The tree property and the continuum function below $\aleph_\omega$

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Abstract: We say that a regular cardinal κ, $\kappa > \aleph_0$, has the tree property if there are no κ-Aronszajn trees; we say that κ has the weak tree property if there are no special κ-Aronszajn trees. Starting with infinitely many weakly compact cardinals, we show that the tree property at every even cardinal $\aleph_{2n}$, $0 < n < \omega$, is consistent with an arbitrary continuum function below $\aleph_\omega$ which satisfies $2^{\aleph_{2n}} > \aleph_{2n+1}$, $n < \omega$. Next, starting with infinitely many Mahlo cardinals, we show that the weak tree property at every cardinal $\aleph_n$, $1 < n < \omega$, is consistent with an arbitrary continuum function below $\aleph_\omega$ which satisfies $2^{\aleph_n} > \aleph_{n+1}$, $n < \omega$. Thus the tree property has no provable effect on the continuum function below $\aleph_\omega$ except for the trivial requirement that the tree property at κ^{++} implies $2^\kappa > \kappa^+$ for every infinite κ.

Keywords: The tree property, continuum function.

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1 Introduction

It is known that the usual large cardinals[1] do not have any effect on the continuum function on small cardinals: in particular, if κ is a large cardinal, all we can say in general is that $2^{\aleph_n}$, for any $n < \omega$, is smaller than κ. Things may be different if we consider a cardinal κ which shares some properties with large cardinals (typically some sort of reflection), but it is not required to be inaccessible: it may even be smaller than $\aleph_\omega$, and therefore may have an effect on the continuum function below $\aleph_\omega$.

In this paper, we are interested in the *tree property* at κ: we say that a regular cardinal $\kappa > \aleph_0$ has the tree property if there are no κ-Aronszajn trees. Thus to have the tree property is the same as to be weakly compact without the requirement for the inaccessibility of κ. If there are no special κ-Aronszajn trees, we refer to this property as the *weak tree property*. By a result of Jensen, the weak tree property at κ^+ is equivalent to the failure of weak square at κ.

If $2^\kappa = \kappa^+$, $\kappa \geq \aleph_0$, then by Specker's result ([14]) there are special κ^{++}-Aronszajn trees. Thus the tree property at $\aleph_2$ implies the failure of CH. It seems natural to ask whether the tree property at κ^{++} puts more restrictions on the continuum function in addition to requiring $2^\kappa > \kappa^+$. We answer this question negatively for the continuum function below

[1] We include the assumption of inaccessibility in the definition of a large cardinal.

$\aleph_\omega$.[2]

The structure of the paper is as follows. First, in Theorem 2.5, we deal for simplicity with a single cardinal and show that the tree property at $\aleph_2$ is compatible with $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ (we use "gap three" for concreteness, there is nothing particular about it).[3] Theorem 2.5 is generalised in Theorem 3.1 where we show (starting with infinitely many weakly compact cardinals) that the tree property at every even cardinal larger than $\aleph_0$ below $\aleph_\omega$ is compatible with any continuum function which satisfies $2^{\aleph_{2n}} \geq \aleph_{2n+2}$, $n < \omega$. In Theorem 3.8, we formulate an analogous result for the weak tree property: starting with infinitely many Mahlo cardinals, we show that the weak tree property at every $\aleph_n$, $1 < n < \omega$, is compatible with any continuum function which satisfies $2^{\aleph_n} \geq \aleph_{n+2}$ for $n < \omega$. We focus on the case when $\aleph_\omega$ is a strong limit cardinal in the resulting model, but the method of the proof is not limited to that configuration.

Note that we use only modest large cardinal assumptions, i.e. weakly compact cardinals and Mahlo cardinals, and therefore we cannot get two successive cardinals with the tree property as this requires large cardinals on the level of one Woodin cardinal (see [6]); note however, that we can get two successive cardinals with the weak tree property from just two Mahlo cardinals. A generalisation of this paper to successive cardinals requires larger cardinals; see Section 4 for more open questions in this direction.

1.1 Basic facts and notation

In general, if P is a forcing notion,[4] we write $V[P]$ to denote a generic extension by P whenever the exact generic filter is not relevant (in particular when we say that some statement holds in $V[P]$, this means that the weakest condition in P forces the statement). We indicate a name for a forcing using the "dot notation", as in $P * \dot{Q}$. We abuse notation and write x instead of $\check{x}$ to denote a ground model object in a forcing language; in particular if P and Q are forcing notions in V, we write that P forces a certain statement about Q, and not about $\check{Q}$. If there is a projection from P onto Q, we denote by P/Q the quotient such that P is equivalent to $Q * P/Q$ (in this case, we do not put a dot over P/Q as it is clear from the context that P/Q is a name and not a ground model object).

We will often use the fact that if P is κ-cc, and $\dot{Q}$ is a P-name for a forcing notion, then $P * \dot{Q}$ is κ-cc if and only if P forces that $\dot{Q}$ is κ-cc. If we consider a product $P \times Q$, it holds that if $P \times Q$ is κ-cc, then P forces that Q is κ-cc in $V[P]$. However, it is not true in general that if P and Q are κ-cc, then their product is κ-cc (consider for instance a Souslin tree T at $\aleph_1$ as a forcing notion; then T is $\aleph_1$-cc, but $T \times T$ has an antichain of size $\aleph_1$). It is therefore useful to consider a strengthening of the κ-cc condition which is preserved by products:

Definition 1.1 *Let κ be a regular cardinal. We say that a forcing notion P is κ-Knaster if every set of conditions in P of size κ has a subset of size κ of pairwise compatible conditions.*

[2] The study of the behaviour of the tree property below and close to $\aleph_\omega$ seems to be the standard test case for many of the results concerning the tree property (see for instance [4] or [13]).

[3] This result for $2^{\aleph_0}$ already follows from the "indestructibility" results presented in [15].

[4] We abuse notation and write P instead of $(P, \leq)$ whenever the ordering is understood from the context.

Being κ-Knaster is a strengthening of κ-cc which has better properties with respect to taking products: we will often use that if P is κ-Knaster and Q is κ-cc, then $P \times Q$ is κ-cc, and that if P and Q are both κ-Knaster, then $P \times Q$ is κ-Knaster; in particular the product $P \times P$ of a κ-Knaster forcing P is still κ-Knaster. See Cummings [3], Section 5, and Jech [9], Section 15, for more details and proofs of the facts mentioned in the two previous paragraphs.

For a regular cardinal κ, we denote by $\mathrm{Add}(\kappa, \alpha)$ the Cohen forcing which adds α-many subsets of κ (we identify $p \in \mathrm{Add}(\kappa, \alpha)$ with a function of size $< \kappa$ from $\kappa \times \alpha$ to 2). If $A \subseteq \alpha$, we write $\mathrm{Add}(\kappa, A)$ for the forcing whose conditions only use coordinates in A:

$$\mathrm{Add}(\kappa, A) = \{p \in \mathrm{Add}(\kappa, \alpha) \,|\, \mathrm{dom}(p) \subseteq \kappa \times A\}. \tag{1.1}$$

If $\beta < \alpha$, we write $\mathrm{Add}(\kappa, \alpha - \beta)$ for $\mathrm{Add}(\kappa, [\beta, \alpha))$. An easy Δ-lemma argument shows that with GCH, $\mathrm{Add}(\kappa, \alpha)$ is κ^+-Knaster.

Let κ be a regular cardinal, and $\lambda > \kappa$ be an inaccessible cardinal. The following presentation of the Mitchell forcing in [12] is introduced in Abraham [1].

Definition 1.2 *We denote by $\mathbb{M}(\kappa, \lambda)$ the forcing which is defined as follows: (p, q) is in $\mathbb{M}(\kappa, \lambda)$ if p is a condition in $\mathrm{Add}(\kappa, \lambda)$ and q is a function of size at most κ with $\mathrm{dom}(q) \subseteq \lambda$ and for all $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name for a condition in $\mathrm{Add}(\kappa^+, 1)^{V[\mathrm{Add}(\kappa,\alpha)]}$. The ordering is defined as follows: $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ and for all $\alpha \in \mathrm{dom}(q_2)$, $q_1|\alpha \Vdash_{\mathrm{Add}(\kappa,\alpha)} q_1(\alpha) \leq q_2(\alpha)$.*

By an analysis of Abraham [1], there is a κ^+-closed term forcing which we denote ${}^1\mathbb{M}(\kappa, \lambda)$ which is defined as follows:

$${}^1\mathbb{M}(\kappa, \lambda) = \{(\emptyset, q) \,|\, (\emptyset, q) \in \mathbb{M}(\kappa, \lambda)\}, \tag{1.2}$$

ordered as a suborder of $\mathbb{M}(\kappa, \lambda)$, and a projection π,

$$\pi : \mathrm{Add}(\kappa, \lambda) \times {}^1\mathbb{M}(\kappa, \lambda) \to \mathbb{M}(\kappa, \lambda) \tag{1.3}$$

which sends $(p, (\emptyset, q))$ to (p, q).

This projection analysis carries over to quotients: for $\alpha < \lambda$, the quotient $\mathbb{M}(\kappa, \lambda)/\mathbb{M}(\kappa, \alpha)$ is a projection in $V[\mathbb{M}(\kappa, \alpha)]$ of $\mathrm{Add}(\kappa, \lambda - \alpha) \times {}^1\mathbb{M}(\kappa, \lambda - \alpha)$ where ${}^1\mathbb{M}(\kappa, \lambda - \alpha)$ is κ^+-closed in $V[\mathbb{M}(\kappa, \alpha)]$.

If κ is regular, we write $\mathrm{TP}(\kappa)$ to denote that the *tree property* holds at κ, i.e. every κ-tree has a cofinal branch (that is, there are no κ-Aronszajn trees). We write $\mathrm{wTP}(\kappa)$ to denote that the *weak tree property* holds at κ, i.e. there are no special κ-Aronszajn trees. Note that by a result of Jensen [10], $\mathrm{wTP}(\kappa^+)$ is equivalent to $\neg\square^*_\kappa$.

For the proof of the following result, see Mitchell [12], and also Abraham [1] for additional details.

Fact 1.3 *Assume GCH holds and let $\kappa < \lambda$ be cardinals such that κ is regular and λ inaccessible. The the following hold:*

(i) $\mathbb{M}(\kappa,\lambda)$ *is* λ*-Knaster.*
(ii) $\mathbb{M}(\kappa,\lambda)$ *collapses exactly the cardinals in the open interval* (κ^+,λ) *and forces* $2^\kappa = \lambda = \kappa^{++}$.
(iii) If λ *is weakly compact, then in* $V[\mathbb{M}(\kappa,\lambda)]$, $\mathrm{TP}(\lambda)$ *holds.*
(iv) If λ *is Mahlo, then in* $V[\mathbb{M}(\kappa,\lambda)]$, $\mathrm{wTP}(\lambda)$ *holds.*

We shall use the following facts for arguments that certain forcings do not add cofinal branches to λ^+-trees. Fact 1.4 is due to Silver (see [1] for more details; a proof with $\lambda = \omega$ is in [11, Chapter VIII, §3]), and Fact 1.5 is due to Baumgartner (see [2]).

Fact 1.4 *Suppose* μ *and* λ *are infinite cardinals such that* $2^\mu \geq \lambda^+$. *Let* T *be a* λ^+*-tree and* P *a* μ^+*-closed forcing notion. Then in* $V[P]$ *there are no new cofinal branches in* T.

Fact 1.5 *Suppose* λ *is an infinite cardinal and* T *is a tree of height* λ^+. *If* P *is a* λ^+*-Knaster forcing notion, then in* $V[P]$, *there are no new cofinal branches in* T.

These facts can be generalised as follows (see Unger [15]):

Fact 1.6 *If* $P \times P$ *is* λ^+*-cc, and* T *is a tree of height* λ^+, *then in* $V[P]$, *there are no new cofinal branches in* T.

Fact 1.7 *Suppose* $\mu < \nu \leq \lambda$ *are infinite cardinals,* ν *regular, such that* $2^\mu \geq \lambda^+$. *Let* P *be a* ν*-cc forcing,* Q *a* ν*-closed forcing, and* $\dot{T}$ *a* P*-name for a* λ^+*-tree. Then in* $V[P]$, Q *cannot add a cofinal branch to* $\dot{T}$.

Remark 1.8 Note that Facts 1.5 and 1.6 easily generalise when we consider trees of heights which are not cardinals: if T is a tree of cofinality κ, where κ is some regular uncountable cardinal, then a κ-Knaster forcing (or a forcing whose square is κ-cc) does not add new cofinal branches to T. This follows from the fact that the proofs in Baumgartner [2] and Unger [15] can be formulated for levels cofinal in T, and the fact that any cofinal branch in T is uniquely determined by its nodes on cofinally many levels.

In the proof, we will often need Easton's lemma (see Jech [9]):

Lemma 1.9 (Easton) *Let* $\kappa > \aleph_0$ *be a regular cardinal and assume that* P *and* Q *are forcing notions and* P *is* κ*-cc and* Q *is* κ*-closed. Then the following hold:*

(i) Q *forces that* P *is* κ*-cc.*
(ii) P *forces that* Q *is* κ*-distributive.*

The following generalisation of Easton's lemma will also be useful:

Lemma 1.10 *Assume* $\kappa \geq \aleph_0$ *is a regular cardinal such that* $\kappa^{<\kappa} = \kappa$ *and that* $\lambda > \kappa$ *is inaccessible. Assume* P *is* κ^+*-cc and* Q *is* κ^+*-closed. Then* $P \times \mathbb{M}(\kappa,\lambda)$ *forces that* Q *is* κ^+*-distributive.*

Proof. It is easy to check that the projection π in (1.3) extends to the projection π^*,

$$\pi^* : P \times Q \times \mathrm{Add}(\kappa,\lambda) \times {}^1\mathbb{M}(\kappa,\lambda) \to P \times Q \times \mathbb{M}(\kappa,\lambda), \tag{1.4}$$

which sends $(r_1, r_2, p, (\emptyset, q))$ to $(r_1, r_2, (p, q))$. It follows that $P \times Q \times \mathrm{Add}(\kappa,\lambda) \times {}^1\mathbb{M}(\kappa,\lambda)$ is forcing equivalent to

$$[P \times Q \times \mathbb{M}(\kappa,\lambda)] * \dot{S} \tag{1.5}$$

for some quotient forcing $\dot{S}$.

Let $G \times g \times F$ be an arbitrary $P \times \mathbb{M}(\kappa,\lambda) \times Q$-generic filter over V. We will show that every sequence x of ordinals of length less than κ^+ which is in $V[G \times g \times F]$ is in $V[G \times g]$ which shows that Q is forced to be κ^+-distributive as required.

Let x as above be fixed. Let h be any $\dot{S}$-generic filter over $V[G \times g \times F]$. It follows by (1.5) that $V[G \times g \times F][h]$ can be written as $V[G \times g_0 \times g_1 \times F]$ where $g_0 \times g_1$ is $\mathrm{Add}(\kappa,\lambda) \times {}^1\mathbb{M}(\kappa,\lambda)$-generic, and the following hold:

(i) $V[G \times g \times F] \subseteq V[G \times g_0 \times g_1 \times F]$,
(ii) $V[G \times g_0] \subseteq V[G \times g]$,

where (ii) holds because g_0 is the Cohen part of g. In particular x is in $V[G \times g_0 \times g_1 \times F]$.

By Easton's lemma, $P \times \mathrm{Add}(\kappa,\lambda)$ (which is κ^+-cc as the Cohen forcing is κ^+-Knaster by $\kappa^{<\kappa} = \kappa$) forces that ${}^1\mathbb{M}(\kappa,\lambda) \times Q$ (which is κ^+-closed) is κ^+-distributive. It follows that x is already in $V[G \times g_0]$, and hence in $V[G \times g]$ as desired. □

2 Large $2^{\aleph_0}$ and $2^{\aleph_1}$ with $\mathrm{TP}(\aleph_2)$

In this section we provide a proof of a special case of Theorem 3.1. It illustrates the main idea behind the construction with more clarity than the proof of Theorem 3.1, which needs to deal with infinitely many cardinals.

We assume that the reader is familiar with the usual argument which shows that $\mathbb{M}(\kappa,\lambda)$ forces the tree property at λ, whenever κ is regular, $\kappa < \lambda$ and λ is weakly compact. For the proof see [12] or [1].

For concreteness of the construction in this section we will force "gap three" on $\aleph_0$ and $\aleph_1$, i.e. get $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ with the tree property at $\aleph_2$. Other values of the continuum functions are easily obtainable; see Theorem 3.1.

Let κ be a weakly compact cardinal. Denote

$$\mathbb{P} = \mathbb{M}(\aleph_0,\kappa) \times \mathrm{Add}(\aleph_0,\kappa^+) \times \mathrm{Add}(\aleph_1,\kappa^{++}). \tag{2.6}$$

Remark 2.1 Note that $\mathbb{M}(\aleph_0,\kappa)$ forces $2^{\aleph_0} = \aleph_2$ (see [12]), and therefore to increase the value of $2^{\aleph_1}$, we need to use some kind of product because the forcing $\mathrm{Add}(\aleph_1, 1)$ defined in $V[\mathbb{M}(\aleph_0,\kappa)]$ collapses $2^{\aleph_0}$ to $\aleph_1$ (by a density argument, every subset of ω occurs as a segment in a generic filter g for $\mathrm{Add}(\aleph_1, 1)$, and therefore g yields a surjection from $\aleph_1$ onto $2^{\aleph_0}$).

Lemma 2.2 *(GCH). In $V[\mathbb{P}]$, $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_3$, $2^{\aleph_1} = \aleph_4$.*

Proof. Let I denote the open interval of cardinals between $\aleph_1$ and κ. It suffices to show that in $V[P]$ the cardinals in I are collapsed, and no other cardinals are collapsed. By Fact 1.3, $\mathbb{M}(\aleph_0, \kappa)$ collapses cardinals in I, but no other cardinals. The forcing $\mathrm{Add}(\aleph_0, \kappa^+)$ is ccc, and therefore no more cardinals are collapsed in $V[\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa^+)]$. By Lemma 1.10, the forcing $\mathrm{Add}(\aleph_1, \kappa^{++})$ is $\aleph_1$-distributive in $V[\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa^+)]$. Since the whole product $\mathrm{Add}(\aleph_0, \kappa^+) \times \mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa^{++})$ is κ-Knaster, $\mathrm{Add}(\aleph_0, \kappa^+) \times \mathbb{M}(\aleph_0, \kappa)$ forces that $\mathrm{Add}(\aleph_1, \kappa^{++})$ is κ-cc (see the paragraph before Definition 1.1). It follows that the forcing $\mathrm{Add}(\aleph_1, \kappa^{++})$ applied over $V[\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa^+)]$ preserves all cardinals not in I. This finishes the proof. □

Lemma 2.3 *Assume $\mathbb{P}$ forces that $\dot{T}$ is a κ-Aronszajn tree, where $\dot{T}$ is a nice name for a subset of κ.*[5] *Then there are $\kappa \subseteq A \subseteq \kappa^+$ and $\kappa \subseteq B \subseteq \kappa^{++}$ both of size κ such that $\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, A) \times \mathrm{Add}(\aleph_1, B)$ forces that $\dot{T}$ is a κ-Aronszajn tree.*

Proof. Since all forcings composing $\mathbb{P}$ are κ-Knaster, it follows that the product $\mathbb{P}$ is κ-cc (in fact κ-Knaster; see the notes after Definition 1.1). It follows that $\dot{T}$ is of the form $\{\{\check{\alpha}\} \times A_\alpha \,|\, \alpha < \kappa\}$, where each A_α is an antichain of size less than κ. Let us write a condition in $\mathbb{P}$ as $\bar{p} = ((p, q), p_0, p_1)$, where (p, q) is in $\mathbb{M}(\aleph_0, \kappa)$, p_0 is in $\mathrm{Add}(\aleph_0, \kappa^+)$ and p_1 is in $\mathrm{Add}(\aleph_1, \kappa^{++})$. The set A is defined by $\kappa \cup A'$ where A' contains all β such there is some $\alpha < \kappa$ and some $\bar{p}$ in A_α such that (n, β), for some $n < \omega$, is in the domain of the condition p_0 which is in $\bar{p}$. Similarly for B and p_1's. □

Corollary 2.4 *If $\mathbb{P}$ adds a κ-Aronszajn tree, then so does*

$$\mathbb{P}|\kappa =_{df} \mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa).$$

Proof. Any bijection between A, B and κ determines an isomorphism between $\mathrm{Add}(\aleph_0, A)$ and $\mathrm{Add}(\aleph_0, \kappa)$, and similarly for B. □

Theorem 2.5 *(GCH). Assume κ is weakly compact and $\mathbb{P}$ is as in (2.6). Then in $V[\mathbb{P}]$, $2^{\aleph_0} = \aleph_3$, $2^{\aleph_1} = \aleph_4$, and $\mathrm{TP}(\aleph_2)$.*

Proof. The fact that $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ is proved in Lemma 2.2. It remains to verify the tree property at $\aleph_2$. By Corollary 2.4, it suffices to show that $\mathbb{P}|\kappa$ cannot add a κ-Aronszajn tree. Suppose for contradiction there is a condition (r_1, r_2) in $\mathbb{P}|\kappa = \mathbb{M}(\aleph_0, \kappa) \times [\mathrm{Add}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa)]$ which forces there is a κ-Aronszajn tree $\dot{T}$.

Let $j : \mathscr{M} \to \mathscr{N}$ be a weakly compact embedding with critical point κ where $\mathscr{M}$ and $\mathscr{N}$ are transitive models of ZFC^- closed under $< \kappa$-sequences, and $\mathscr{M}$ contains all parameters required for the argument (in particular, the forcing $\mathbb{P}|\kappa$ and name $\dot{T}$).

[5] We will identify κ-trees with subsets of κ (every κ-tree is isomorphic to (κ, R) for some some binary relation R).

Let $G * (H_1 \times H_2)$ denote a generic filter over V for

$$\mathbb{M}(\aleph_0, \kappa) * [\mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times {}^1\dot{\mathbb{M}}(\aleph_0, j(\kappa) - \kappa)],$$

where the product $\mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times {}^1\dot{\mathbb{M}}(\aleph_0, j(\kappa) - \kappa)$ projects to $j(\mathbb{M}(\aleph_0, \kappa))/\mathbb{M}(\aleph_0, \kappa)$. Denote by $G * H$ the $j(\mathbb{M}(\aleph_0, \kappa))$-generic filter obtained from $G * (H_1 \times H_2)$, using the projection (1.3); note that we have automatically $j\text{"}G \subseteq G * H$ because j is the identity on the conditions in $G \subseteq \mathbb{M}(\aleph_0, \kappa)$. Assume further that $r_1 \in G$.

Now we can lift j in $V[G * (H_1 \times H_2)]$ to

$$j : \mathscr{M}[G] \to \mathscr{N}[G * H].$$

Let $x^* \times y^*$, with $x^* = x_0 \times x_1$ and $y^* = y_0 \times y_1$, be $V[G * (H_1 \times H_2)]$-generic for $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa))$, with $x_0 \times y_0$ being $\mathrm{Add}(\aleph_1, \kappa) \times \mathrm{Add}(\aleph_0, \kappa)$-generic over $V[G * (H_1 \times H_2)]$ so that

$$j\text{"}(x_0 \times y_0) \subseteq x^* \times y^*. \tag{2.7}$$

The inclusion (2.7) is possible because j is the identity on the conditions in $x_0 \times y_0$. Assume further that $r_2 \in x_0 \times y_0$.

Remark 2.6 It is worth noting that $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa))$ lives in $V[G]$ (actually already in V), so $x^* \times y^* \times H_1 \times H_2$ is a generic filter over $V[G]$ for the product forcing $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times {}^1\mathbb{M}(\aleph_0, j(\kappa) - \kappa)$, and therefore x^*, y^*, H_1, and H_2 are mutually generic over $V[G]$.

Now we can lift j in $V[G * (H_1 \times H_2)][x^* \times y^*]$ to

$$j : \mathscr{M}[G][x_0 \times y_0] \to \mathscr{N}[G][H][x^* \times y^*].$$

Recall that we have put the name $\dot{T}$ into $\mathscr{M}$; we can assume that $\dot{T}$ is a nice name for a subset of κ, and is therefore present also in $\mathscr{N}$. Since (r_1, r_2) is in $G * (x_0 \times y_0)$, $T = \dot{T}^{G*(x_0 \times y_0)}$ is a κ-Aronszajn tree in $\mathscr{M}[G][x_0 \times y_0]$, and also in $\mathscr{N}[G][x_0 \times y_0]$. As $j(T)$ is a $j(\kappa)$-tree, it has nodes of height κ. Since $T = j(T)|\kappa$ (the restriction of $j(T)$ to κ), the last sentence implies that T has a cofinal branch in $\mathscr{N}[G][H][x^* \times y^*]$.

By Remark 2.6, the relevant filters are mutually generic over $V[G]$, and hence also over $\mathscr{N}[G]$, and therefore we can write

$$\mathscr{N}[G][H][x^* \times y^*] \subseteq \mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2]. \tag{2.8}$$

We finish the proof by showing that the generic filter $x_1 \times y_1 \times H_1 \times H_2$ cannot add a cofinal branch to T, and therefore any such branch existing in the models in (2.8) must already exist in $\mathscr{N}[G][x_0 \times y_0]$, which contradicts our initial assumption that T is a κ-Aronszajn tree in $\mathscr{N}[G][x_0 \times y_0]$.

Let P_1 denote the forcing $\mathrm{Add}(\aleph_1, j(\kappa) - \kappa) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa)$ which adds the generic filter $x_1 \times y_1 \times H_1$. As the square of P_1 is isomorphic to P_1, it suffices to show by Fact 1.6 that $\mathbb{P}|\kappa$ forces P_1 to be κ-cc. This follows from the fact that both $\mathbb{P}|\kappa$ and P_1 are κ-Knaster, and therefore $\mathbb{P}|\kappa \times P_1$ is κ-cc (in fact κ-Knaster), and so $\mathbb{P}|\kappa$ forces that P_1 is κ-cc (see the paragraph before Definition 1.1). Hence there are no new cofinal branches in T in

$$\mathscr{N}[G][y_0][x_0][x_1][y_1][H_1].$$

Now we show that H_2 cannot add a cofinal branch either, which will finish the proof. The quotient forcing ${}^1\mathbb{M}(\aleph_0, j(\kappa) - \kappa)$ is $\aleph_1$-closed in $\mathscr{N}[G]$, and by Lemma 1.10 (with P being trivial), the forcing $\mathrm{Add}(\aleph_1, j(\kappa))$ (which adds $x_0 \times x_1$) is $\aleph_1$-distributive in $\mathscr{N}[G]$, and therefore does not add new countable sequences; this implies that that ${}^1\mathbb{M}(\aleph_0, j(\kappa) - \kappa)$ is still $\aleph_1$-closed in $\mathscr{N}[G][x_0][x_1]$. We can therefore apply Fact 1.7 over the the model $\mathscr{N}[G][x_0][x_1]$ with $P = \mathrm{Add}(\aleph_0, j(\kappa))$ (note that P is isomorphic to the $\aleph_0$-Cohen forcing which adds $y_0 \times y_1 \times H_1$) and $Q = {}^1\mathbb{M}(\aleph_0, j(\kappa) - \kappa)$. Thus there are no new cofinal branches in T in the model

$$\mathscr{N}[G][x_0][x_1][y_0][y_1][H_1][H_2] = \mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2].$$

This finishes the proof. □

3 Main theorems

In this section, we prove a more general version of Theorem 2.5, both for the tree property (Theorem 3.1), and the weak tree property (Theorem 3.8).

3.1 The tree property

Let $\kappa_1 < \kappa_2 < \cdots$ be an ω-sequence of weakly compact cardinals with limit λ. Let κ_0 denote $\aleph_0$. In Theorem 3.1, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the tree property at all even aleph's.

Let A denote the set $\{\kappa_i \,|\, i < \omega\} \cup \{\kappa_i^+ \,|\, i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+1}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)).

Theorem 3.1 *Assume GCH and let $\langle \kappa_i \,|\, i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is an $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_{2n}$, and $\kappa_n^+ = \aleph_{2n+1}$.
(ii) For all $0 < n < \omega$, the tree property holds at $\aleph_{2n}$.
(iii) The continuum function on $A = \{\aleph_n \,|\, n < \omega\}$ is controlled by f.

Proof. Let $\mathbb{P}$ be a reverse Easton iteration of the Cohen forcings $\mathrm{Add}(\alpha, 1)$ for every inaccessible $\alpha < \lambda$. We will see that $\mathbb{P}$ ensures that the weak compactness of the κ_i's is preserved at a certain stage of the argument (see the paragraph after (3.18)).

Let $\dot{\mathbb{M}}(\kappa_n, \kappa_{n+1})$ denote a $\mathbb{P}$-name for the Mitchell forcing which makes $2^{\kappa_n} = \kappa_{n+1}$ and forces the tree property at κ_{n+1}. Let $\dot{\mathbb{Q}}$ be a name for the full-support product of the Mitchell forcings in $V[\mathbb{P}]$:

$$\dot{\mathbb{Q}} \text{ is a name for } \prod_{n<\omega} \dot{\mathbb{M}}(\kappa_n, \kappa_{n+1}).$$

Finally, let $\dot{\mathbb{R}}$ be a $\mathbb{P}$-name for the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to the cardinals in A):

$$\dot{\mathbb{R}} \text{ is a name for } \prod_{n<\omega} \big(\dot{\mathrm{Add}}(\kappa_n, f(\kappa_n)) \times \dot{\mathrm{Add}}(\kappa_n^+, f(\kappa_n^+))\big). \tag{3.9}$$

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{P} * (\dot{\mathbb{Q}} \times \dot{\mathbb{R}}). \tag{3.10}$$

We leave it as an exercise for the reader to verify that $\mathbb{S}$ preserves all cardinals in A and forces the prescribed continuum function (it is a routine generalisation of Lemma 2.2 using the product analysis of $\mathbb{M}(\kappa, \lambda)$ in (1.3) and Lemma 1.10). We will check that the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$.

Let us work in $V[\mathbb{P}]$ for simplicity of notation (so that we can remove all "dots" from the forcing notions).

Let us write $\mathbb{R}^0(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$, $\mathbb{R}^1(n) = \mathrm{Add}(\kappa_n^+, f(\kappa_n^+))$, and $\mathbb{R}(n) = \mathbb{R}^0(n) \times \mathbb{R}^1(n)$. Thus $\mathbb{R} = \prod_{n<\omega} \mathbb{R}(n)$.

Let us denote for $0 < n < \omega$:

$$\mathbb{T}(n) = \mathbb{R}^0(n+1) \times \prod_{m \le n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m), \tag{3.11}$$

and

$$\mathbb{T}(n)_{\mathrm{tail}} = \mathbb{R}^1(n+1) \times \prod_{m > n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m > n+1} \mathbb{R}(m), \tag{3.12}$$

so that $\mathbb{Q} \times \mathbb{R} = \mathbb{T}(n) \times \mathbb{T}(n)_{\mathrm{tail}}$.

Suppose for contradiction that $\mathbb{S}$ adds a κ_{n+1}-Aronszajn tree T (for simplicity, we assume that the weakest condition forces the existence of such a tree; otherwise we would work below a condition which forces it). Then T is added by

$$\mathbb{P} * \dot{\mathbb{T}}(n) \tag{3.13}$$

because $\mathbb{T}(n)_{\rm tail}$ is κ_{n+1}^+-closed in $V[\mathbb{P}]$, and using Lemma 1.10, viewing $\mathbb{T}(n)$ as a product of a κ_{n+1}^+-cc forcing and $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2})$, it follows that $\mathbb{T}(n)_{\rm tail}$ is still κ_{n+1}^+-distributive in $V[\mathbb{P}]$ over $\mathbb{T}(n)$, and hence does not add any κ_{n+1}-trees.

The forcing $\mathbb{T}(n)$ is κ_{n+2}-Knaster in $V[\mathbb{P}]$ (because all the forcings making up the product are κ_{n+2}-Knaster, and this property is preserved by products), and therefore T has in $V[\mathbb{P}]$ a $\mathbb{T}(n)$-name $\dot{T}$ which can be taken to be a $< \kappa_{n+2}$-sequence of elements in $V[\mathbb{P}]$ (without loss of generality, a nice name for a subset of κ_{n+1}). This name – having size less than κ_{n+2} in $V[\mathbb{P}]$ – is already present in $\mathbb{P}(< \kappa_{n+2})$ (the iteration $\mathbb{P}$ below κ_{n+2}) because $\mathbb{P}(< \kappa_{n+2})$ forces its tail in $\mathbb{P}$ to be κ_{n+2}-closed. It follows that

$$\mathbb{P}(< \kappa_{n+2}) * \dot{\mathbb{T}}(n) \tag{3.14}$$

already adds T (note that $\dot{\mathbb{T}}(n)$ can be taken to be a $\mathbb{P}(< \kappa_{n+2})$-name as all the conditions in this forcing have size less than κ_{n+2}, so the expresion (3.14) is meaningful).

Let us define in $V[\mathbb{P}(< \kappa_{n+2})]$:

$$\mathbb{T}(n)^- = \prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m).$$

Thus we can write the forcing in (3.14) as

$$\mathbb{P}(< \kappa_{n+2}) * (\dot{\mathbb{M}}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1) \times \dot{\mathbb{T}}(n)^-). \tag{3.15}$$

This forcing is equivalent to

$$\mathbb{P}(< \kappa_{n+2}) * \big(\dot{\mathbb{M}}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1)\big) * \dot{\mathbb{T}}(n)^-. \tag{3.16}$$

because $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1)$ does not change $H(\kappa_{n+1})$ where the conditions in the rest of the forcing live.

We claim that T is in fact added by

$$\mathbb{P}(< \kappa_{n+2}) * \dot{\rm Add}(\kappa_{n+1}, 1) * \dot{\mathbb{T}}(n)^-. \tag{3.17}$$

This is true because T has a name in the forcing

$$\mathbb{P}(< \kappa_{n+2}) * \big(\dot{\rm Add}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1)\big) * \dot{\mathbb{T}}(n)^- \tag{3.18}$$

of size at most κ_{n+1} (using the product analysis of $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2})$ which we discussed after Definition 1.2) and therefore a name in the forcing (3.17).

$\mathbb{P}(< \kappa_{n+2}) * \dot{\rm Add}(\kappa_{n+1}, 1)$ preserves the weak compactness of κ_{n+1} (since in $\mathbb{P}(< \kappa_{n+2})$ we prepared by the Cohen forcing at inaccessibles below κ_{n+1}; this may be shown for instance by lifting a weakly compact embedding, see [3] for more details), so it remains to show that $\mathbb{T}(n)^-$ forces the tree property at κ_{n+1} for a weakly compact κ_{n+1}.

Work in $V[\mathbb{P}(< \kappa_{n+2})]$ and let us write $\mathbb{T}(n)^-$ as:

$$\mathbb{T}(n)^- = \mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3, \tag{3.19}$$

where

- $\mathbb{T}_1 = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1})$,
- $\mathbb{T}_2 = \mathbb{R}^0(n) \times \prod_{m<n} \mathbb{R}(m)$, and
- $\mathbb{T}_3 = \mathbb{R}^1(n)$.

These forcings have the following basic properties which are relevant for the proof:

(a) $\mathbb{M}(\kappa_n, \kappa_{n+1})$ is κ_{n+1}-Knaster, and there is a projection onto it from the product forcing $\mathrm{Add}(\kappa_n, \kappa_{n+1}) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1})$, where ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1})$ is a κ_n^+-closed term forcing.
(b) $\mathbb{T}_1$ is κ_n-Knaster, and bounded in $H(\kappa_{n+1})$.
(c) $\mathbb{T}_2$ is κ_n^+-Knaster.
(d) $\mathbb{T}_3$ is κ_n^+-closed.

Denote $\kappa_{n+1} = \kappa$. Exactly as in the proof of Theorem 2.5, using the fact that the whole product $\mathbb{T}(n)^-$ is κ-cc (using the productivity of the Knaster property), if $\mathbb{T}(n)^-$ adds a κ-Aronszajn tree, then so does the forcing

$$\mathbb{M}(\kappa_n, \kappa) \times \mathbb{T}_1 \times \mathbb{T}_2|\kappa \times \mathbb{T}_3|\kappa, \tag{3.20}$$

where $\mathbb{T}_2|\kappa$ and $\mathbb{T}_3|\kappa$ denote the restrictions of all of the Cohen products in $\mathbb{T}_2$ and $\mathbb{T}_3$ to length κ. In more detail, for $\mathbb{R}^i(m)$, $i < 2, m \le n$, let us write $\mathbb{R}^0(m)|\kappa = \mathrm{Add}(\kappa_m, \kappa)$ and $\mathbb{R}^1(m)|\kappa = \mathrm{Add}(\kappa_m^+, \kappa)$, and $\mathbb{R}(m)|\kappa = \mathbb{R}^0(m)|\kappa \times \mathbb{R}^1(m)|\kappa$. Then $\mathbb{T}_2|\kappa$ denotes the forcing $\mathbb{R}^0(n)|\kappa \times \prod_{m<n} \mathbb{R}(m)|\kappa$, and $\mathbb{T}_3|\kappa$ denotes $\mathbb{R}^1(n)|\kappa$. The fact that already the forcing in (3.20) adds the tree follows exactly as in Lemma 2.3 and Corollary 2.4 with appropriate reformulations.

Recall that we work in $V[\mathbb{P}(< \kappa_{n+2})]$ where by our assumption we have a name $\dot{T}$ for a κ-Aronszajn tree in the forcing (3.20). Let

$$j : \mathscr{M} \to \mathscr{N}$$

be a weakly compact embedding with critical point κ, where $\mathscr{M}$ contains all the relevant parameters (in particular, the forcing from (3.20) and the name $\dot{T}$ (which we view as a nice name for a subset of κ – and is therefore also in $\mathscr{N}$ as it is a subset of $H(\kappa)$)). Note that $\mathscr{M}$ and $\mathscr{N}$ are closed under sequences of length $< \kappa$ in $V[\mathbb{P}(< \kappa_{n+2})]$, so in particular the forcings $j(\mathbb{T}_2|\kappa)$ and $j(\mathbb{T}_3|\kappa)$ mean in $\mathscr{N}$ and in $V[\mathbb{P}(< \kappa_{n+2})]$ the same thing as the conditions in these forcings are sequences of length $< \kappa$ (for the same reason, $\mathbb{T}_2|\kappa$ and $\mathbb{T}_3|\kappa$ denote the same forcing in $\mathscr{M}$, $\mathscr{N}$ and $V[\mathbb{P}(< \kappa_{n+2})]$).

Pursuing the analogy with Theorem 2.5, and the notation in that proof, consider the model $\mathscr{N}[G][x_0][y_0][x_1][H_2][H_1][y_1]$, where in our case we have:

(a) $G = G_0 \times G_1$ is $\mathbb{M}(\kappa_n, \kappa) \times \mathbb{T}_1$-generic.
(b) x_0 is $\mathbb{T}_3|\kappa$-generic.
(c) x_1 is such that $x_0 \times x_1$ is $j(\mathbb{T}_3|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{T}}_3$: $j(\mathbb{T}_3|\kappa) = \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3$.
(d) y_0 is $\mathbb{T}_2|\kappa$-generic.
(e) y_1 is such that $y_0 \times y_1$ is $j(\mathbb{T}_2|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{T}}_2$: $j(\mathbb{T}_2|\kappa) = \mathbb{T}_2|\kappa \times \hat{\mathbb{T}}_2$.
(f) H_1 is $\mathrm{Add}(\kappa_n, j(\kappa) - \kappa)$-generic.
(g) H_2 is ${}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa)$-generic, where ${}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa)$ is the term forcing which is κ_n^+-closed in $\mathscr{N}[G_0]$.

Recall that we assume for simplicity that the weakest condition forces that $\dot{T}$ is a κ-Aronszajn tree; otherwise we would need to choose $G_0 \times G_1 \times x_0 \times y_0$ below a condition wich forces it.

Remark 3.2 Note that the product $\mathbb{T}_1 \times j(\mathbb{T}_3|\kappa) \times j(\mathbb{T}_2|\kappa)$ lives in $V[\mathbb{P}(<\kappa_{n+2})[G_0]$ (actually already in $V[\mathbb{P}(<\kappa_{n+2})])$, so that $G_1 \times x_0 \times x_1 \times y_0 \times y_2 \times H_1 \times H_2$ is a generic filter for a product forcing over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$, and therefore all these generic filters are mutually generic over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$.

Let us write

$$V^* = V[\mathbb{P}(<\kappa_{n+2})][G][x_0 \times x_1 \times y_0 \times y_1 \times H_1 \times H_2].$$

The conditions (a)–(g) above guarantee we can lift j in V^*:

$$j : \mathscr{M}[G][x_0][y_0] \to \mathscr{N}[G][x_0][y_0][x_1][y_1][H], \tag{3.21}$$

where H is generic for the quotient $j(\mathbb{M}(\kappa_n,\kappa))]/\mathbb{M}(\kappa_n,\kappa)$ over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$ and $j\text{"}G_0 \subseteq H$ (this is ensured exactly as in the proof of Theorem 2.5).

Since $\dot{T}$ is also present in $\mathscr{N}$, the κ-Aronszajn tree $T = \dot{T}^{G\times x_0\times y_0}$ is present in $\mathscr{N}[G][x_0][y_0]$; since $j(T)$ restricted to κ is T, the embedding (3.21) ensures that $j(T)$ has a node of height κ, and therefore T has a cofinal branch, in $\mathscr{N}[G][x_0][y_0][x_1][y_1][H]$. We will argue this is not possible as the larger model $\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2]$ cannot obtain a new cofinal branch over $\mathscr{N}[G][x_0][y_0]$.

First note that $P_1 = \hat{\mathbb{T}}_3 \times \hat{\mathbb{T}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa)-\kappa)$ (which adds the generic $x_1 \times y_1 \times H_1$) is isomorphic to its square. By Fact 1.6 it therefore suffices to show that $P_0 = \mathbb{M}(\kappa_n,\kappa) \times \mathbb{T}_1 \times \mathbb{T}_3|\kappa \times \mathbb{T}_2|\kappa$ (which adds the generic $G \times x_0 \times y_1$) forces that P_1 is κ-cc to conclude that there are no new cofinal branches in T in

$$\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1].$$

This follows by the productivity of Knaster forcings as both P_0 and P_1 are κ-Knaster.

It remains to show that H_2 cannot add a cofinal branch to T either. Denote $P = \mathbb{T}_1 \times \mathbb{T}_2 \times \hat{\mathbb{T}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa)-\kappa)$.

Claim 3.3 *(i)* ${}^1\mathbb{M}(\kappa_n, j(\kappa)-\kappa)$ *is* κ_n^+*-closed in* $\mathscr{N}[G_0][x_0][x_1]$.
(ii) P *is* κ_n^+*-cc in* $\mathscr{N}[G_0][x_0][x_1]$.

Proof. (i). ${}^1\mathbb{M}(\kappa_n, j(\kappa)-\kappa)$ is κ_n^+-closed in $\mathscr{N}[G_0]$, and by Lemma 1.10 (with P being trivial), $\mathscr{N}[G_0]$ and $\mathscr{N}[G_0][x_0][x_1]$ have the same $<\kappa_n^+$-sequences of ordinals. Now the claim follows.

(ii). We will show that P is forced to be κ_n^+-cc by

$$\mathrm{Add}(\kappa_n,\kappa) \times {}^1\mathbb{M}(\kappa_n,\kappa) \times \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3. \tag{3.22}$$

This suffices as there is a projection from the forcing (3.22) to the forcing $\mathbb{M}(\kappa_n,\kappa) \times \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3$ (which adds $G_0 \times x_0 \times x_1$).

We use Easton's lemma: $P_2 = {}^1\mathbb{M}(\kappa_n, \kappa) \times \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3$ is κ_n^+-closed and $\mathrm{Add}(\kappa_n, \kappa) \times P$ is κ_n^+-cc, and therefore P_2 forces $\mathrm{Add}(\kappa_n, \kappa) \times P$ to be κ_n^+-cc, and so $P_2 \times \mathrm{Add}(\kappa_n, \kappa)$ forces P to be κ_n^+-cc. In some detail, P_2 forces that $\mathrm{Add}(\kappa_n, \kappa) \times P$ is κ_n^+-cc if and only if P_2 forces that $\mathrm{Add}(\kappa_n, \kappa)$ forces that P is κ_n^+-cc, which is equivalent to $P_2 \times \mathrm{Add}(\kappa_n, \kappa)$ forcing that P is κ_n^+-cc. □

Recall that our tree T is in $\mathscr{N}[G][x_0][y_0]$. Denote

$$P = \mathbb{T}_1 \times \mathbb{T}_2 \times \hat{\mathbb{T}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa) - \kappa) \text{ and } Q = {}^1\mathbb{M}(\kappa, j(\kappa) - \kappa). \tag{3.23}$$

Consider now the P-generic filter $G_1 \times y_0 \times y_1 \times H_1$ over the model $\mathscr{N}[G_0][x_0][x_1]$. It gives rise to the model $\mathscr{N}[G_0][x_0][x_1][G_1][y_0][y_1][H_1]$ which is equal to $\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1]$, and extends $\mathscr{N}[G][x_0][y_0]$, and therefore contains the tree T. Now let us apply Fact 1.7 over the model $\mathscr{N}[G_0][x_0][x_1]$ with P and Q fixed in (3.23) (note that in this model $2^{\kappa_n} = \kappa$, so the cardinal assumptions of Fact 1.7 are satisfied with $\mu = \kappa_n$). By Claim 3.3, P is κ_n^+-cc and Q is κ_n^+-closed in $\mathscr{N}[G_0][x_0][x_1]$. It follows that that H_2 (the generic filter for Q) does not add new cofinal branches to T over

$$\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1].$$

This finishes the proof. □

A more succinct formulation of Theorem 3.1 is as follows:

Corollary 3.4 *(GCH) Assume there are infinitely many weakly compact cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \rightarrow f(m) \leq f(n)$.
(ii) $f(2n) \geq 2n + 2$ for all $n < \omega$.

Then there is a model where the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Remark 3.5 Let us remark that the technique in the proof of Theorem 3.1 is not limited to having $\aleph_\omega$ strong limit: the values of $2^{\aleph_n}$ for $n < \omega$ can be bigger than $\aleph_\omega$ in the resulting model (subject to the usual restrictions on the continuum function). This follows from the fact that the argument for the tree property at $\kappa_{n+1} = \kappa$ reduces to the forcing in (3.20) which uses only a portion of $\mathbb{R}$ of size κ; thus $\mathbb{R}$ can be chosen to force $2^{\aleph_n}$ arbitrarily high without a material change in the argument.

3.2 The weak tree property

For the sake of completeness, we also address the question of the weak tree property and the continuum function below $\aleph_\omega$.

Let $\kappa_2 < \kappa_3 < \cdots$ be an ω-sequence of Mahlo cardinals with limit λ. Let κ_0 denote $\aleph_0$, and κ_1 denote $\aleph_1$. In Theorem 3.8, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the weak tree property at all $\aleph_n$, $n \geq 2$.

Let A denote the set $\{\kappa_i \mid i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+2}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)).

The following natural modification of the Mitchell forcing first appeared in [16].

Definition 3.6 *Let $0 \leq n < \omega$ be given. We define $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ as the collection of pairs (p, q) which satisfy the same conditions as in $\mathbb{M}(\kappa_n, \kappa_{n+2})$ with the difference that instead of* $\mathrm{Add}(\kappa_n^+, 1)$ *for collapsing, we use* $\mathrm{Add}(\kappa_{n+1}, 1)$*, and the size of the domain of q is now $< \kappa_{n+1}$.*

In particular, $\mathbb{M}(\kappa_n, \kappa_{n+2})$ is equal to $\mathbb{M}(\kappa_n, \kappa_n^+, \kappa_{n+2})$. By an analysis similar to Abraham [1], one can show that $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ is a projection of the product

$$\mathrm{Add}(\kappa_n, \kappa_{n+2}) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2}), \tag{3.24}$$

where ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ is a term forcing which is κ_{n+1}-closed and κ_{n+2}-cc (see [16], Lemma 4.7; see also [9], Theorem 16.30; it is easy to see that the forcing is actually κ_{n+2}-Knaster by the same argument).

We get the following analogue of Lemma 1.10.

Lemma 3.7 *Assume $\aleph_0 \leq \kappa < \mu < \lambda$ are regular cardinals, $\nu^{<\kappa} < \mu$ for all $\nu < \mu$, and λ is inaccessible. Assume P is μ-cc and Q is μ-closed. Then $P \times \mathbb{M}(\kappa, \mu, \lambda)$ forces that Q is μ-distributive.*

PROOF. The proof is the same as in Lemma 1.10 with the modification that $P \times \mathrm{Add}(\kappa, \lambda)$ is now μ-cc (since by our cardinal-arithmetic assumption, $\mathrm{Add}(\kappa, \lambda)$ is μ-Knaster by the usual Δ-system argument) and the term part of the Mitchell forcing is μ-closed. ☐

Note that in the paper Lemma 3.7 is used with GCH, so the cardinal-arithmetic assumptions are automatically satisfied.

The following theorem is a generalisation of Theorem 4.11 in [16].

Theorem 3.8 *Assume GCH and let $\langle \kappa_i \mid i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is an $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_n$,
(ii) The continuum function on $A = \{\aleph_n \mid n < \omega\}$ is controlled by f.
(iii) The weak tree property holds at every $\aleph_n$, $2 \leq n < \omega$.

PROOF. Set $\mathbb{Q}$ to be the full support product

$$\mathbb{Q} = \prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2}).$$

Let $\mathbb{R}$ be the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to cardinals in A):

$$\mathbb{R} = \prod_{n<\omega} \mathrm{Add}(\kappa_n, f(\kappa_n)).$$

For simplicity of notation, let us write $\mathbb{R}(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$.

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{Q} \times \mathbb{R}. \tag{3.25}$$

Again, we leave it as an exercise for the reader to verify that the cardinals in A are preserved, $\kappa_n = \aleph_n$, and the continuum function below $\aleph_\omega$ is controlled by f. The proof is basically the same as in [16] using the usual Easton-style analysis, the product analysis of the forcing $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ in (3.24) and Lemma 3.7.

Let $n < \omega$ be fixed. We show that there are no special κ_{n+2}-Aronszajn trees in $V[\mathbb{S}]$.

Let us denote:

$$\mathbb{T}(n) = \prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n+1} \mathbb{R}(m), \tag{3.26}$$

and

$$\mathbb{T}(n)_{\mathrm{tail}} = \prod_{m > n+2} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m > n+2} \mathbb{R}(m), \tag{3.27}$$

so that

$$\mathbb{S} = \mathbb{T}(n) \times \mathbb{R}(n+2) \times \mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\mathrm{tail}}. \tag{3.28}$$

Suppose for contradiction $\mathbb{S}$ adds a special κ_{n+2}-Aronszajn tree (we assume for simplicity that the weakest condition forces it; otherwise we would work below an appropriate condition). Then also the forcing

$$\begin{aligned} \mathbb{T}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \\ \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \times {}^1\mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\mathrm{tail}} \end{aligned} \tag{3.29}$$

adds a special κ_{n+2}-Aronszajn tree because it projects onto $\mathbb{S}$. Denote the tree T.

Then T is added by

$$\mathbb{T}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \tag{3.30}$$

because ${}^1\mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\mathrm{tail}}$ is κ_{n+3}-closed in V, and therefore by Easton's lemma κ_{n+3}-distributive over the forcing (3.30) which is κ_{n+3}-cc by the productivity of the Knaster property.

We finish the proof by arguing that the forcing in (3.30) cannot add T (and its specialising function g which maps T to the cardinal predecessor of κ_{n+2} (i.e. κ_{n+1}) in the generic extension by (3.30)). In the interest of further simplification of notation, we will not introduce variables for generic filters, but we will use the convention that $V[P]$ denotes a generic extension by a forcing P whenever the exact generic filter is irrelevant (recall that we assume that the weakest condition forces that $\dot{T}$ is a κ_{n+2}-Aronszajn tree).

Let us work in

$$V^* = V[\mathbb{R}(n+2) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})].$$

V^* still satisfies that κ_{n+2} is a Mahlo cardinal and $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$ is κ_{n+2}-cc (this follows by Easton's lemma since $\mathbb{R}(n+2) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})$ is κ_{n+2}-closed). Note that since κ_{n+2} is a Mahlo cardinal in V^* (and in particular inaccessible), T together with its specialising function g cannot be present already in V^*.

Since T is in the generic extension over V by the forcing in (3.30), there are in V^* some $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$-names $\dot{T}$ and $\dot{g}$ for the tree T and the function g witnessing its specialisation.[6] We can identify both $\dot{T}$ and $\dot{g}$ with a name for a subset of κ_{n+2}. Since the forcing $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$ is κ_{n+2}-cc, we may assume that already $\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})$ adds the tree and the specialising function, where $\mathbb{T}(n)|\kappa_{n+2}$ is the forcing

$$\prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n+1} \mathbb{R}(m)|\kappa_{n+2}, \tag{3.31}$$

where $\prod_{m \le n+1} \mathbb{R}(m)|\kappa_{n+2}$ is the restriction of the Cohen forcings to length κ_{n+2} (see (3.20) and what follows for more details on the notation). This follows from the fact that the names $\dot{T}$ and $\dot{g}$ refer to up to κ_{n+2}-many Cohen coordinates in the forcing $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$, so we can proceed as in Lemma 2.3 with suitable modifications.

Since κ_{n+2} is a Mahlo cardinal, there is a V-inaccessible δ, $\kappa_{n+1} < \delta < \kappa_{n+2}$, such that $T|\delta, g|\delta$ (the restrictions of T and g to the subtree of T of height δ) are added over V^* by the following forcing

$$\prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \ \times \ \mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \ \times \ \prod_{m \le n+1} \mathbb{R}(m)|\delta \ \times \ \mathrm{Add}(\kappa_{n+1}, \delta). \tag{3.32}$$

Notice that $\mathbb{R}(n+1)|\delta$ denotes the forcing $\mathrm{Add}(\kappa_{n+1}, \delta)$, and since $\mathrm{Add}(\kappa_{n+1}, \delta)$ is isomorphic to its square, we may replace $\prod_{m \le n+1} \mathbb{R}(m)|\delta$ by $\prod_{m \le n} \mathbb{R}(m)$ in (3.32):

$$\mathbb{T} \ = \ \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \times \prod_{m \le n} \mathbb{R}(m)|\delta \times \mathrm{Add}(\kappa_{n+1}, \delta). \tag{3.33}$$

The forcing $\mathbb{T}$ in (3.33) is δ-cc – and therefore in particular preserves the regularity of δ – using the productivity of the Knaster property. The inaccessible δ is found as follows: as T and g restricted to $\beta < \kappa_{n+2}$ have size $< \kappa_{n+2}$, there is a closed unbounded subset in κ_{n+2} of ordinals $\sigma(\beta)$ such that $T|\beta$ and $g|\beta$ are added by the forcing restricted to $\sigma(\beta)$, where β ranges over ordinals between κ_{n+1} and κ_{n+2}. By the Mahloness of κ_{n+2}, this closed unbounded set has an inaccessible fixed point (which we denote δ).

[6] We assume for simplicity that the weakest condition forces that there is a special κ_{n+2}-Aronszajn tree; if not, choose the generic filter to contain this condition.

In (3.37) below, we will work with the product

$$\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{3.34}$$

With the current definition of $\mathbb{T}(n)|\kappa_{n+2}$ in (3.31), the product (3.34) is equal to

$$\prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n} \mathbb{R}(m)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{3.35}$$

Since the Cohen forcing is isomorphic to its square, the forcing in (3.35) is isomorphic to (3.36):

$$\prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n} \mathbb{R}(m)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{3.36}$$

In order to simplify notation, we will from now on identify (3.34) with (3.36).

We finish the proof by arguing that the forcing from the model $V^*[\mathbb{T}]$ to

$$V^*[\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})] \tag{3.37}$$

cannot add a cofinal branch to $T|\delta$. This will be a contradiction for the following reason. $T|\delta$ has a cofinal branch in the final model since it is a tree of height κ_{n+2} and therefore has nodes of height $\delta < \kappa_{n+2}$. If the forcing from $V^*[\mathbb{T}]$ to $V^*[\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})]$ cannot add such a branch, it must be present already in $V^*[\mathbb{T}]$, but this is impossible: by the choice of δ and the properties of the specialisation function g, g restricted to such a branch yields an injective function from δ into the cardinal predecessor of δ (i.e. κ_{n+1}) in the forcing extension $V^*[\mathbb{T}]$ and therefore the cardinal δ would be collapsed in $V^*[\mathbb{T}]$ (which we argued below (3.33) is not the case). See also [12], Section 5, for more details.

Let us denote by ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta)$ the term forcing which is κ_{n+1}-closed in the extension $V[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$, and hence also in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$ using the κ_{n+2}-closure of the forcing from V to V^*, such that in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$,

$$\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})/\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)$$

is a projection of $\mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta)$.

Thus it suffices to show that over $V^*[\mathbb{T}]$, the forcing

$$\mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta) \times \prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta) \tag{3.38}$$

does not add a cofinal branch to $T|\delta$, where $\prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta)$ is the restriction of the Cohen forcings to the interval $[\delta, \kappa_{n+2})$ (see (3.20) and what follows for details on the notation).

We can now finish the proof analogously to Theorem 3.1.[7] Denote

$$P_1 = \mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta), \tag{3.39}$$

and

$$Q = {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta). \tag{3.40}$$

Note that $P_1 \times Q$ is the forcing from (3.38).

P_1 is isomorphic to its square and is δ-cc in $V^*[\mathbb{T}]$ by the productivity of the Knaster property. It follows that P_1 cannot add a cofinal branch to $T|\delta$ by Fact 1.6. It remains to show that Q cannot add a cofinal branch to $T|\delta$ over the model $V^*[\mathbb{T}][P_1]$.

Let

$$P = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n} \mathbb{R}(m)|\delta \times \mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta). \tag{3.41}$$

Now we state the analogue of Claim 3.3 (we explicitly spell out all the relevant forcings for better orientation even though the expression could be simplified: in particular, $\mathrm{Add}(\kappa_{n+1}, \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)$ is isomorphic to $\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})$).

Claim 3.9 *(i) Q in (3.40) is κ_{n+1}-closed in*

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]. \tag{3.42}$$

(ii) P in (3.41) is κ_{n+1}-cc in

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]. \tag{3.43}$$

Proof. (i). By Lemma 3.7, $\mathrm{Add}(\kappa_{n+1}, \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)$ is κ_{n+1}-distributive over the model $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$, where Q is κ_{n+1}-closed, and therefore Q stays κ_{n+1}-closed in the model (3.42).

(ii). As in the proof of Claim 3.3(ii), use the product analysis of $\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)$ and show the κ_{n+1}-cc of P using the productivity of the Knaster property and Easton's lemma. □

Now the proof can be finished by applying Fact 1.7 to P and Q from Claim 3.9 over the model $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]$ (notice that in this model it is true that $\delta = \kappa_{n+1}^+$ and $2^{\kappa_n} = \delta$, so the cardinal assumptions in Fact 1.7 are satisfied). In more detail: the tree $T|\delta$ is in the generic extension $V^*[\mathbb{T}]$, and hence also in

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \delta)][\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)][P] = V^*[\mathbb{T}][P_1] \tag{3.44}$$

[7] It is immaterial to the argument whether we work in a generic extension of M as in Theorem 3.1, and discuss the ordinals $\kappa < j(\kappa)$, or work in a generic extension of V, and discuss the ordinals $\delta < \kappa_{n+2}$. Note that the proof of Theorem 3.1 could also have been formulated with some $\delta < \kappa$ without mentioning an elementary embedding.

which extends $V^*[\mathbb{T}]$ (note that (3.44) holds by the fact that the forcings over the model V^* on the left-hand side and the right-hand side of the equation are composed of the product of the same forcing notions, just suitably regrouped).

Now the proof is finished by applying Q over the model (3.44) using Fact 1.7. □

As with Theorem 3.1, we may formulate a more succinct version of Theorem 3.8:

Corollary 3.10 *(GCH) Assume there are infinitely many Mahlo cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \to f(m) \leq f(n)$,
(ii) $f(n) > n + 1$ for all $n < \omega$.

Then there is a model where the weak tree property holds at every $\aleph_n$, $1 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Note that Remark 3.5 also applies in this context.

4 Open questions

Q1. Is it possible to get the results of Theorems 3.1 and 3.8 with the failure of SCH at $\aleph_\omega$, i.e. with a strong limit $\aleph_\omega$ satisfying $2^{\aleph_\omega} > \aleph_{\omega+1}$? Or even stronger, with the tree property at $\aleph_{\omega+2}$?

Note that with the failure of SCH at $\aleph_\omega$, the situation is much more complex because we can no longer use a simple product construction. In [8], Gitik and Merimovich show that an arbitrary continuum function below $\aleph_\omega$ is compatible with the failure of SCH at $\aleph_\omega$. However in their model they do not discuss the tree property. Note that in the paper [7], a model is constructed with the tree property at the $\aleph_{2n}$'s, $0 < n < \omega$, with the failure of SCH at $\aleph_\omega$, but in that model, $2^{\aleph_{2n}} = \aleph_{2n+2}$ for every $n < \omega$, and GCH holds at the remaining cardinals below $\aleph_\omega$. The construction in [7] does not seem to admit an easy generalisation along the lines of Theorem 3.1.

Cummings and Foreman [4] proved that starting with infinitely many supercompact cardinals, there is a model where the tree property holds at every $\aleph_n$, $2 \leq n < \omega$, with $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$.

Q2. Starting with infinitely many supercompact cardinals, is it consistent that the tree property holds at every $\aleph_n$, $2 \leq n < \omega$, and the continuum function is arbitrary below $\aleph_\omega$ such that $2^{\aleph_n} \geq \aleph_{n+2}$, $n < \omega$?

There is a notion of a *super tree property* which captures the combinatorial essence of a supercompact cardinal (see for instance [17], or [5], for definitions). Weiss noticed that Mitchell's forcing over a supercompact cardinal yields the super tree property. Later, Fontanella [5] proved that starting with infinitely many supercompact cardinals, the super tree property can hold at every $\aleph_n$, $2 \leq n < \omega$.

Q3. Starting with infinitely many supercompact cardinals, is it consistent that the super tree property holds at every $\aleph_n$, $2 \leq n < \omega$, and the continuum function is arbitrary below $\aleph_\omega$ such that $2^{\aleph_n} \geq \aleph_{n+2}$, $n < \omega$?

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