Exercises to *Properties of axiomatic theories* (January 2, 2023)

Exercises

1. Let P and Q be unary and R a binary predicate. Prove that the following sentences are logically valid, but reverting the outermost implication yields (in all cases) a formula that is not logically valid:

   $\exists x(P(x) \,\&\, Q(x)) \to \exists xP(x) \,\&\, \exists xQ(x)$,
   $\forall xP(x) \vee \forall xQ(x) \to \forall x(P(x) \vee Q(x))$,
   $\exists x\forall yR(x,y) \to \forall y\exists xR(x,y)$,
   $\forall x(P(x) \to Q(x)) \to (\forall xP(x) \to \forall xQ(x))$,
   $\forall x(P(x) \to Q(x)) \to (\exists xP(x) \to \exists xQ(x))$.

2. Which of $\forall x(P(x) \to \forall yP(y))$, $\exists x(P(x) \to \forall yP(y))$ and $\exists x(\exists yP(y) \to P(x))$ are logically valid sentences?

3. For every sentence from the previous two exercises that is logically valid prove its provability in the Hilbert-style calculus. Use tautological consequences and the fact that all tautologies are provable, but avoid using the predicate completeness theorem (otherwise there would be nothing to do).

4. Show that $\Delta, \psi \models \varphi$ if and only if $\Delta \models \psi \to \varphi$ for any formulas φ and ψ and any set Δ of formulas.

5. Theories T and S are *equivalent* if every axiom of S is a consequence of T, and at the same time every axiom of T os a consequence of S. Prove that T and S are equivalent if and only if they have the same models (that is, every model of T is a model of S and vice versa).

6. Let φ be a formula in a language L. Consider the conditions (i) there exist terms $t_1, .., t_n$ of L such that $\varphi_x(t_1) \vee .. \vee \varphi_x(t_n)$ is a logically valid formula, and (ii) the formula $\exists x\varphi$ is logically valid. Show that (ii) is a consequence of (ii) but (ii) $\Rightarrow$ (i) is not necessarily true.

   Hint. Let L be $\{P\}$ and let φ be the formula $P(x) \to \forall vP(v)$. Since there are no function symbols, $t_1, .., t_n$ must be variables. However, no disjunction of the form $\bigwedge_i P(x_i) \to \forall vP(v)$ is logically valid.

7. The claim that if φ is open, then condition (i) in the previous exercise is a consequence of (ii) is true and is known as the Hilbert-Ackermann theorem.

This theorem was omitted in the course. Explain that one term would not be enough: if φ is an open formula in L and $\exists x\varphi$ is logically valid, then there may not exist a single term t of L such that $\varphi_x(t)$ is logically valid.

Hint. Pick the language $\{P, F\}$ containing a unary predicate and a unary function and consider the formula $P(x) \vee \neg P(F(x))$. The term t must have the form $F^{(m)}(z)$ where z is a variable.

8. For the formula φ from the above hint find an m and terms $t_1, \ldots, t_n$ of L such that $\varphi_x(t_1) \vee \ldots \vee \varphi_x(t_n)$ is logically valid.

9. Let the language of T be $\{\in\}$ and let its axioms be

   $\forall x\forall y(\forall v(v \in x \equiv v \in y) \to x = y)$,
   $\exists x\forall v\neg(v \in x)$,
   $\forall x\forall y\exists z\forall v(v \in x \vee v = y \to v \in z)$.

   (a) Use finite models to show that $\forall x(x \notin x)$ and $\neg\exists x\forall v(v \in x)$ cannot be proved in T.
   (b) Prove that none of the axioms of T is provable from the remaining two.

10. Let T be a theory with an empty language and no axioms. Describe all models of T. Find an extension S of T formulated in the same (empty) language such that S is consistent and has no finite models.

11. For each of the structures $\langle \mathrm{N}, <\rangle$, $\langle \mathrm{Z}, <\rangle$ a $\langle \mathrm{Q}, <\rangle$ find a sentence that is valid in it but is not valid in the remaining two structures. Can also the structures $\mathbb{R}$ and $\mathbb{Q}$ be distinguished by the validity of some sentence? And what about $\langle \mathrm{Z}, +\rangle$ and $\langle \mathrm{Q}, +\rangle$?

12. Show that the structures $\langle \mathrm{R}, <\rangle$ a $\langle \mathrm{R} - \{0\}, <\rangle$ are not isomorphic. Prove that they are elementarily equivalent.

    Hint. In $\langle \mathrm{R}, <\rangle$, every nonempty subset has the least upper bound. This is not true about $\langle \mathrm{R} - \{0\}, <\rangle$. The two structures are models of the same complete theory.

13. Use Vaught test to show that the theory S from Exercise 10 is complete.

14. Show that if T is equivalent (in the sense of Exercise 5) to some finite set of sentences, then it is equivalent to its own finite subset. Conclude that the theory S from Exercise 10 is not finitely axiomatizable. The theory SUCC is not finitely axiomatizable either.

15. Prove that if a class $\mathcal{C}$ of structures for a language L is axiomatizable and its complement $-\mathcal{C}$ (that is, the class of all structures for L that are not in $\mathcal{C}$) is axiomatizable as well, then both $\mathcal{C}$ and $-\mathcal{C}$ are finitely axiomatizable.

16. Show that the class of all connected graphs, understood as structures for a language with a binary predicate, is not axiomatizable.

17. Consider the class of all structures $\langle D, P\rangle$ for a language with a unary predicate such that both P and $D - P$ are infinite. Prove that this class is axiomatizable. Is it finitely axiomatizable? For which κ is the corresponding theory κ-categorical?

18. Show that the theory whose language is $\{+, 0, \mathrm{S}\}$ and whose axioms are Q1–Q5 is a conservative extension of the theory with axioms Q1–Q3.

19. Use the same method to prove that adding the axioms Q4 a Q5 to SUCC yields a theory that is a conservative extension of SUCC. Prove the same using the following fact: *every consistent extension of a complete theory is a conservative extension.* Explain that this fact is true. Prove that also $\mathrm{Th}(\langle \mathrm{N}, +, 0, \mathrm{s}\rangle)$ is a conservative extension of SUCC. Explain that this last claim cannot be proved using the method from the previous exercise: no expansion of the structure $\langle \mathrm{N}, 0, \mathrm{s}\rangle + \langle \mathrm{Z}, \mathrm{s}\rangle$ is a model of $\mathrm{Th}(\langle \mathrm{N}, +, 0, \mathrm{s}\rangle)$.

    Hint. There is no realization of the symbol $+$ such that the sentences $\forall x \exists y (x = y + y \ \vee\ x = \mathrm{S}(y + y))$ and $\forall x \forall y \forall z (z + x = z + y \ \rightarrow\ x = y)$ are valid.

20. Show that every natural number is a definable element of $\langle \mathrm{N}, <\rangle$. Furthermore, let R be the relation $\{\, [a, b] \,;\, |a - b| = 1 \,\}$. Prove that every natural number is a definable element of $\langle \mathrm{N}, R\rangle$.

21. Let M be the set $\mathrm{N} \cup \{a, b\}$ obtained by adding two new elements a and b to the set of all natural numbers, and consider the prolongation of the successor function $\mathbb{N}$ defined by the conditions $\mathrm{S}(a) = b$ and $\mathrm{S}(b) = a$. Show that there are ways how to define addition and multiplication on M so that the resulting structure is a model of Robinson aritmetic Q.

22. Find out which of the following sentences are provable in Q:

$\forall x (x \leq x)$	$\forall x \forall y (x + y = 0 \ \rightarrow\ x = 0 \ \&\ y = 0)$
$\forall x (x \leq 0 \ \rightarrow\ x = 0)$	$\forall x \forall y (x \leq y \ \equiv\ \mathrm{S}(x) \leq \mathrm{S}(y))$
$\forall x (0 \leq x)$	$\forall x \forall y (x < y \ \rightarrow\ x < \mathrm{S}(y))$
$\forall x (0 \cdot x = 0)$	$\forall x \forall y (\mathrm{S}(x) < y \ \rightarrow\ x < y)$
$\forall x (x \cdot \overline{1} = x)$	$\forall x \forall y (x \cdot y = 0 \ \rightarrow\ x = 0 \ \vee\ y = 0)$
$\forall x \forall y \exists z (x \leq z \ \&\ y \leq z)$	$\forall x (x \leq \overline{1} \ \rightarrow\ x = 0 \ \vee\ x = \overline{1})$
$\forall x \neg (x < x)$	$\forall x \forall y \forall z ((z + y) + x = z + (y + x))$.
$\forall x \forall y (x \leq y \ \rightarrow\ x < y \ \vee\ x = y)$	

Hint. Unprovability can be proved by a suitable choice of operations in the previous exercise, and just two models are sufficient.

23. Use Post's theorem to prove that if $X \subseteq \mathrm{N}^q$ and $Y \subseteq \mathrm{N}^q$ are *RE* sets such that $X \cup Y$ is recursive and $X \cap Y = \emptyset$, then both X and Y are recursive.

24. Prove that if $R \subseteq \mathrm{N}^2$ is an equivalence having only finitely many classes (classes of equivalence) and R is *RE*, then R must be recursive.

    Hint. Let $A_1 \ldots, A_n$ be a list of all classes of equivalence R. Explain in detail the following facts. Every A_i is *RE*. Its complement is *RE* as well, and so Post's theorem is applicable. The relation R can be defined in terms of the list $A_1 \ldots, A_n$ via a recursive condition.

25. Prove that $\mathrm{Thm}(T) = \bigcap\{\, \mathrm{Thm}(S)\ ;\ S \text{ is a complete extension of } T \,\}$ holds for any theory T. Conclude that if T has only finitely many extensions formulated in the language of T and all of them are decidable, then T is decidable. It follows that the theory obtained from DNO by removing the axioms postulating the existence of the greatest and the least individual is decidable.

26. Let T be a recursively axiomatizable extension of Robinson arithmetic formulated in the arithmetic language. such that $\mathbb{N}$ is a model (one of the models) of T. Find out whether the following claims are true.
    (a) If φ and ψ are sentences such that $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.
    (b) if φ and ψ are Σ_1-sentences such that $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.

    Hint. In (a), use Gödel's first incompleteness theorem. In (b) apply the Σ-completeness theorem separately to φ and to ψ.

27. In the same situation find out whether the following claims are true.
    (a) If $\exists x\varphi(x)$ is an arithmetic sentence such that $T \vdash \exists x\varphi(x)$, then there exists a number n such that $T \vdash \varphi(\overline{n})$.
    (b) If $\exists x\varphi(x)$ is an arithmetic sentence such that $T \vdash \exists x\varphi(x)$ and $\varphi \in \Delta_0$, then there exists a number n such that $T \vdash \varphi(\overline{n})$.

    Hint. In (a) pick a formula $\psi(y) \in \Delta_0$ such that $\mathbb{N} \models \forall y\psi(y)$ and $T \nvdash \forall y\psi(y)$. The existence of a formula like that is guaranteed by Gödel's first incompleteness theorem. Then consider the sentence $\exists x\forall y(\psi(y) \vee \neg\psi(x))$.