Infinite natural numbers: unwanted phenomenon, or a useful concept?

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Outline

Non-standard model of Peano arithmetic, some history

Definable initial segments of natural numbers

A connection to non-standard analysis

Non-standard model of Peano arithmetic

is a model of PA non-isomorphic to the *standard model* **N**.
That is, a non-standard model is a model containing a number e such that

$$0 < e, \qquad 1 < e, \qquad 2 < e, \qquad \ldots$$

A non-standard model is usually depicted like this:

because there must be many non-standard numbers. The order structure of the model is $\omega + (\omega^* + \omega) \cdot \eta$, where η is a dense linear order without endpoints. However, this is *not a construction*, just a reasoning about the structure once its existence is proved.

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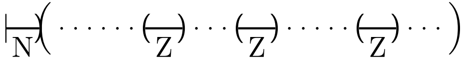

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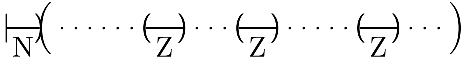

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Some history

T. Skolem (1887–1963) A construction of a non-standard model, 1934.

Ladislav Svante Rieger (1916–1963) A thesis advisor of Petr Hájek, inventor of Rieger-Nishimura lattice (1949), worked with non-standard models of set theory.

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Definable cuts

The non-standard models defined above may or may not be elementarily equivalent with the standard model, but they do satisfy induction. Hájek: every model of PA thinks about itself that it is standard.

Definition

A formula $J(x)$ is a *cut* in a theory T if $T \vdash J(0)$ and $T \vdash \forall x(J(x) \rightarrow J(x+1))$.
We informally write $J = \{\, x \,;\, J(x) \,\}$.

Example

In Robinson arithmetic Q, take $J(x) \equiv 0 + x = x$. (Note that $\forall x(x + 0 = x)$ and $\forall x \forall y(y + \mathrm{S}(x) = \mathrm{S}(y + x))$ are axioms, but $\forall x(0 + x = x)$ is unprovable in Q).

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Truth relations in Gödel-Bernays set theory

	$\dots$	e_1	$\dots$	$\dots$	e_2	$\dots$	$\dots$	$\dots$
$\vdots$		$\vdots$			$\vdots$			
φ_1	$\dots$	1	$\dots$	$\dots$	1	$\dots$	$\dots$	$\dots$
$\vdots$		$\vdots$			$\vdots$			
φ_2	$\dots$	0	$\dots$	$\dots$	1	$\dots$	$\dots$	$\dots$
$\vdots$		$\vdots$			$\vdots$			
$\varphi_1 \mathbin{\&} \varphi_2$	$\dots$	0	$\dots$	$\dots$	1	$\dots$	$\dots$	$\dots$

Definition (in GB)

A *truth relation* on n is a relation between set formulas (formulas of ZF set theory) having Gödel numbers less than n, and evaluations of free variables, satisfying the Tarski's conditions:

$$[\varphi_1 \mathbin{\&} \varphi_2, e] \in R \Leftrightarrow [\varphi_1, e] \in R \text{ and } [\varphi_2, e] \in R, \quad \text{etc.},$$

$$[\forall x \varphi, e] \in R \Leftrightarrow \text{for each set } a, [\varphi, e(x/a)] \in R, \quad \text{etc.},$$

where $e(x/a)$ evaluates x by a, and is identical to e everywhere else.

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	...	e_1	...	...	e_2	...	...	...
$\vdots$		$\vdots$			$\vdots$			
φ_1	...	1	...	...	1	...	...	...
$\vdots$		$\vdots$			$\vdots$			
φ_2	...	0	...	...	1	...	...	...
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Occupable numbers

Lemma If both R_1 and R_2 are truth relations on n then $R_1 = R_2$.

Definition $\mathrm{Ocp} = \{\, n\,;\, \exists R(R \text{ is a truth relation on } n) \,\}$.

Lemma $0 \in \mathrm{Ocp}$. If $n \in \mathrm{Ocp}$ then $n + 1 \in \mathrm{Ocp}$.

Theorem $\mathrm{GB} \nvdash \forall n(n \in \mathrm{Ocp})$.

Some consequences and remarks

- There are reasonably defined formulas of GB that do not determine a class.
- The Tarski's definition of first-order semantics is not absolute; it is developed in some sort of set theory, and it needs some strength of axioms to work.
- A connection to Gödel 2nd theorem: $\mathrm{GB} \vdash \mathrm{Con}^{\mathrm{Ocp}}(\mathrm{ZF})$.

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The concept of infinitesimals

If x is a non-standard number then $1/x$ is infinitely small, i.e. it is infinitesimal.

Example definition

A function f is continuous in a if, for every infinitesimal $\mathrm{d}x$, the value $f(x+\mathrm{d}x)$ is infinitely close to $f(x)$.
That is, if $|f(x+\mathrm{d}x)-f(x)|$ is infinitesimal.

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References

T. Skolem. Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzhlbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. *Fundamenta Mathematicae*, 23:150–161, 1934.

R. M. Solovay. Interpretability in set theories. Unpublished letter to P. Hájek, Aug. 17, 1976, `http://www.cs.cas.cz/~hajek/RSolovayZFGB.pdf`, 1976.

P. Vopěnka and P. Hájek. Existence of a generalized semantic model of Gödel-Bernays set theory. *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.*, XXI(12), 1973.

P. Vopěnka. *Mathematics in the Alternative Set Theory*. Teubner, Leipzig, 1979.