\eta)$, then $x \in W_t(f(\eta))$.

Proof: We have that $Y(\eta)$ is the downward closure of $W_p(\eta)$ in T_t. By Definition 9.2(a), $W_p(\eta) = W_p(f(\eta))$. So x is in the downward closure of $W_p(f(\eta))$ in T_t. As $t \leq \pi_\theta(p)$, $W_p(f(\eta)) = W_{\pi_\theta(p)}(f(\eta)) \subseteq W_s(f(\eta))$. Since $W_s(f(\eta))$ is downwards closed in T_s, $x \in W_s(f(\eta))$. This completes the proof of the claim.

Assume that one of η or ξ is in θ and the other is not. Without loss of generality, assume that $\eta \in \theta$ and $\xi \notin \theta$. Since f and ξ are in $\mathrm{Sk}(N)$, $f(\xi) \in N$. By the claim, $x \in W_t(f(\xi))$. So we have that η and $f(\xi)$ are in $N \cap \theta$, $N \cap \theta \in A_t$, and $x \in W_t(\eta) \cap W_t(f(\xi))$. Since W_t is A_t-separated, $x \in N \cap \theta \subseteq N$. Finally, assume that η and ξ are both not in θ. By the claim,

$x \in W_s(f(\eta)) \cap W_s(f(\xi))$. Since f, η, and ξ are in $\mathrm{Sk}(N)$, $f(\eta)$ and $f(\xi)$ are in $N \cap \theta$. As $N \cap \theta \in A_t$ and W_t is A_t-separated, it follows that $x \in N \cap \theta \subseteq N$. □

12. Quotients Are Indestructibly Y-Proper

We are finally ready to prove that quotient forcings of $\mathbb{P}$ are Y-proper on a stationary set.

Theorem 12.1. *Let $\theta \in \Sigma$. Suppose that H is a generic filter on $\mathbb{P}_\theta$. Assume that W is a transitive model of ZFC with the same ordinals as V satisfying:*

- *$V[H] \subseteq W$;*
- *$\omega_1^V = \omega_1^W$;*
- *T_H is an Aronszajn tree in W;*
- *κ is a regular cardinal in W;*
- *$\mathcal{X}(H)$ is a stationary subset of $[\kappa]^\omega$ in W.*

Then in W, $\mathbb{P}/H$ is Y-proper on a stationary set.

For our purposes, we are primarily interested in the special cases that either $W = V[H]$, or W is a generic extension of $V[H]$ by a Y-proper forcing. Recall that Y-proper forcings do not add new cofinal branches of ω_1-trees and are proper, and therefore preserve stationary subsets of $[\kappa]^\omega$.

Corollary 12.2. *Let $\theta \in \Sigma$. Suppose that H is a generic filter on $\mathbb{P}_\theta$. Then in $V[H]$, $\mathbb{P}/H$ is Y-proper on a stationary set.*

Corollary 12.3. *Let $\theta \in \Sigma$, let H be a generic filter on $\mathbb{P}_\theta$, and let $\mathbb{Q}$ be a Y-proper forcing in $V[H]$. Then $\mathbb{Q}$ forces over $V[H]$ that $\mathbb{P}/H$ is Y-proper on a stationary set.*

For the remainder of the section, fix θ, H, and W as in the statement of Theorem 12.1. All of the results in this section are intended to take place in W. A key point in what follows is that many of the properties related to the compatibility of conditions in $\mathbb{P}$ or $\mathbb{P}/H$ are absolute between V or $V[H]$ and W.

Theorem 12.4. *In W, let $\chi > \kappa$ be regular and let M be a countable elementary substructure of $\mathcal{B} = (H(\chi), \in, \psi, \mathbb{P}, \theta, H, \mathbb{P}/H, D_\theta, w_\theta)$ such that $N = M \cap \kappa \in \mathcal{X}(H)$. Then for any $u \in M \cap D_\theta \cap (\mathbb{P}/H)$ such that A_u is non-empty, $u + N$ is in $\mathbb{P}/H$, $u + N \leq u$, and $u + N$ is $(M, \mathbb{P}/H)$-generic.*

Proof. Throughout the proof we work in W. The short proof of Lemma 5.7 is easily adjusted to show that $M \cap H(\kappa)^V = \mathrm{Sk}(N)$. By Lemma 11.9, $u + N$ is in $\mathbb{P}/H$ and extends u. To show that $u+N$ is $(M, \mathbb{P}/H)$-generic, fix $q \leq u$ in $\mathbb{P}/H$ and fix $\mathcal{D} \in M$ which is a dense open subset of $\mathbb{P}/H$. By extending further if necessary using Proposition 9.4 and Lemma 11.1(4), we may assume that q is in $\mathcal{D} \cap D_\theta$ and A_q is N-closed. Then $(q, N) \in \mathrm{dom}(w_\theta)$. Let $\vec{z} = w_\theta(q, N)$.

We claim that there exists some $(\bar{q}, \bar{N}) \in M$ such that $\bar{q} \in \mathcal{D} \cap D_\theta$, $w_\theta(\bar{q}, \bar{N}) = \vec{z}$, and for all $x \in T_q \setminus T_{\bar{q}}$ and for all $y \in T_{\bar{q}} \setminus T_q$, x and y are incomparable in T_H. Suppose not. Let $\mathcal{I}$ be the set of all ordered pairs $(\bar{q}, \bar{N})$ such that $\bar{q} \in \mathcal{D} \cap D_\theta$ and $w_\theta(\bar{q}, \bar{N}) = \vec{z}$. Note that $(q, N) \in \mathcal{I}$ and $\mathcal{I} \in M$ by elementarity.

Define in M by induction a sequence $\langle (q_\alpha, N_\alpha) : \alpha < \omega_1 \rangle$ of members of $\mathcal{I}$ so that for all $\alpha < \beta < \omega_1$, $(q_\alpha, N_\alpha) \in \mathrm{Sk}(N_\beta)$ and there exist $x \in T_{q_\alpha} \setminus T_{q_\beta}$ and $y \in T_{q_\beta} \setminus T_{q_\alpha}$ such that x and y are comparable in T_H. If the induction fails, then there exists $\delta \in M \cap \omega_1$

and $\langle (q_\alpha, N_\alpha) : \alpha < \delta \rangle \in M$ satisfying the required properties, but this sequence cannot be extended any further. But then (q, N) is a witness that this sequence can be extended further, which is a contradiction.

Let $q_\alpha = (T_\alpha, W_\alpha, D_\alpha, A_\alpha)$ and $\delta_\alpha = N_\alpha \cap \omega_1$ for all $\alpha < \omega_1$. By Lemma 8.7, for all $\alpha < \beta < \omega_1$, $T_\alpha \cap T_\beta = T_\alpha \cap \delta_\alpha = T_\beta \cap \delta_\beta$. Hence, $\{T_\alpha \setminus \delta_\alpha : \alpha < \omega_1\}$ is a disjoint family of finite subsets of the Aronszajn tree T_H. By the theorem of Baumgartner-Malitz-Reinhardt stated at the end of the introduction, there exist $\alpha < \beta < \omega_1$ such that every element of $T_\alpha \setminus \delta_\alpha$ is incomparable in T_H with every element of $T_\beta \setminus \delta_\beta$. But $T_\alpha \setminus \delta_\alpha = T_\alpha \setminus T_\beta$ and $T_\beta \setminus \delta_\beta = T_\beta \setminus T_\alpha$, and we have a contradiction to the definition of the sequence.

So indeed, there exists some $(\bar{q}, \bar{N}) \in M$ such that $\bar{q} \in \mathcal{D} \cap D_\theta$, $w_\theta(\bar{q}, \bar{N}) = \vec{z}$, and for all $x \in T_q \setminus T_{\bar{q}}$ and for all $y \in T_{\bar{q}} \setminus T_q$, x and y are incomparable in T_H. Then $(\bar{q}, \bar{N}) \in M \cap H(\kappa)^V = \mathrm{Sk}(N)$. By Lemma 8.6, $\bar{q} \oplus q$ a condition in $\mathbb{P}$ which extends $\bar{q}$ and q. By Lemma 11.6, $\bar{q} \oplus q$ is in $\mathbb{P}/H$. □

Definition 12.5. *Let $\vec{z}$ be in the range of w_θ. A set $R \subseteq D_\theta \cap (\mathbb{P}/H)$ is said to be $\vec{z}$-*robust *if the set*

$$\{N \in \mathcal{X}(H) : \exists q \in R\ (w_\theta(q, N) = \vec{z})\}$$

is stationary in $[\kappa]^\omega$.

Proposition 12.6. *For any $\vec{z}$ in the range of w_θ, the collection*

$$\left\{ \sum R : R \subseteq D_\theta \cap (\mathbb{P}/H) \text{ is } \vec{z}\text{-robust} \right\}$$

is a centered subset of $\mathcal{B}(\mathbb{P}/H)$.

Proof. Let $d < \omega$ and let $R_0, \ldots, R_{d-1}$ be $\vec{z}$-robust subsets of $D_\theta \cap (\mathbb{P}/H)$. We show that there exists some $r \in \mathbb{P}/H$ such that for all $i < d$, $r \leq \sum R_i$.

By induction we construct a sequence of finite sequences

$$\langle \, \langle \, (p^i_\alpha, N^i_\alpha) \; : \; i < d \rangle \; : \; \alpha < \omega_1 \, \rangle$$

so that the following are satisfied:

(1) for all $\alpha < \omega_1$ and $i < d$, $p^i_\alpha \in R_i$ and $N^i_\alpha \in \mathcal{X}(H)$;
(2) for all $\alpha < \omega_1$ and $i < d$, $w_\theta(p^i_\alpha, N^i_\alpha) = \vec{z}$;
(3) for all $\alpha < \beta < \omega_1$ and for all $i, j < d$, p^i_α is in $\mathrm{Sk}(N^j_\beta)$.

Suppose that $\beta < \omega_1$ and for each $\alpha < \beta$ and each $i < d$, p^i_α and N^i_α are defined. Let $j < d$. Using the fact that R_j is $\vec{z}$-robust we can pick some $p^j_\beta \in R_j$ and some $N^j_\beta \in \mathcal{X}(H)$ such that $w_\theta(p^j_\beta, N^j_\beta) = \vec{z}$ and for all $\alpha < \beta$ and $i < d$, p^i_α is in $\mathrm{Sk}(N^j_\beta)$. This completes the induction.

For each $\alpha < \omega_1$ and $i < d$, write $p^i_\alpha = (T^i_\alpha, W^i_\alpha, D^i_\alpha, A^i_\alpha)$ and $\delta^i_\alpha = N^i_\alpha \cap \omega_1$. Consider $\alpha < \beta < \omega_1$ and $i, j < d$. By Lemma 8.7 applied to p^i_α, N^i_α, p^j_β, and N^j_β, we have that $T^i_\alpha \cap T^j_\beta = T^i_\alpha \cap \delta^i_\alpha = T^j_\beta \cap \delta^j_\beta$. In particular, $T^i_\alpha \setminus \delta^i_\alpha$ and $T^j_\beta \setminus \delta^j_\beta$ are disjoint.

For each $\alpha < \omega_1$, define $J_\alpha = \bigcup \{T^i_\alpha \setminus \delta^i_\alpha : i < d\}$. By the previous paragraph, for all $\alpha < \beta < \omega_1$, J_α and J_β are disjoint. By the theorem of Baumgartner-Malitz-Reinhardt stated at the end of the introduction and the fact that T_H is Aronszajn, fix $\alpha_0 < \cdots < \alpha_{d-1} < \omega_1$ so that for all $i < j < d$, for all $x \in J_{\alpha_i}$ and for all $y \in J_{\alpha_j}$, x and y are incomparable in T_H. By Lemma 8.6, $q = p^0_{\alpha_0} \oplus \cdots \oplus p^{d-1}_{\alpha_{d-1}}$ is a condition which extends each of $p^0_{\alpha_0}, \ldots, p^{d-1}_{\alpha_{d-1}}$.

Note that q is in E_θ. By Lemma 11.6, q is in $\mathbb{P}/H$. For all $i < d$, $p^i_{\alpha_i} \in R_i$, and hence $q \leq p^i_{\alpha_i} \leq \sum R_i$. □

Proof of Theorem 12.1. Working in W, fix a regular cardinal $\chi > \kappa$. Define $\mathcal{S}$ to be the set of all $M \in [H(\chi)]^\omega$ such that M is an elementary substructure of the structure $\mathcal{B} = (H(\chi), \in, \psi, \mathbb{P}, \theta, H, \mathbb{P}/H, D_\theta, w_\theta)$ and $M \cap \kappa \in \mathcal{X}(H)$. Since $\mathcal{X}(H)$ is stationary in $[\kappa]^\omega$, the set $\mathcal{S}$ is stationary in $[H(\chi)]^\omega$.

Let $M \in \mathcal{S}$ and define $N = M \cap \kappa$. Consider $u \in M \cap (\mathbb{P}/H)$. By extending u further, we may assume that $u \in D_\theta$ and A_u is non-empty. By Theorem 12.4, $u + N$ is in $\mathbb{P}/H$, is an extension of u, and is $(M, \mathbb{P}/H)$-generic. Now consider a condition $q \leq u + N$. We will find a filter $\mathcal{F}$ on $\mathcal{B}(\mathbb{P}/H)$ in M such that for every $s \in M \cap \mathcal{B}(\mathbb{P}/H)$, if $q \leq s$ then $s \in \mathcal{F}$.

Using Proposition 9.4 and Lemma 11.1(4), extend q to some r in $D_\theta \cap (\mathbb{P}/H)$ such that A_r is N-closed. Then (r, N) is in the domain of w_θ. Let $\vec{z} = w_\theta(q, N)$. Then $\vec{z} \in \mathrm{Sk}(N) \subseteq M$. By Proposition 12.6, the collection $\mathcal{F}_0 = \{\sum R : R \subseteq D_\theta \cap (\mathbb{P}/H) \text{ is } \vec{z}\text{-robust}\}$ is a centered subset of $\mathcal{B}(\mathbb{P}/H)$. By elementarity, $\mathcal{F}_0 \in M$. So $\mathcal{F} = \{b \in \mathcal{B}(\mathbb{P}/H) : \exists c \in \mathcal{F}_0 \ (c \leq b)\}$ is a filter on $\mathcal{B}(\mathbb{P}/H)$ which is also in M.

To complete the proof, suppose that $q \leq s$ and $s \in M \cap \mathcal{B}(\mathbb{P}/H)$, and we show that $s \in \mathcal{F}$. Define $R = \{t \in D_\theta \cap (\mathbb{P}/H) : t \leq s\}$. Clearly, $s = \sum R$, $R \in M$, and $r \in R$. We claim that R is $\vec{z}$-robust, and hence $s = \sum R \in \mathcal{F}_0 \subseteq \mathcal{F}$. Let C be a club subset of $[\kappa]^\omega$ in M. Then $N \in C$. So $N \in C$, $N \in \mathcal{X}(H)$, $r \in R$, and $w_\theta(r, N) = \vec{z}$. By elementarity, it follows that the set of $K \in \mathcal{X}(H)$ for which there exists some $t \in R$ such that $w_\theta(t, K) = \vec{z}$ is stationary in $[\kappa]^\omega$. □

13. The Main Theorems: Part 2

We are now prepared to prove the remaining main theorems of the article. We start by answering the question which originally motivated this work.

Theorem 13.1. *The forcing poset $\mathbb{P}$ forces that there exists a strongly non-saturated normal infinitely splitting Aronszajn tree and there does not exist a weak Kurepa tree.*

Proof. The first part was established in Theorem 10.13. For the second part, suppose that $\dot{T}$ is a $\mathbb{P}$-name for a tree with height and size ω_1. Without loss of generality, assume that $\dot{T}$ is forced to have underlying set ω_1 and $\dot{T}$ is a nice name. Since $\mathbb{P}$ is κ-c.c., we can find some $\theta \in \Sigma$ such that $\dot{T}$ is a nice $\mathbb{P}_\theta$-name. By Lemma 11.2, $\mathbb{P}$ forces that $\dot{T}$ is in $V^{\mathbb{P}_\theta}$. By Theorem 8.2 and Corollary 12.2, $\mathbb{P}$ forces that every cofinal branch of $\dot{T}$ in $V^{\mathbb{P}}$ lies in $V^{\mathbb{P}_\theta}$. As κ is inaccessible in $V^{\mathbb{P}_\theta}$ and κ equals ω_2 in $V^{\mathbb{P}}$, $\mathbb{P}$ forces that $\dot{T}$ has fewer than ω_2-many cofinal branches. □

Since by a result of Solovay, the non-existence of a Kurepa tree implies that ω_2 is inaccessible in L, we have the following corollary.

Corollary 13.2. *The following statements are equiconsistent.*

(1) *There exists an inaccessible cardinal.*
(2) *There exists a strongly non-saturated Aronszajn tree and there does not exist a weak Kurepa tree.*

The next proposition follows from Theorem 8.2 and Corollary 12.2 by standard methods for constructing models satisfying the tree property ([Mit73]).

Proposition 13.3. *If κ is a Mahlo cardinal, then $\mathbb{P}$ forces that there does not exist a special ω_2-Aronszajn tree. If κ is a weakly compact cardinal, then $\mathbb{P}$ forces that there does not exist an ω_2-Aronszajn tree.*

Recall the principle ISP of Weiss, which asserts the existence of an ineffable branch for every slender $P_{\omega_2}(\lambda)$-list, for any regular cardinal $\lambda \geq \omega_2$ ([Wei12]). Viale-Weiss [VW11] proved that this principle is equivalent to the statement that for any regular cardinal $\lambda \geq \omega_2$, there exist stationarily many guessing models in $[H(\lambda)]^{\omega_1}$. In [CK17] we denote this last statement by GMP, which stands for the *guessing model principle*.

For the next two results, we assume that the reader is familiar with constructing models of GMP (see, for example, [CK16, Section 7]). In particular, if κ is a supercompact cardinal, then by standard arguments the Y-properness of the quotient together with Theorem 8.2 imply the existence of stationarily many guessing models.

Theorem 13.4. *Assuming that κ is a supercompact cardinal, $\mathbb{P}$ forces that GMP holds. So the existence of a strongly non-saturated Aronszajn tree is consistent with GMP.*

In [CK17], the idea of an indestructible guessing model is introduced together with the principle IGMP, which stands for the *indestructible guessing model principle*. An *indestructible guessing model* is a guessing model which remains guessing in any ω_1-preserving generic extension. And IGMP states that for any regular cardinal $\lambda \geq \omega_2$, there exist stationarily many indestructible guessing models in $[H(\lambda)]^{\omega_1}$. By [CK16, Corollary 4.5] and [Kru19, Theorem 1.4], IGMP follows from the conjunction of GMP and the statement that every tree of height and size ω_1 which has no cofinal branches is special.

We provide a sketch of a proof for how to use the indestructibility of the Y-properness of the quotient to obtain a model of IGMP together with a strongly non-saturated Aronszajn tree. For any tree T with no uncountable branches, the standard forcing for specializing T with finite conditions is Y-c.c. ([CZ15, Corollary 3.3]). And any finite support forcing iteration of Y-c.c. forcings is Y-c.c. ([CZ15, Theorem 6.2]). Consequently, there exists a Y-c.c. finite support forcing iteration of length $(2^{\omega_1})^+$ which forces that every tree with underlying set ω_1 which has no cofinal branches is special.

Consider a generic filter G on $\mathbb{P}$. In $V[G]$, we have that $2^{\omega_1} = \omega_2 = \kappa$, so we can fix a finite support forcing iteration $\langle \mathbb{Q}_i : i \leq \kappa \rangle$ as described in the previous paragraph. Let K be a $V[G]$-generic filter on $\mathbb{Q}_\kappa$. Consider any $\theta \in \Sigma$ such that $\langle \mathbb{Q}_i : i \leq \theta \rangle$ is in $V[G \cap \mathbb{P}_\theta]$. Let $G_\theta = G \cap \mathbb{P}_\theta$ and $K_\theta = K \cap \mathbb{Q}_\theta$. By the product lemma, $V[G][K] = V[G_\theta][K_\theta][G][K]$. Now in $V[G_\theta]$, $\mathbb{Q}_\theta$ is a finite support forcing iteration of Y-c.c. forcings, and hence is Y-c.c. and therefore Y-proper. So by Corollary 12.3, $\mathbb{P}/G_\theta$ is Y-proper on a stationary set in $V[G_\theta][K_\theta]$, and hence has the ω_1-approximation property in $V[G_\theta][K_\theta]$ by Theorem 8.2. In $V[G_\theta][K_\theta][G] = V[G][K_\theta]$, $\mathbb{Q}_\kappa / K_\theta$ is forcing equivalent to a finite support forcing iteration of Y-c.c. forcings, and hence has the ω_1-approximation property by Theorem 8.2. It follows that $V[G][K]$ is a generic extension of $V[G_\theta][K_\theta]$ by the forcing $(\mathbb{P}/G_\theta) * (\mathbb{Q}_\kappa / K_\theta)$ which has the ω_1-approximation property.

The above analysis combined with standard methods shows that if κ is a supercompact cardinal, then in $V[G][K]$ we have that GMP holds and every tree of height and size ω_1 which has no cofinal branches is special. (More specifically, we apply the arguments of the previous paragraph to $j(\mathbb{P})$, $j(\mathbb{P})/G$, and $\theta = \kappa \in j(\Sigma)$, where $j : V \to M$ is an elementary embedding witnessing the supercompactness of κ and G is a generic filter on $\mathbb{P} = j(\mathbb{P})_\kappa$.) So IGMP holds in $V[G][K]$.

Theorem 13.5. *Suppose that κ is a supercompact cardinal. Then there exists a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a finite support forcing iteration of Y-c.c. forcings of length κ such that $\mathbb{P} * \dot{\mathbb{Q}}$ forces that there exists a strongly non-saturated Aronszajn tree and IGMP holds.*

REFERENCES

[Bau76] J. E. Baumgarter, *Almost-disjoint sets, the dense set problem and the partition calculus*, Ann. Math. Logic **9** (1976), no. 4, 401–439.

[Bau85] J. E. Baumgartner, *Bases for Aronszajn trees*, Tsukuba J. Math. **9** (1985), no. 1, 31–40.

[BMR70] J. E. Baumgartner, J. Malitz, and W. Reinhardt, *Embedding trees in the rationals*, Proc. Nat. Acad. Sci. **67** (1970), no. 4, 1748–1753.

[CK16] S. Cox and J. Krueger, *Quotients of strongly proper forcings and guessing models*, J. Symb. Log. **81** (2016), no. 1, 264–283.

[CK17] ———, *Indestructible guessing models and in the continuum*, Fund. Math. **239** (2017), no. 3, 221–258.

[CZ15] D. Chodounský and J. Zapletal, *Why Y-c.c.*, Ann. Pure Appl. Logic **166** (2015), no. 11, 1123–1149.

[JS90] R. Jensen and K. Schlechta, *Results on the generic Kurepa hypothesis*, Arch. Math. Logic **30** (1990), 13–27.

[KLMV08] B. König, P. Larson, J. T. Moore, and B. Veličković, *Bounding the consistency strength of a five element linear basis*, Israel J. Math. **164** (2008), 1–18.

[KMM24] J. Krueger and E. Martinez Mendoza, *An almost Kurepa Suslin tree with strongly non-saturated square*, 2024, preprint.

[Kru17] J. Krueger, *Forcing with adequate sets of models as side conditions*, Math. Log. Q. **63** (2017), no. 1-2, 124–149.

[Kru19] ———, *Guessing models imply the singular cardinal hypothesis*, Proc. Amer. Math. Soc. **147** (2019), no. 12, 5427–5434.

[KS24] J. Krueger and Š. Stejskalová, *Forcing over a free Suslin tree*, 2024, preprint.

[Mit73] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972/73), 21–46.

[Moo06] J. T. Moore, *A five element basis for the uncountable linear orders*, Ann. of Math. (2) **163** (2006), no. 2, 669–688.

[Moo08] ———, *Structural analysis of Aronszajn trees*, Lect. Notes Log., vol. 28, Assoc. Symbol. Logic, Urbana, IL, 2008, pp. 85–106.

[Sil71] J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Axiomatic Set Theory (D. S. Scott, ed.), Proc. Sympos. Pure Math., vol. XIII, Part 1, Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390.

[Tod87] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), no. 3-4, 261–294.

[Tod91] ———, *Remarks on Martin's axiom and the continuum hypothesis*, Can. J. Math. **43** (1991), no. 4, 832–851.

[VW11] M. Viale and C. Weiss, *On the consistency strength of the proper forcing axiom*, Adv. Math. **228** (2011), no. 5, 2672–2687.

[Wei12] C. Weiss, *The combinatorial essence of supercompactness*, Ann. Pure Appl. Logic **163** (2012), no. 11, 1710–1717.

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