Exercises to *Modal and Non-Classical Logics* (January 11, 2025)

Excercises

1. Extend the classical propositional calculus GK for the case where the equivalence connective $\equiv$ is considered a basic symbol. An antecedent and a succedent rule are needed, both are binary and they have to be sound and satisfy the *subformula property*. The rules also have to be reversely sound (*invertible*). The completeness and cut-elimination theorems should thus hold for the resulting calculus.

2. Show that if $\Box A \vee \Box B$ is provable in K, then at least one of the formulas A and B is provable in K. Find both a semantic and a proof-theoretic proof.

3. The formula $\Box(p \equiv \neg\Box p) \to (\neg\Box\bot \to \neg\Box p)$ is a modal version of Gödel's first incompleteness theorem: if p is provably equivalent to the claim that it is unprovable, then, under the assumption of consistency, p is unprovable. The formula $\Box(p \equiv \neg\Box p) \to (\neg\Box\bot \to \neg\Box\neg\Box\bot)$ relates to (the proof of) Gödel's second incompleteness theorem. Prove the two formulas in K4. Find out whether replacing $\equiv$ with $\to$ yields formulas provable in K4. Show that $\neg\Box\bot \to \neg\Box\neg\Box\bot$ is not provable in K4, but it is provable in GL.

4. Modify the completeness proof (the decision procedure) for K so that it works for GL. The sequent calculus for GL is thus complete and satisfies the cut-elimination theorem, GL has the *finite model property* and is decidable.

   Hint. For a given sequent $\langle\, \Sigma \Rightarrow \Omega \,\rangle$, let n be the number of formulas $\Box A$ such that $\Box A$ is a subformula of some formula in $\Sigma \cup \Omega$ but $\Box A \notin \Sigma$. By induction on n and by inner induction on the total number of logical symbols in $\Sigma \cup \Omega$, show that $\langle\, \Sigma \Rightarrow \Omega \,\rangle$ either has a cut-free proof, or it has a finite transitive and irreflexive counterexample.

5. Add an antecedent rule to the sequent calculus for K to obtain a calculus for the logic T. Adding the same rule to the calculus for K4 yields a calculus for S4. Show that the sequent and the Hilbert-style calculi for T simulate each other: if A is provable in the Hilbert-style calculus, then $\langle \Rightarrow A \rangle$ is provable in the sequent calculus, and if $\langle\, \Gamma \Rightarrow \Delta \,\rangle$ is provable in the sequent calculus, then $\& \Gamma \to \bigvee \Delta$ is provable in the Hilbert-style calculus. Show that the same simulability result is true also for K4 and for S4.

6. Can S5 prove $\Box(A \vee B) \rightarrow \Box A \vee \Box B$? Here and in all other exercises we assume that & and $\vee$ have higher priority than $\rightarrow$ (and so $\Box A \vee \Box B$ in the above formula does not have to be parenthesized), and also that $\rightarrow$ has higher priority than $\equiv$. Can T prove $\Box(\Box A \vee \Box B) \equiv \Box A \vee \Box B$? Can S4 prove the latter formula?

7. Consider the following proof of the completeness of the sequent calculus for K4. Fill in all missing details. The proof is not based on an analysis of an algorithm making recursive calls to itself; nevertheless, it yields both cut eliminability and the finite model property.

   Assume that $\neg$ is not a basic symbol, i.e. that $\neg A$ is defined as $A \rightarrow \bot$. A sequent $\langle\, \Gamma \Rightarrow \Delta \,\rangle$ is *saturated*, if it satisfies the following conditions:

   - if $B \,\&\, C \in \Gamma$ (or $B \vee C \in \Delta$), then B and C are in Γ (in Δ),
   - if $B \vee C \in \Gamma$ (or $B \,\&\, C \in \Delta$), then B or C is in Γ (in Δ),
   - if $B \rightarrow C \in \Gamma$, then $B \in \Delta$ or $C \in \Gamma$,
   - if $B \rightarrow C \in \Delta$, then $B \in \Gamma$ and $C \in \Delta$.

   Lemma 1 says that *for every sequent $\langle\, \Gamma \Rightarrow \Delta \,\rangle$ that has no cut-free proof in the sequent calculus for K4 there exists a saturated sequent $\langle\, \Pi \Rightarrow \Lambda \,\rangle$ such that $\Gamma \subseteq \Pi$ and $\Delta \subseteq \Lambda$, the sequent $\langle\, \Pi \Rightarrow \Lambda \,\rangle$ has no cut-free proof and is saturated, and each formula in $\Pi \cup \Lambda$ is built up from subformulas of formulas in $\Gamma \cup \Delta$.* Let $\langle\, \Sigma \Rightarrow \Omega \,\rangle$ be (from now on fixed) sequent that has no cut-free proof. Let W be defined as the set of all saturated sequents that are built from subformulas of formulas in $\Sigma \cup \Omega$ and have no cut-free proof. Then $W \neq \emptyset$ (why?) and W is finite. Let a relation R on W be defined as follows: $\langle\, \Gamma \Rightarrow \Delta \,\rangle \; R \; \langle\, \Pi \Rightarrow \Lambda \,\rangle$ if whenever $\Box B \in \Gamma$, then both $\Box B$ and B are in Π. Clearly (?) R is transitive. Then the condition $\langle\, \Gamma \Rightarrow \Delta \,\rangle \Vdash p \Leftrightarrow p \in \Gamma$ defines a truth relation $\Vdash$ on $\langle W, R \rangle$. Lemma 2 says that *the implications*

   $$A \in \Gamma \Rightarrow \langle\, \Gamma \Rightarrow \Delta \,\rangle \Vdash A \quad \textit{and} \quad A \in \Delta \Rightarrow \langle\, \Gamma \Rightarrow \Delta \,\rangle \nVdash A$$

   hold for any modal formula A and every node $\langle\, \Gamma \Rightarrow \Delta \,\rangle$ in W. Prove this lemma by induction on the complexity of A and notice that even the base case deserves some thinking. The resulting model $\langle W, R, \Vdash \rangle$ contains a node at which all formulas in Σ are satisfied and all formulas in Ω are refuted (why?) and thus it is a counterexample for $\langle\, \Sigma \Rightarrow \Omega \,\rangle$.

8. Modify the proof in the preceding exercise for the logic T, and also for the logic S4.

9. Find the characteristic class of the formula $\neg\Box\bot \rightarrow \neg\Box\neg\Box\bot$. Show that K4 enhanced by this formula is not sufficient to prove $\Box(\Box p \rightarrow p) \rightarrow \Box p$.

10. Find the characteristic class of the schema $\Box\Box A \to \Box A$. Show that K plus 5 plus this schema is not sufficient to prove the schema 4.

11. Decide about each of the following classes of frames whether it is the characteristic class of some modal logic:

    (a) $\{\,[W,R]\,;\ R = \emptyset\,\}$,

    (b) $\{\,[W,R]\,;\ R \neq \emptyset\,\}$,

    (c) $\{\,[W,R]\,;\ R = W^2\,\}$,

    (d) $\{\,[W,R]\,;\ \forall x \exists y (x\ R\ y)\,\}$,

    (e) $\{\,[W,R]\,;\ \forall x \neg (x\ R\ x)\,\}$,

    (f) $\{\,[W,R]\,;\ \forall x \forall y \neg (x\ R\ y \Rightarrow \neg (y\ R\ x))\,\}$.

12. Determine which of the following schemas are intuitionistically tautological. Construct the corresponding proofs or counterexamples:

$(A \to B) \to (\neg B \to \neg A)$,	$A \vee \neg A \to (\neg\neg A \to A)$,
$(\neg B \to \neg A) \to (A \to B)$,	$\neg\neg A \vee (\neg\neg A \to A)$,
$(\neg\neg A \to B) \to (\neg B \to \neg A)$,	$(A \to \neg\neg B) \to (\neg\neg A \to \neg\neg B)$,
$(A \to B) \vee (B \to A)$,	$(A \to \neg\neg B) \to \neg\neg(A \to B)$,
$\neg(A \to B) \to \neg B$,	$\neg\neg(A \to B) \to (A \to \neg\neg B)$,
$\neg(A \to B) \to A$,	$A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$,
$\neg(A \to B) \to \neg\neg A$,	$A \vee (B \,\&\, C) \to (A \vee B) \,\&\, (A \vee C)$,
$(A \to B) \to ((\neg A \to B) \to B)$,	$A \to (B \vee C) \to (A \to B) \vee (A \to C)$,
$(A \to B) \to (\neg\neg A \to \neg\neg B)$,	$\neg\neg(A \,\&\, B) \equiv (\neg\neg A \,\&\, \neg\neg B)$,
$\neg A \vee \neg\neg A$,	$\neg\neg(\neg\neg A \to A)$.

13. A propositional formula A is a *Harrop formula* if each occurrence of disjunction in it is in the scope of some negation or in the left scope of some implication. Thus for example, $B \to C$ is a Harrop formula if and only if C is a Harrop formula. Consider $n \geq 1$ and models $K_1, .., K_n$ with roots $b_1, .., b_n$, where *root* is a node from which every (other) node is accessible. Let $\langle W, \leq \rangle$ be obtained from $K_1, .., K_n$ by *amalgamation*, i.e. by taking disjoint copies of the frames of $K_1, .., K_n$ and adding a node a that sees everything (is a root of the new frame). Show that there exists a truth relation $\Vdash$ that extends the truth relation of every K_i and such that the resulting model $K = \langle W, \leq, \Vdash \rangle$ has the property that every Harrop formula is satisfied at a if and only if it is satisfied at all $b_1, .., b_n$.

14. Use the previous exercise to show the following. If a sequent $\langle\, \Gamma \Rightarrow A \vee B\,\rangle$ is intuitionistically tautological (which happens if and only if $\langle\, \Gamma \Rightarrow A, B\,\rangle$

is intuitionistically tautological) and Γ is a set of Harrop formulas, then one of the sequents $\langle \Gamma \Rightarrow A \rangle$ and $\langle \Gamma \Rightarrow B \rangle$ is intuitionistically tautological. This claim, or sometimes also the weaker claim saying that if $A \vee B$ is an intuitionistic tautology, then at least one of the formulas A and B is an intuitionistic tautology, is the *disjunction property* of intuitionistic logic.

15. Prove Glivenko's theorem: a formula $B \to A$ is a (classical) tautology if and only if $B \to \neg\neg A$ is an intuitionistic tautology.

    Hint. Let $B \to A$ be a tautology, let $p_1, .., p_m$ be all atoms that occur in it, and consider a Kripke model $\langle W, \leq, \Vdash \rangle$ and a node $x \in W$ such that $x \Vdash B$. To prove $x \Vdash \neg\neg A$, let $y \geq x$ be given. Construct nodes $y_0, .., y_m$ by recursion: $y_0 = y$, and if $i \neq 0$ and y_{i-1} is already constructed, pick y_i so that $y_i \geq y_{i-1}$ and the atom p_i has the same truth value at all $s \geq y_i$. Put $z = y_m$ and define a classical truth valuation v as follows: $v(p) = 1$ iff $z \Vdash p$. Verify that $v(C) = 1$ iff $\forall s \geq z(s \Vdash C)$ for every subformula C of $B \to A$. Explain that from $v(B \to A) = 1$ it follows that $z \Vdash A$.

16. Consider a function f from propositional formulas to modal formulas defined by the following recursion:

$$f(p) = \Box p \quad \text{if } p \text{ is an atom}, \qquad f(\bot) = \bot,$$
$$f(A \mathbin{\&} B) = f(A) \mathbin{\&} f(B), \qquad f(A \vee B) = f(A) \vee f(B),$$
$$f(A \to B) = \Box(f(A) \to f(B)).$$

    Prove that each formula $\Box f(A)$ is S4-equivalent to $f(A)$. Show, perhaps by induction on the depth of a (cut-free) proof in GJ, that the left-to-right implication in the following condition is true:

$$A \in \mathit{IntTaut} \Leftrightarrow \mathrm{S4} \vdash f(A). \qquad (*)$$

    Prove semantically that $\Leftarrow$ holds as well. This equivalence is described as an *embedding* of intuitionistic logic to S4.

    Hint. Let $K = \langle W, \leq, \Vdash \rangle$ be an intuitionistic Kripke counterexample for A. Then the values of $\Vdash$ on atoms uniquely determine a *modal* valuation $\Vdash_{\mathrm{m}}$. Verify that $x \Vdash A \Leftrightarrow x \Vdash_{\mathrm{m}} f(A)$ for each $x \in W$ and every propositional formula A. Thus $K = \langle W, \leq, \Vdash_{\mathrm{m}} \rangle$ is an S4-counterexample for $f(A)$.

References

[1] P. Blackburn, J. F. A. K. van Benthem, and F. Wolter, editors. *Handbook of Modal Logic.* Elsevier, 2007.

[2] D. H. J. de Jongh and F. Veltman. Intensional Logic. Lecture notes, Philosophy Department, University of Amsterdam, Amsterdam, 1988.